On a Kaehler hypersurface of a complex space form with the recurrent second fundamental form

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Abstract. The purpose of the present paper is to prove: Let M be a Kaehler hypersurface of a complex space form $\tilde{M}(c)$ of constant holomorphic sectional curvature c. Then the following conditions are equivalent: (A) M has the recurrent second fundamental form. (B) M has the birecurrent second fundamental form. (C) the derivation $R(X,Y)$ and the second fundamental form are commutative. (D) M is totally geodesic in $\tilde{M}(c)$.

M.S.C. 2000: 53C40, 53B25.
Key words: Kaehler hypersurface, the recurrent second fundamental form, totally geodesic.

1 Introduction

Let $\tilde{M}(c)$ be a complex space form (i.e. a complete, simply-connected Kaehler manifold) of the complex dimension $n+1$ with constant holomorphic sectional curvature c. If c is positive, then $\tilde{M}(c)$ is the complex projective space $P^{n+1}(c)$ with the Fubini-Study metric of constant holomorphic sectional curvature c. If c is negative, then $\tilde{M}(c)$ is the complex hyperbolic space $D^{n+1}(c)$ with the Bergman metric of constant holomorphic sectional curvature c. If c is zero, then $\tilde{M}(c)$ is the complex Euclidean space C^{n+1} (See [1]). Let M be a connected manifold of complex dimension $n(\geq 2)$ isometrically and holomorphically immersed in $\tilde{M}(c)$. Then we call M a Kaehler hypersurface of $\tilde{M}(c)$. Let R and ∇ be the curvature tensor of M and the covariant differentiation in M, respectively. Furthermore, $\tilde{S}$ and S denote the Ricci tensors of $\tilde{M}(c)$ and M, respectively. Nomizu and Smyth [3] proved: The following conditions are equivalent: (i) The normal connection of M is trivial. (ii) $S=\tilde{S}$ on M. (iii) $S=0$ on M. (iv) $c=0$ and M is totally geodesic in $\tilde{M}(c)$.

Now, let A be the second fundamental form of M in $\tilde{M}(c)$ defined on a neighborhood of each point of M. M is called a Kaehler hypersurface with the recurrent second fundamental form (resp. the birecurrent second fundamental form) if there exists a 1-form α (resp. a covariant tensor field α of order 2) such that the second fundamental form A of M satisfies $(\nabla_X A)Y=\alpha(X)AY$ (resp. $(\nabla^2_{X,Y}A)Z=$

Differential Geometry - Dynamical Systems, Vol.10, 2008, pp. 221-225.

$\nabla_X\nabla_Y A - \nabla_{\nabla_X Y} A)Z = \alpha(X,Y)AZ)$ for any X, Y and $Z \in T_xM$ (See [6]).

The purpose of the paper is to prove the following:

Theorem *Let M be a Kaehler hypersurface in a complex space form $\tilde{M}(c)$. The following conditions are equivalent:*
(A) *M has the recurrent second fundamental form.*
(B) *M has the birecurrent second fundamental form.*
(C) $(R(X,Y)A)Z = 0$ *on a neighborhood of each point of M.*
(D) *M is totally geodesic in $\tilde{M}(c)$.*

2 Preliminaries

Let M be a connected manifold of complex dimension $n(\geq 2)$ isometrically and holomorphically immersed in a complex space form $\tilde{M}(c)$ of complex dimension $n+1$. Then we call M a Kaehler hypersurface of $\tilde{M}(c)$. The complex structure $\tilde{J}$ and the Kaehler metric $\tilde{g}$ of $\tilde{M}(c)$ induce a complex structure J and Kaehler metric g on M, respectively. Let ∇ (resp. $\tilde{\nabla}$) denotes the covariant differentiation in M (resp. $\tilde{M}(c)$). Extend ξ to a normal vector field defined in a neighborhood U of $x \in M$ and define $-AX$ to be the tangential component of $\tilde{\nabla}_X\xi$ for $X \in T_xM$. AX depends only on ξ at x and X, and we call A the second fundamental form. Let R be the curvature tensor of M and X,Y and Z be the tangent vectors on M. Then we have the following relationships(See [2], [3], [4] and [5]):

$$
\begin{aligned}
&(2.1) & \tilde{\nabla}_X Y &= \nabla_X Y + g(AX,Y)\xi + g(JAX,Y)\tilde{J}\xi,\\
&(2.2) & \tilde{\nabla}_X \xi &= -AX + s(X)\tilde{J}\xi,\\
&(2.3) & AJ &= -JA,\\
&(2.4) & R(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX\\
&&&\quad -g(JX,Z)JY + 2g(X,JY)JZ\} + g(AY,Z)AX\\
&&&\quad -g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY,\\
&(2.5) & (\nabla_X A)Y &- s(X)JAY = (\nabla_Y A)X - s(Y)JAX,
\end{aligned}
$$

where s denotes the connection form. Furthermore, we call (2.4) *Gauss equation* and (2.5) *Codazzi equation*, respectively.

Here we recall the definition of the recurrent second fundamental form and the birecurrent second fundamental form. The second fundamental form A is called the recurrent second fundamental form if there exists a 1-form α such that $\nabla A = A \otimes \alpha$. Similarly, the second fundamental form A is called the birecurrent second fundamental form if there exists a covariant tensor field α of order 2 such that $\nabla^2 A = A \otimes \alpha$.

Then we have the following propositions.

Proposition 1. *If M has the recurrent second fundamental form, then M has the birecurrent second fundamental form.*

Proof. Suppose that M is a Kaehler hypersurface with the recurrent second fundamental form. Here we consider the equation:

$$(\nabla^2_{X,Y}A)Z = (\nabla_X(\nabla_Y A))Z - (\nabla_{\nabla_X Y}A)Z.$$

It yealds

$$(\nabla^2_{X,Y}A)Z = \nabla_X((\nabla_Y A)Z) - (\nabla_Y A)\nabla_X Z - (\nabla_{\nabla_X Y}A)Z.$$

From the assumption of Proposition we obtain

$$\begin{aligned}(\nabla^2_{X,Y}A)Z &= \nabla_X(\alpha(Y)AZ) - \alpha(Y)A\nabla_X Z + \alpha(\nabla_X Y)AZ \\ &= X(\alpha(Y))AZ + \alpha(Y)(\nabla_X A)Z + \alpha(Y)A\nabla_X Z \\ &\quad -\alpha(Y)A\nabla_X Z - \alpha(\nabla_X Y)AZ \\ &= (X(\alpha(Y)) + \alpha(Y)\alpha(X) - \alpha(\nabla_X Y))AZ\end{aligned}$$

for any tangent vectors X, Y and Z. This equation means M is a Kaehler hypersurface with the birecurrent second fundamental form. □

Proposition 2. *If M has the birecurrent second fundamental form, then M satisfies $(R(X,Y)A)Z = 0$.*

Proof. Assume that M is a Kaehler hypersurface with the birecurent second fundamental form. We consider the following equation:

$$(R(X,Y)A^2)Z = (R(X,Y)A)AZ + A(R(X,Y)A)Z.$$

Noting that

$$(R(X,Y)A)Z = (\nabla^2_{X,Y}A)Z - (\nabla^2_{Y,X}A)Z,$$

and using the assumption of Proposition, we get

$$(R(X,Y)A^2)Z = 2(\alpha(X,Y) - \alpha(Y,X))A^2 Z. \tag{2.6}$$

From (2.6) and the commutativity of the trace and the covariant differentiation we have

$$\begin{aligned}&2(\alpha(X,Y) - \alpha(Y,X))\mathrm{trace}A^2 \\ =\ &(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})\mathrm{trace}A^2 \\ =\ &(XY - YX - [X,Y])\mathrm{trace}A^2 \\ =\ &0.\end{aligned}$$

Therefore we can see that $\alpha(X,Y) = \alpha(Y,X)$ or trace $A^2 = 0$. In both cases we have

$$(R(X,Y)A)Z = 0 \tag{2.7}$$

for any vector fields X, Y and Z. □

3 Proof of Theorem

Theorem *Let M be a Kaehler hypersurface in a complex space form $\tilde{M}(c)$ of constant holomorphic sectional curvature c. The following conditions are equivalent:*
(A) *M has the recurrent second fundamental form.*
(B) *M has the birecurrent second fundamental form.*
(C) *$(R(X,Y)A)Z = 0$ on a neighborhood of each point of M.*
(D) *M is totally geodesic in $\tilde{M}(c)$.*

Proof. From the assumption of Theorem and the results of Propositions 1-2 we see that they hold (A) $\Rightarrow$ (B) and (B) $\Rightarrow$ (C). Assume that $(R(X,Y)A)Z = 0$. Equation (2.7) implies that

$$R(X,Y)AZ - AR(X,Y)Z = 0. \tag{3.1}$$

If we suppose that rank of $A \geq 2$, then we can choose orthonormal vectors X and Y which satisfy $AX = \lambda X (\lambda \neq 0)$ and $AY = \mu Y$. Then the equation (3.1) reduces to

$$R(Y,JY)AX - AR(Y,JY)X \quad = \quad (\lambda I - A)R(Y,JY)X = 0$$

We compute $R(Y,JY)X$. From (2.4) we have

$$\begin{aligned} R(Y,JY)X &= \frac{c}{4}\{g(JY,X)Y - g(Y,X)JY + g(J^2Y,X)JY - g(JY,X)J^2Y \\ &\quad +2g(Y,J^2Y)JX\} + g(AJY,X)AY - g(AY,X)AJY \\ &\quad +g(JAJY,X)JAY - g(JAY,X)JAJY \\ &= -\frac{c}{2}JX. \end{aligned}$$

Thus we obtain

$$\lambda c = 0.$$

Hence we see that if M is not totally geodesic, then $c = 0$. Thus we obtain that if $c \neq 0$, then M is totally geodesic.

If $c = 0$ and rank of $A \geq 2$, then we use the next equation. The equation (3.1) reduces to

$$R(X,JX)(AJX) - AR(X,JX)JX = (-\lambda I - A)R(X,JX)JX = 0 \tag{3.2}$$

We compute $R(X,JX)JX$. From (2.4) we have

$$\begin{aligned} R(X,JX)JX &= g(AJX,JX)AX - g(AX,JX)AJX \\ &\quad +g(JAJX,JX)JAX - g(JAX,JX)JAJX \\ &= -2\lambda^2 X. \end{aligned} \tag{3.3}$$

Combining (3.2) with (3.3) , we get

$$\lambda^3 = 0, \tag{3.4}$$

which is a contradiction. Thus we see that M is also totally geodesic in the case of $c = 0$. Assme that M is totally geodesic. Then we know that M has the recurrent second fundamental form. This proves theorem.

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