

# A characterization of minimal surfaces in the Heisenberg group $\mathbb{H}^3$

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**Abstract.** In this paper we establish equations for the Gaussian Curvature and for the Laplacian of a minimal surface in the Heisenberg Group  $\mathbb{H}^3$ . Using Gauss-Codazzi equations we prove that the contact angle ( $0 < \beta < \frac{\pi}{2}$ ) is constant for compact minimal surfaces in  $\mathbb{H}^3$ . Also, using Codazzi equation we give a congruence theorem for minimal surfaces in  $\mathbb{H}^3$ .

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## 1 Introduction

The study of minimal surfaces played a formative role in the development of mathematics over the last two centuries. Today, minimal surfaces appear in various guises in diverse areas of mathematics, physics, chemistry and computer graphics, but have also been used in differential geometry to study basic properties of immersed surfaces in contact manifolds. We mention for example the two papers, [4] and [7], where the authors classify Legendrian minimal surfaces in  $S^5$  with constant Gaussian curvature. Also, interesting methods of constructions of minimal surfaces in  $\mathbb{H}^3$  are given in [2], [3], and [6].

The scope of this note is to use a geometric invariant in order to study immersed surfaces in the three dimensional Heisenberg group  $\mathbb{H}^3$ . This invariant (the contact angle ( $\beta$ )) is the complementary angle between the contact distribution and the tangent space of the surface.

We show that the Gaussian curvature  $K$  of a minimal surface in  $\mathbb{H}^3$  with contact angle  $\beta$  is given by:

$$K = -3 \sin \beta - |\nabla \beta + e_1|^2$$

Moreover, the contact angle satisfies the following Laplacian equation

$$\Delta(\beta) = -\tan(\beta)(|\nabla \beta + 2e_1|^2 + \frac{\cos^2 \beta}{\sin \beta})$$

where  $e_1$  is the characteristic field defined in section 2 and introduced by Bennequin, in [1] pages 190 - 206.

Using the equations of Gauss and Codazzi, we have proved the following two theorems:

**Theorem 1.** *The contact angle is constant for compact minimal surfaces in  $\mathbb{H}^3$  with contact angle  $(0 < \beta < \frac{\pi}{2})$ .*

**Theorem 2.** *Gaussian curvature is non-positive for minimal surfaces in  $\mathbb{H}^3$  with contact angle  $(0 < \beta < \pi)$ .*

More in general, we have the following congruence result:

**Theorem 3.** *Consider  $S$  a Riemannian surface,  $\mathbf{e}$  a vector field on  $S$ , and  $\beta : S \rightarrow ]0, \frac{\pi}{2}[$  a function over  $S$  that verifies the following equation:*

$$\Delta(\beta) = -\tan(\beta)(|\nabla\beta + 2e|^2 + \frac{\cos^2\beta}{\sin\beta})$$

*then there exist one minimal immersion of  $S$  into  $\mathbb{H}^3$  such that  $\mathbf{e}$  is the characteristic vector field, and  $\beta$  is the contact angle of this immersion.*

## 2 The contact angle for minimal surfaces in the Heisenberg Group $\mathbb{H}^3$

Consider the following objects:

- The 3-dimensional Heisenberg group  $\mathbb{H}^3$  can be viewed as  $R^3$  endowed with the metric

$$(2.1) \quad dx_1^2 + dx_2^2 + \left(\frac{1}{2}(x_2 dx_1 - x_1 dx_2) + dx_3\right)^2$$

- the Reeb vector field in  $\mathbb{H}^3$ , given by:  $\xi(z) = iz$ ;
- the contact distribution in  $\mathbb{H}^3$ , which is orthogonal to  $\xi$ :

$$\Delta_z = \{v \in T_z\mathbb{H}^3 \mid \langle \xi, v \rangle = 0\}.$$

We identify  $\mathbb{H}^3$  as  $\mathbb{R}^3$  given by the following frame:

$$(2.2) \quad \begin{aligned} f_1 &= \frac{\delta}{\delta x_1} + x_2 \frac{\delta}{\delta x_3} \\ f_2 &= \frac{\delta}{\delta x_2} - x_1 \frac{\delta}{\delta x_3} \\ f_3 &= \frac{\delta}{\delta x_3} \end{aligned}$$

Let  $(\omega^1, \omega^2, \omega^3)$  be the coframe associated to  $(f_1, f_2, f_3)$ . Thus, from equations (2.2), it follows that:

$$(2.3) \quad \begin{aligned} \omega^1 &= dx_1 \\ \omega^2 &= dx_2 \\ \omega^3 &= dx_3 - x_2 dx_1 + x_1 dx_2 \end{aligned}$$

Therefore, we get:

$$(2.4) \quad \begin{aligned} d\omega^1 - \omega^3 \wedge \omega^2 - \omega^2 \wedge \omega^3 &= 0 \\ d\omega^2 + \omega^3 \wedge \omega^1 + \omega^1 \wedge \omega^3 &= 0 \\ d\omega^3 + \omega^2 \wedge \omega^1 - \omega^1 \wedge \omega^2 &= 0 \end{aligned}$$

It follows that:

$$(2.5) \quad \begin{aligned} \omega_2^1 &= -\omega^3; \omega_3^2 = \omega^1; \omega_1^3 = \omega^2; \\ d\omega_2^1 &= -2\omega^1 \wedge \omega^2; d\omega_3^2 = 0; d\omega_1^3 = 0 \end{aligned}$$

We have the following curvature forms:

$$(2.6) \quad \begin{aligned} \Omega_2^1 &= d\omega_2^1 + \omega_3^1 \wedge \omega_2^3 = -3\omega^1 \wedge \omega^2 \\ \Omega_3^2 &= d\omega_3^2 + \omega_1^2 \wedge \omega_3^1 = \omega^2 \wedge \omega^3 \\ \Omega_1^3 &= d\omega_1^3 + \omega_2^3 \wedge \omega_1^2 = \omega^3 \wedge \omega^1 \end{aligned}$$

Therefore:

$$(2.7) \quad \begin{aligned} \Omega_2^1(e_1, e_2) &= -3 \\ \Omega_3^2(e_2, e_3) &= 1 \\ \Omega_1^3(e_3, e_1) &= 1 \end{aligned}$$

Let now  $S$  be an immersed orientable surface in  $\mathbb{H}^3$ .

**Definition 1.** *The Contact angle  $\beta$  is the complementary angle between the contact distribution  $\Delta$  and the tangent space  $TS$  of the surface.*

Let  $(e_1, e_2)$  be a local frame of  $TS$ , where  $e_1 \in TS \cap \Delta$ . Then  $\cos \beta = \langle \xi, e_2 \rangle$ . Let  $e_1$  be an unitary vector field in  $TS \cap \Delta$ , where  $\Delta$  is the contact distribution. Thus follows that:

$$(2.8) \quad \begin{aligned} e_1 &= f_1 \\ e_2 &= \sin(\beta) f_2 + \cos(\beta) f_3 \\ e_3 &= -\cos(\beta) f_2 + \sin(\beta) f_3 \end{aligned}$$

where  $\beta$  is the angle between  $f_3$  and  $e_2$ ,  $(e_1, e_2)$  are tangent to  $S$  and  $e_3$  is normal to  $S$

### 3 Equations for the Gaussian curvature and for the Laplacian of a minimal surface in $\mathbb{H}^3$

In this section, we will give formulas for the Laplacian and for the Gaussian curvature of a minimal surface immersed in  $\mathbb{H}^3$ .

The reader can see [5], and [8] for further details.

Let  $(\theta^1, \theta^2, \theta^3)$  be the coframe associated to  $(e_1, e_2, e_3)$ .

We know that  $\theta^3 = 0$  on  $S$ , then we obtain the following equation:

$$(3.1) \quad \sin(\beta) w^3 = \cos(\beta) w^2$$

we have also

$$\begin{aligned} w^2 &= \sin \beta \theta^2 \\ w^3 &= \cos \beta \theta^2 \end{aligned}$$

It follows from (2.3) that:

$$\begin{aligned} d\theta^1 &+ \sin(\beta)(w_2^1 - \cos(\beta)\theta^2) \wedge \theta^2 = 0 \\ d\theta^2 &+ \sin(\beta)(w_1^2 + \cos(\beta)\theta^2) \wedge \theta^1 = 0 \\ d\theta^3 &= d\beta \wedge \theta^2 - \cos(\beta)w_2^1 \wedge w^1 + (1 + \sin^2(\beta))\theta^1 \wedge \theta^2 \end{aligned}$$

Therefore the connection form of  $S$  is given by

$$(3.2) \quad \theta_2^1 = \sin(\beta)(w_2^1 - \cos(\beta)\theta^2)$$

Differentiating  $e_3$  at the basis  $(e_1, e_2)$ , we have fundamental second forms coefficients

$$De_3 = \theta_3^1 e_1 + \theta_3^2 e_2$$

where

$$\begin{aligned} \theta_3^1 &= -\cos(\beta)w_2^1 - \sin^2(\beta)\theta^2 \\ \theta_3^2 &= d\beta + \theta^1 \end{aligned}$$

It follows from  $d\theta^3 = 0$ , that

$$(3.3) \quad w_2^1(e_2) = -\frac{\beta_1}{\cos \beta} - \frac{(1 + \sin^2 \beta)}{\cos \beta}$$

onde  $d\beta(e_1) = \beta_1$ .

The condition of minimality is equivalent to the following equation

$$\theta_1^3 \wedge \theta^2 - \theta_2^3 \wedge \theta^1 = 0$$

we have

$$(3.4) \quad w_2^1(e_1) = \frac{\beta_2}{\cos(\beta)}$$

where  $d\beta(e_2) = \beta_2$ .

It follows from (3.2), (3.3) and (3.4),

$$\begin{aligned} \theta_2^1 &= \tan(\beta)(\beta_2\theta^1 - (\beta_1 + 2)\theta^2) \\ \theta_3^1 &= -\beta_2\theta^1 + (\beta_1 + 1)\theta^2 \\ \theta_3^2 &= (\beta_1 + 1)\theta^1 + \beta_2\theta^2 \end{aligned}$$

If  $J$  is the complex structure of  $S$  we have  $Je_1 = e_2$  e  $Je_2 = -e_1$ .

Using  $J$ , the forms above reduce to:

$$(3.5) \quad \begin{aligned} \theta_2^1 &= \tan \beta (d\beta \circ J - 2\theta^2) \\ \theta_3^1 &= -d\beta \circ J + \theta^2 \\ \theta_3^2 &= d\beta + \theta^1 \end{aligned}$$

Gauss equation is

$$d\theta_1^2 = \Omega_1^2 + \theta_2^3 \wedge \theta_1^3$$

which implies

$$(3.6) \quad d\theta_1^2 = -(|\nabla\beta|^2 + 2\beta_1) - 1 - 3 \sin \beta (\theta^2 \wedge \theta^1)$$

where:

$$(3.7) \quad \Omega_1^2(e_2, e_1) = -3 \sin \beta$$

and therefore

$$(3.8) \quad K = -3 \sin \beta - |\nabla\beta + e_1|^2$$

Codazzi equations are

$$\begin{aligned} d\theta_1^3 + \theta_2^3 \wedge \theta_1^2 &= \Omega_1^3 \\ d\theta_2^3 + \theta_1^3 \wedge \theta_2^1 &= \Omega_2^3 \end{aligned}$$

A straightforward computation in the first equation gives (1.1) and the second equation is always verified.

$$(3.9) \quad \Delta(\beta) = -\tan(\beta)((\beta_1 + 2)^2 + \beta_2^2) + \frac{\cos^2 \beta}{\sin \beta}$$

Or

$$\Delta(\beta) = -\tan(\beta)(|\nabla\beta + 2e_1|^2 + \frac{\cos^2 \beta}{\sin \beta})$$

where:

$$(3.10) \quad \begin{aligned} \Omega_1^3(e_1, e_2) &= -\cos \beta \\ \Omega_2^3(e_1, e_2) &= -\sin \beta \cos \beta \end{aligned}$$

## 4 Main results

### 4.1 Proof of the Theorem 1

For  $0 \leq \beta < \frac{\pi}{2}$ , we have  $\tan \beta \geq 0$ , hence  $\Delta(\beta) \leq 0$  and using that  $S$  is a compact surface, we conclude by Hopf's Lemma that  $\beta$  is constant, which prove the Theorem 1.  $\square$

### 4.2 Proof of the Theorem 2

Suppose that  $(0 < \beta < \pi)$ , it follows from (3.8) that the Gaussian curvature of  $S$  is  $K < 0$ , which prove the Theorem 2.  $\square$

### 4.3 Proof of the Theorem 3

Let  $S$  be an orientable surface in  $\mathbb{H}^3$ , and let  $\mathbf{e}$  be a unit vector field on  $S$ .

We choose an orthonormal positive basis  $(e_1, e_2)$  with  $e_1 = e$ , and let  $(\theta^1, \theta^2)$  be a coframe on  $S$ .

For each function  $\beta : S \rightarrow ]0, \frac{\pi}{2}[$  that satisfies the following Laplacian equation:

$$\Delta(\beta) = -\tan(\beta)(|\nabla\beta + 2e_1|^2 + \frac{\cos^2 \beta}{\sin \beta})$$

We define the following fundamental second form:

$$\begin{aligned} \theta_1^3 &= (d\beta + \theta^1) \circ J \\ \theta_2^3 &= - (d\beta + \theta^1) \end{aligned} \tag{4.1}$$

Now, the proof follows from Gauss-Codazzi equations.  $\square$

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