S-convexity - foundations for Analysis

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Abstract. In this work, we start building some foundational theory for $S-$convexity. We also define $S-$convex generalizations (for more dimensions, functions).

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Key words: convexity, $S-$convex, function, $s-$convex, s_1-convex, s_2-convex, process, fuzzy.

1 Introduction

From [1], we borrow the following definitions:
1.1 *Definitions*

Definition. The function $f : X \to \mathbb{R}$ is called convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds $\forall \lambda \in [0,1]$; $\forall x, y \in X$.

Further, for $0 < s_n \leq 1$, where $n \in \{1,2\}$, we have

Definition. A function $f : X \to \mathbb{R}$ is said to be s_1-convex if the inequality

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1-\lambda^s)f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$; $X \subset \mathbb{R}_+$.

Definition. f is called s_2-convex, $s \neq 1$, if the graph lies below a 'bent chord' (L) between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, it is true that

$$sup_J(L-f) \geq sup_{\partial J}(L-f).$$

Definition. A function $f : X \to \mathbb{R}$ is said to be s_2-convex[1] if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$; $X \subset \mathbb{R}_+$.

1.2 *Terminology*
We use the same symbols and definitions presented in [1]:

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[1] This is the only proper extension to the concept of convexity in $S-$convexity.

- K_s^1 for the class of $S-$convex functions in the first sense, some s;
- K_s^2 for the class of $S-$convex functions in the second sense, some s;
- K_0 for the class of convex functions;
- s_1 for the constant S, $0 < s_1 \leq 1$, used in the first definition of S-convexity;
- s_2 for the constant S, $0 < s_2 \leq 1$, used in the second definition of S-convexity.

Remark 1. *The class of* 1*-convex functions is just a restriction of the class of convex functions, that is, when* $X = \mathbb{R}_+$*,*

$$K_1^1 \equiv K_1^2 \equiv K_0.$$

All other definitions we extend to $S-$convexity here may be found in [2] in regards to convexity.

2 New definitions

2.1 *S-convex sets*
2.1.1 *s_1-convex sets*

Definition. Let V be a vector space over $\mathbb{R}$. A subset $X \subset V$ is called s_1-convex if every s_1-convex curve, defined by $\lambda^s x_1 + (1 - \lambda^s)x_2$, $\forall x_1, x_2 \in X$, intersects X in an interval, that is:

$$(\lambda^s x_1 + (1 - \lambda^s)x_2) \subset X$$

when $0 < \lambda < 1$ and $x_1, x_2 \in X$.

2.1.2 *s_2-convex sets*

Definition. Let V be a vector space over $\mathbb{R}$. A subset $X \subset V$ is called s_2-convex if every s_2-convex curve, defined by $\lambda^s x_1 + (1 - \lambda)^s x_2$, $\forall x_1, x_2 \in X$, intersects X in an interval, that is:

$$(\lambda^s x_1 + (1 - \lambda)^s x_2) \subset X$$

when $0 < \lambda < 1$ and $x_1, x_2 \in X$.

2.2 *S-convex combinations*
2.2.1 *s_1-convex combinations*

Definition. An s_1-convex combination of a set of vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}$ is a linear combination of those vectors in which the coefficients are all nonnegative, and the individual raise to a power 's' results in their sum being one, that is: $\lambda_1^s \overrightarrow{v_1} + \lambda_2^s \overrightarrow{v_2} + ... + \lambda_n^s \overrightarrow{v_n}$, with $\sum_{p=1}^n \lambda_p^s = 1$ and $\lambda_p \geq 0$, $\forall p/p \in \mathbb{N}, 1 \leq p \leq n$, is an s_1-convex combination of vectors.

Remark 2. *The set of all s_1-convex combinations of the vectors is their s_1-convex span, denoted by* $SCS_1(\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n})$*.*

2.2.2 s_2-*convex combinations*

Definition. An s_2-convex combination of a set of vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}$ is a 'bent' combination[2] of those vectors in which the coefficients are all nonnegative and hold sum one, that is: $\lambda_1^s \overrightarrow{v_1} + \lambda_2^s \overrightarrow{v_2} + ... + \lambda_n^s \overrightarrow{v_n}$, with $\sum_{p=1}^{n} \lambda_p = 1$ and $\lambda_p \geq 0$, $\forall p / p \in \mathbb{N}, 1 \leq p \leq n$, is an s_2-convex combination of vectors.

Remark 3. *The set of all s_2-convex combinations of the vectors is their s_2-convex span, denoted by $SCS_2(\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n})$.*

2.3 $S-$*convexly independent sets*

2.3.1 s_1-*convexly independent sets*

Definition. A finite set, of two or more distinct points, is s_1-convexly independent if none of the points is an s_1-convex combination of the others. A finite set, which is not s_1-convexly independent, is s_1-convexly dependent.

2.3.2 s_2-*convexly independent sets*

Definition. A finite set, of two or more distinct points, is s_2-convexly independent if none of the points is an s_2-convex combination of the others. A finite set, which is not s_2-convexly independent, is s_2-convexly dependent.

2.4 *Closure*

2.4.1 *Closure of an s_1-convex set*

Theorem 1. *The closure of an s_1-convex set is s_1-convex.*

Proof. Take x, y in the closure of A[3], where A is an s_1-convex set. Therefore, in each neighborhood of x and y, it is possible to find some element which belongs to A. We now need to prove that $\lambda^s x + (1 - \lambda^s) y$ is in $cl(A)$. Suppose, by absurd, that there is δ such that $V_\delta(\lambda^s x + (1 - \lambda^s) y) \bigcap A = \emptyset$, some $\lambda / \lambda \in [0, 1]$. Because x, y (both) belong to $cl(A)$, it is possible to take x_1, y_1, as close as wanted, to x, y, and still in A. Because A is s_1-convex, this implies that $(\lambda^s x_1 + (1 - \lambda^s) y_1) \subset A$, for all those x_1, y_1. Now it is just a matter of getting the adequate value for λ so that it is proved that x_1 and y_1 will be in the neighborhood chosen and, therefore, such a statement is absurd, what makes the closure be the same nature as the set. □

2.4.2 *Closure of an s_2-convex set*

Theorem 2. *The closure of an s_2-convex set is s_2-convex.*

Proof. Take x, y in the closure of A[4], where A is an s_2-convex set. Therefore, in each neighborhood of x and y, it is possible to find some element which belongs to A. We now need to prove that $\lambda^s x + (1 - \lambda)^s y$ is in $cl(A)$. Suppose, by absurd, that there is δ such that $V_\delta(\lambda^s x + (1 - \lambda)^s y) \bigcap A = \emptyset$, some $\lambda / \lambda \in [0, 1]$. Because x, y (both) belong to $cl(A)$, it is possible to take x_1, y_1, as close as wanted, to x, y, and still in A. Because A is s_2-convex, this implies that $(\lambda^s x_1 + (1 - \lambda)^s y_1) \subset A$, for all those x_1, y_1. Now it is just a matter of getting the adequate value for λ so that it is proved that x_1 and y_1 will be in the neighborhood chosen and, therefore, such a statement is absurd, what makes the closure be the same nature as the set. □

[2] Simply a choice of name to distinguish between it and the L. C.

[3] cl(A)

[4] cl(A)

2.5 *Generalization of the concept of $S-$convexity*
2.5.1 *s_1-convex functions for several dimensions*

Definition. Let f be a function from $\mathbb{R}^m$ to $\mathbb{R}$. Then f is s_1-convex iff

$$f(\lambda_1 x_1 + ... + \lambda_m x_m) \leq \lambda_1^s f(x_1) + ... + \lambda_m^s f(x_m)$$

whenever $\lambda_1 \geq 0, ..., \lambda_m \geq 0$, $\sum_{p=1}^{m} \lambda_p^s = 1$.

Remark 4. *$f(\lambda_1 x_1 + ... + \lambda_m x_m) \leq \lambda_1^s f(x_1) + ... + \lambda_m^s f(x_m)$ is reduced to $f(\lambda_1 x_1 + (1-\lambda_1^s)^{\frac{1}{s}} x_2) \leq \lambda_1^s f(x_1) + (1-\lambda_1^s) f(x_2)$ when $m = 2$.*

2.5.2 *s_2-convex functions for several dimensions*

Definition. Let f be a function from $\mathbb{R}^m$ to $\mathbb{R}$. Then f is s_2-convex iff

$$f(\lambda_1 x_1 + ... + \lambda_m x_m) \leq \lambda_1^s f(x_1) + ... + \lambda_m^s f(x_m)$$

whenever $\lambda_1 \geq 0, ..., \lambda_m \geq 0$, $\sum_{p=1}^{m} \lambda_p = 1$.

Remark 5. *$f(\lambda_1 x_1 + ... + \lambda_m x_m) \leq \lambda_1^s f(x_1) + ... + \lambda_m^s f(x_m)$ is reduced to $f(\lambda_1 x_1 + (1-\lambda_1) x_2) \leq \lambda_1^s f(x_1) + (1-\lambda_1)^s f(x_2)$ when $m = 2$.*

2.6 *Epigraph*
2.6.1 *Epigraph and s_1-convex functions*

Theorem 3. *$f : X \subset \mathbb{R}_+ \to \mathbb{R}$ is considered s_1-convex iff the epigraph of f is s_1-convex, that is, iff*

$$epi(f) = \{(x,t) \mid x \in X, t \in \mathbb{R},\ t \geq f(x)\}$$

is s_1-convex.

Proof. f is s_1-convex implies

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1-\lambda^s) f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$. Suppose that there were

$$t_1 \geq f(x)$$

$$t_2 \geq f(x)$$

and, therefore, belonging to $epi(f)$, such that

$$(\lambda^s t_1 + (1-\lambda^s) t_2) \not\subset epi(f)$$

when $0 < \lambda < 1$ and $t_1, t_2 \in \mathbb{R}$. Therefore, the interval $(\lambda^s t_1 + (1-\lambda^s) t_2)$ will have at least one point not belonging to $epi(f)$. Therefore, that point would be less than f. Therefore, taking that point as being x_3, there would exist $f(x_3) = \lambda^s t_1 + (1-\lambda^s) t_2 < f(x)$, for some specific allowed value of λ. But $t_1 \geq f(x)$ and $t_2 \geq f(x)$. Therefore, $\lambda^s t_1 + (1-\lambda^s) t_2 \geq (\lambda^s + (1-\lambda^s)) f(x) = f(x)$, contradiction! □

2.6.2 *Epigraph and s_2-convex functions*

Theorem 4. *$f : X \subset \mathbb{R}_+ \to \mathbb{R}$ is considered s_2-convex iff the epigraph of f is s_2-convex, that is, iff*

$$epi(f) = \{(x,t) \mid x \in X, t \in \mathbb{R},\ t \geq f(x)\}$$

is s_2-convex.

Proof. f is s_2-convex implies

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$. Suppose that there were

$$t_1 \geq f(x)$$

$$t_2 \geq f(x)$$

and, therefore, belonging to $epi(f)$, such that

$$(\lambda^s t_1 + (1-\lambda)^s t_2) \not\subset epi(f)$$

when $0 < \lambda < 1$ and $t_1, t_2 \in \mathbb{R}$. Therefore, the interval $(\lambda^s t_1 + (1-\lambda)^s t_2)$ will have at least one point not belonging to $epi(f)$. Therefore, that point would be less than f. Therefore, taking that point as being x_3, there would exist $f(x_3) = \lambda^s t_1 + (1-\lambda)^s t_2 < f(x)$, for some specific allowed value of λ. But $t_1 \geq f(x)$ and $t_2 \geq f(x)$. Therefore, $\lambda^s t_1 + (1-\lambda)^s t_2 \geq (\lambda^s + (1-\lambda)^s) f(x) \geq f(x)$, contradiction! □

2.7 *Interior of an $S-$convex set*
2.7.1 *s_1-convex sets*

Theorem 5. *The interior of an s_1-convex set is also an s_1-convex set.*

Proof. Take x_0, y_0 from the interior of an s_1-convex set A. Then, there is $\delta \geq 0$ such that $V_\delta(x_0) \subset A$ as well as $V_\delta(y_0) \subset A$. Therefore, $x_0(\text{or } y_0) - \delta < x(\text{or } y) < x_0(\text{or } y_0) + \delta$. Therefore, $\lambda^s x_0 + (1-\lambda^s) y_0 - \delta < \lambda^s x + (1-\lambda^s) y < \lambda^s x_0 + (1-\lambda^s) y_0 + \delta$. Therefore, we also have a neighborhood of radius δ entirely contained in A, what proves our thesis. □

2.7.2 *s_2-convex sets*

Theorem 6. *The interior of an s_2-convex set is also an s_2-convex set.*

Proof. Take x_0, y_0 from the interior of an s_2-convex set A. Then, there is $\delta \geq 0$ such that $V_\delta(x_0) \subset A$ as well as $V_\delta(y_0) \subset A$. Therefore, $x_0(\text{or } y_0) - \delta < x(\text{or } y) < x_0(\text{or } y_0) + \delta$. Therefore, $\lambda^s x_0 + (1-\lambda)^s y_0 - \delta < \lambda^s x + (1-\lambda)^s y < \lambda^s x_0 + (1-\lambda)^s y_0 + \delta$. Therefore, we also have a neighborhood of radius δ entirely contained in A, what proves our thesis. □

2.8 *Union of $S-$convex combinations*
2.8.1 *s_1-convex sets*

Theorem 7. *For every $E \subset V$, where V is a vector space, the union, $s_1ch(E)$, of all s_1-convex combinations of elements of E, that is,*

$$s_1ch(E) = \{\sum_0^N \lambda_j^s x_j; \lambda_j \geq 0; \sum_0^N \lambda_j^s = 1; x_j \in E; N = 1, 2, ...\},$$

is an s_1-convex set.

Proof. Trivial. □

2.8.2 *s_2-convex sets*

Theorem 8. *For every $E \subset V$, where V is a vector space, the union, $s_2ch(E)$, of all s_2-convex combinations of elements of E, that is,*

$$s_2ch(E) = \{\sum_0^N \lambda_j^s x_j; \lambda_j \geq 0; \sum_0^N \lambda_j = 1; x_j \in E; N = 1, 2, ...\},$$

is an s_2-convex set.

Proof. Trivial. □

3 Conclusion

In this one more precursor paper, we have established the analytical foundations for $S-$convexity in general. We now believe to have the foundational work almost fully completed. We intend to finally define them for complex numbers and close the foundational work with derivatives.

References

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