On constant isotropic submanifold by generalized null cubic

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Abstract. In this paper we shall be concerned with curves in an Lorentzian submanifold M_1, and give a characterization of each constant isotropic immersion by generalized null cubic with constant curvature on the Lorentzian submanifold.

M.S.C. 2000: 53B25,53C42, 53C40.
Key words: constant isotropic immersion, generalized null cubic, Frenet frame.

1 Introduction

We review fundamental equations in submanifold theory. Let M_1 be an $n-$dimensional Lorentzian submanifold in an $(n+p)-$ dimensional indefinite-Riemannian manifold $\bar{M}_\alpha$ and $f : M_1 \to \left(\bar{M}_\alpha, g\right)$ be a isometric immersion. Throughout this paper we will identify a vector X of M_1 with a vector $f_*(X)$ of $\bar{M}_\alpha$. We denote the induced metric on M_1 by the same letter g.

If $\bar{\nabla}$ and ∇ denotes the metric connection of $\bar{M}_\alpha$ and the induced connection on M_1 from g respectively then we have Gauss' formula

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{1.1}$$

where X and Y are tangent vector fields of M_1 and $h(X,Y)$ is the second fundamental form of the immersion. For a vector field N normal to M_1, we can write

$$\bar{\nabla}_X N = -A^N X + \nabla^\perp_X N, \tag{1.2}$$

where $\nabla^\perp$ is the covariant differentiation with respect to the induced connection in the normal bundle $N(M_1)$, A_N is the shape operator of M_1 and satisfies the relation

$$g(A_N(X), Y) = g(h(X,Y), N).$$

For indefinite-Riemannian manifod we refer to O'Neill [6].

Let $\tilde{\nabla}$ be a covariant differentiation induced on the $T(M_1) \oplus N(M_1)$. The covariant differentiation of the second fundamental form h is denoted by $\tilde{\nabla} h$ which is $N(M_1)$ valued tensor field of type $(0,3)$. $\tilde{\nabla} h$ is also defined by

$$\left(\tilde{\nabla}_X h\right)(Y,Z) = \nabla^\perp_X (h(Y,Z)) - h(\nabla_X Y, Z) - h(X, \nabla_X Z). \tag{1.3}$$

Differential Geometry - Dynamical Systems, Vol.10, 2008, pp. 249-253.

The mean curvature vector field of M_1 is defined by

$$H=\frac{1}{n}\sum_{i=1}^{n}g\left(e_i,e_i\right)h\left(e_i,e_i\right),$$

where $\{e_1,...e_n\}$ is an orthonormal frame at each point of M_1. H is said to be parallel if $\nabla^{\perp}H=0$ holds. If the second fundamental form h satisfies

$$h\left(X,Y\right)=g\left(X,Y\right)H$$

for any tangent vector field X and Y of M_1, then $M_1$is called totally umbilical submanifold. If the second fundamental form vanishes identically on M_1 then M_1 is said to be totally geodesic. A totally umbilical submanifold with the parallel mean curvature vector field is said to be an extrinsic sphere.

An isometric immersion $f:M_1\rightarrow\bar{M}_\alpha$ is said to be *isotropic* at $p\in M_1$ if $\|h\left(X,X\right)\|/\|X\|^2\left(=\lambda\left(p\right)\right)$ does not depend on the choice of $X\in T_p\left(M\right)$. If the immersion is isotropic at every point, then the immersion is said to be isotropic. When the function λ is constant on M_1, we call M_1 a constant $(\lambda-)$ isotropic submanifold. Note that totally umbilic immersion is isotropic , but the converse is not true.

There are many examples of isotropic submanifolds which are not totally umbilic in standard spheres [7].

The following is well known [5].

Lemma 1. *Let f be isometric immersion of M into $\left(\bar{M},g\right)$. Then f is isotropic at $p\in M$ if and only if the second fundamental form h of f satisfies $g\left(h\left(u,u\right),h\left(u,v\right)\right)=0$ for an orthogonal pair u ,$v\in T_p\left(M\right)$.*

Theory of isotropic submanifolds is one of the most interesting objects in differential geometry. In particular Maeda in [4] obtained that the following theorem by studying the first curvature of each circle on the Riemannian submanifold.

Theorem 1. *Let M^n be an n-dimensional connected Riemannian submanifold of an $(n+p)-$ dimensional Riemannian manifold $\tilde{M}^{n+p}$through an isometric immersion f. Then the following are equivalent:*

(i) M^n is a constant $(\lambda-)$ isotropic submanifold of $\tilde{M}^{n+p}$.

(ii) There exist $K>0$ satisfying that for each circle γ of curvature K on the submanifold M^n the curve $f\circ\gamma$ in $\tilde{M}^{n+p}$ has constant first curvature K_1 along this curve.

In this paper we are interested in generalized null cubic in an Lorentzian submanifold M_1 and give a characterization that these submanifolds are constant $(\lambda-)$ isotropic in terms of constant first curvature K_1 along this curve.

The starting point to study Lorentzian submanifold consisting of investigate curves lies on the submanifold. In this sense, circles or helices in semi-Riemannian submanifold has been studied by several authors (see,for example [1], [2],[3]).

Here we recall the definition of circles, helix and generalized null cubic in Lorentzian submanifold.

A regular curve in Lorentzian manifold M_1 is a smooth mapping $\gamma:I\rightarrow M_1$. If $\gamma=\gamma\left(t\right)$ is a spacelike or timelike curve, we can reparametrize it such that $g\left(\acute{\gamma}\left(t\right),\acute{\gamma}\left(t\right)\right)=$

ε, where $\varepsilon = +1$ if γ spacelike and $\varepsilon = -1$ if γ is timelike, respectively. We assume that the spacelike or timelike curve $\gamma(t)$ has an arc length parametrization.

Let $\gamma = \gamma(t)$ be a timelike curve in M_1. By $k_j(t)$, we denote the j-th curvature of γ. If $k_j(t) = 0$ for $j > 2$ and if the principal vector field Y and the binormal vector field Z are spacelike, then we have the following Frenet formulas along γ [2].

$$\begin{aligned} \acute{\gamma}(t) &= X \\ \nabla_X X &= k_1(t) Y \\ \nabla_X Y &= k_1(t) X + k_2(t) Z \\ \nabla_X Z &= -k_2(t) Y, \end{aligned} \tag{1.4}$$

where ∇ denotes the covariant differentiation in M_1. A curve γ is called a circle if $k_2(t) = 0$ and $k_1(t)$ is positive constant along γ. If both $k_1(t)$ and $k_2(t)$ are positive constant along γ then γ is called helix.

We consider a null curve in a Lorentzian manifold M_1.By a Cartan Frame $\{X, Y, Z\}$ of a null curve γ we mean a family of vector fields $X = x(t)$, $Y = Y(t)$, $Z = Z(t)$ along the curve γ satisfying the following conditions

$$\begin{gathered} \acute{\gamma}(t) = X, \ \ g(X, X) = g(Y, Y) = 0, \\ g(X, Y) = -1, \ g(X, Z) = g(Y, Z) = 0, \ g(Z, Z) = 1 \\ \nabla_X X = k_1(t) Z, \ \nabla_X Y = k_2(t) Z, \ \nabla_X Z = k_2(t) X + k_1(t) Y, \end{gathered} \tag{1.5}$$

where $k_1(t)$ and $k_2(t)$ are functions defined along the curve γ. If $k_1(t)$ and $k_2(t)$ are positive along γ then, we call the curve γ a Cartan framed null curve with constant curvatures. On the definition of the Cartan frame of a null curve γ, if $k_2(t) = 0$ then γ is called a generalized null cubic. Moreover if k_1 is constant, then γ is called a generalized null cubic with constant curvature. For any point p of a Lorentzian manifold, any constants k_1 and k_2, and any Cartan frame$\{X, Y, Z\}$ at p, there exist locally a Cartan framed null curve γ whit constant curvatures passing through p with velocity vector $\acute{\gamma}(t) = X(p)$, which satisfies certain conditions.

Abe in [1] showed that, for $\varepsilon_0 = +1$ or -1 $(-\alpha \leq \varepsilon_0 \leq n - \alpha)$, every geodesic γ in M_α with $g(\acute{\gamma}, \acute{\gamma}) = \varepsilon_0$ is a circle in $\bar{M}_\beta$ iff M_α is constant isotropic and $\tilde{\nabla} h(X, X, X) = 0$, for any $X \in T_p(M)$.

Ikawa proved that in [2] M_1 is a totally geodesic submanifold in semi-Riemannian manifold $\bar{M}_\alpha$ if every Cartan framed null curve with constant curvatures in M_1 is also a Cartan framed null curve with constant curvature in $\bar{M}_\alpha$. Moreover, he also obtained that M_1 is a totally geodesic in $\bar{M}_\alpha$ if every generalized null cubic in M_1 is also a generalized null cubic in $\bar{M}_\alpha$.

2 Main results

Our aim here is to prove the following theorem which is related to the generalized null cubic.

Theorem 2. *Let M_1 be a Lorentzian submanifold of an indefinite-Riemannian manifold $\bar{M}_\alpha$ through an isometric immersion f. Then the following are equivalent.*

(i) M_1 is a constant $(\lambda-)$ isotropic submanifold of $\bar{M}_\alpha$.

(ii) There exist $K>0$ satisfying that for each generalized null cubic γ of curvature K on the submanifold M_1, the curve $f\circ\gamma$ in $\bar{M}_\alpha$ has constant first curvature K_1 along this curve.

Proof. $(i)\Rightarrow(ii)$: Let $f : M_1 \to \bar{M}_\alpha$ be a $(\lambda-)$ isotropic immersion. In the following , for simplicity we also denote $f\circ\gamma$ by γ.It follows from the equations(1.1)and(1.5)

$$\bar{\nabla}_{\acute{\gamma}(t)}\acute{\gamma}\left(t\right)=KZ\left(t\right)+h\left(\acute{\gamma}\left(t\right),\acute{\gamma}\left(t\right)\right),t\in I. \tag{2.1}$$

Here $\bar{\nabla}$ is the Riemannian connection of the ambient space $\bar{M}_\alpha$ and I is the some open interval on $\mathbb{R}$. Then from (2.1) we can see that the first curvature $K_1=\left\|\bar{\nabla}_{\acute{\gamma}}\acute{\gamma}\left(t\right)\right\|$ of the curve $f\circ\gamma$ is equal to $\sqrt{K^2+\lambda^2}$, which is constant on the inteval I.

$(ii)\Rightarrow(i)$: Let $f : M_1 \to \bar{M}_\alpha$ be a isotropic immersion satisfying the condition (ii). We take a point $p\in M$ and choose an arbitrary orthonormal pair vectors u and $v\in T_p\left(M\right)$. Let γ be a generalized null cubic of curvature K on the submanifold M_1 with initial condition that $\gamma\left(0\right)=p$, $\acute{\gamma}\left(0\right)=u$ and $\nabla_{\acute{\gamma}}\acute{\gamma}\left(0\right)=Kv$. By condition (ii) the first curvature $K_1=\left\|\bar{\nabla}_{\acute{\gamma}}\acute{\gamma}\right\|$ of the curve $f\circ\gamma$ is constant, so that the equation (2.1) implies $\left\|h\left(\acute{\gamma},\acute{\gamma}\right)\right\|$ is constant on I. Hence, denoting by $\nabla^{\perp}$the connection of the normal bundle of M_1 in $\bar{M}_\alpha$, from (1.5) we obtain

$$\begin{aligned}0 &= \frac{d}{dt}\left\|h\left(\acute{\gamma},\acute{\gamma}\right)\right\|^2=2g\left(\nabla^{\perp}\left(h\left(\acute{\gamma},\acute{\gamma}\right)\right),h\left(\acute{\gamma},\acute{\gamma}\right)\right)\\ &= 2g\left(\left(\tilde{\nabla}_{\acute{\gamma}}h\right)\left(\acute{\gamma},\acute{\gamma}\right)+2h\left(\nabla_{\acute{\gamma}}\acute{\gamma},\acute{\gamma}\right),h\left(\acute{\gamma},\acute{\gamma}\right)\right)\\ &= 2g\left(\left(\tilde{\nabla}_{\acute{\gamma}}h\right)\left(\acute{\gamma},\acute{\gamma}\right),h\left(\acute{\gamma},\acute{\gamma}\right)\right)+4Kg\left(h\left(Z,\acute{\gamma}\right),h\left(\acute{\gamma},\acute{\gamma}\right)\right).\end{aligned}$$

by virtue of the equation (1.3). Evaluating this equation at $t=0$, we obtain

$$g\left(\left(\tilde{\nabla}_u h\right)\left(u,u\right),h\left(u,u\right)\right)+2Kg\left(h\left(u,v\right),h\left(u,u\right)\right)=0. \tag{2.2}$$

On the other hand, for another generalized null cubic $\rho=\rho\left(t\right)$ of the same curvature

K on the submanifold $M_1$with the initial condition that$\rho\left(0\right)=p,\acute{\rho}\left(0\right)=u$, and $\nabla_{\acute{\rho}}\acute{\rho}\left(0\right)=-Kv$, we have

$$g\left(\left(\tilde{\nabla}_u h\right)\left(u,u\right),h\left(u,u\right)\right)-2Kg\left(h\left(u,v\right),h\left(u,u\right)\right)=0 \tag{2.3}$$

which corresponds to equation (2.2). Thus from (2.2) and (2.3) we can see that

$$g\left(h\left(u,v\right),h\left(u,u\right)\right)=0,$$

for any orthonormal pair of vectors u and v at each point p of M_1, so that the submanifold M_1 is $(\lambda-)$ isotropic in $\bar{M}_\alpha$ through the isometric immersion f by Lemma 1.1.

Next, we shall show that $\lambda : M_1 \to \mathbb{R}$ is constant. It follows from (2.2) and (2.3) that

$$g\left(\left(\tilde{\nabla}_u h\right)\left(u,u\right),h\left(u,u\right)\right)=0$$

for every unit vector u at each point p of M_1.

Then for every generalized null cubic τ on the submanifold M_1 we see that $\lambda=\lambda\left(t\right)$ is constant along τ. Therefore we obtained that λ is constant on M_1.□

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