Conformally and quasi-conformally conservative curvature tensors on a trans-Sasakian manifold with respect to semi-symmetric metric connections

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Abstract. In this paper we study the conservative conformal curvature tensor and the conservative quasi-conformal curvature tensor on a trans-Sasakian manifold with respect to a semi-symmetric metric connection. It is shown that a trans-Sasakian manifold with conservative conformal curvature tensor is an Einstein manifold, and with quasi-conformal curvature tensor is an η-Einstein manifold.

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1 Introduction

In 1924, Friedman and Schouten [10] introduced the notion of semi-symmetric linear connection on a differentiable manifold. Then in 1932, Hayden [13] introduced the idea of metric connection with torsion on Riemannian manifold. A systematic study of semi-symmetric metric connections on a Riemannian manifold has been given by K. Yano [17] in 1970 and later studied by K.S.Amur and S.S. Pujar [2], C.S. Bagewadi [3], U.C. De et al [9], Sharafuddin and Hussain [15] and others.

In the papers ([9], [11] and [16]), the authors have obtained results on the conservativeness of projective, pseudo projective, conformal, concircular, quasi-conformal curvature tensors on K-contact, Kenmotsu and trans-Sasakian manifolds. C.S. Bagewadi et al [6] have studied conservative projective curvature tensors on a trans-Sasakian manifold with respect to a semi-symmetric metric connection.

In this paper we study the conservativeness of conformal and quasi-conformal curvature tensors on a trans-Sasakian manifold under the condition $\phi(grad\alpha) = (n-2)grad\beta$, admitting a semi-symmetric metric connection. The paper is organized as follows: after preliminaries in section 2, we study in section 3 basic results on a trans-Sasakian manifold under the above condition with respect to a semi-symmetric metric connection. The sections 4 and 5 consider the study of conformal and quasi-conformal curvature tensors with respect to semi-symmetric metric connection on trans-Sasakian manifolds.

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2 Preliminaries

Let M be an almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a compatible Riemannian metric g such that ([4], [5], [7])

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta.\phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in TM$.

An almost contact metric structure (ϕ, ξ, η, g) in M is called a trans-Sasakian structure [14] if $(M \times \mathbf{R}, J, G)$ belongs to the class w_4 [12], where J is the almost complex structure on $M \times \mathbf{R}$ defined by $J(X, \lambda d/dt) = (\phi X - \lambda \xi, \eta(X) d/dt)$ for all vector fields X on M and smooth functions λ on $M \times \mathbf{R}$ and G is the product metric on $M \times \mathbf{R}$. This may be expressed by the condition [8]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.4}$$

for some smooth functions α and β on M, and we say that the trans-Sasakian structure is of type (α, β).

From (2.4) we see that

$$\nabla_X \xi = -\alpha \phi X + \beta(X - \eta(X)\xi), \tag{2.5}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.6}$$

In an n-dimensional trans-Sasakian manifold, we obtain

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X), \tag{2.7}$$

$$2\alpha\beta + \xi\alpha = 0, \tag{2.8}$$

$$\begin{aligned} S(X, \xi) = {} & ((n-1)(\alpha^2 - \beta^2) - \xi\beta)\eta(X) \\ & -(n-2)X\beta - (\phi X)\alpha. \end{aligned} \tag{2.9}$$

In an n-dimensional trans-Sasakian manifold of type (α, β), we have

$$\phi(grad\alpha) = (n-2)grad\beta. \tag{2.10}$$

Under the condition (2.10), the equations (2.7) and (2.9) reduce to

$$R(\xi, X)\xi = (\alpha^2 - \beta^2)(\eta(X)\xi - X), \tag{2.11}$$

$$S(X, \xi) = (n-1)(\alpha^2 - \beta^2)\eta(X). \tag{2.12}$$

In this paper we study trans-Sasakian manifolds under the condition (2.10).

Let (M, g) be an n-dimensional Riemannian manifold of class C^∞ with metric tensor g and let ∇ be the Levi-Civita connection on M. A linear connection $\widetilde{\nabla}$ on (M, g) is said to be semi-symmetric [17] if the torsion tensor T of the connection $\widetilde{\nabla}$ satisfies

$$T(X, Y) = \pi(Y)X - \pi(X)Y, \tag{2.13}$$

where π is a 1-form on M with ρ as associated vector field, i.e., $\pi(X) = g(X, \rho)$ for any differentiable vector field X on M.

A semi-symmetric connection $\widetilde{\nabla}$ is called semi-symmetric metric connection [13] if it further satisfies $\widetilde{\nabla} g = 0$.

In an almost contact manifold semi-symmetric metric connection is defined by identifying a 1-form π of (2.13) with a contact-form η, i.e., by setting [15]

$$T(X,Y) = \eta(Y)X - \eta(X)Y \tag{2.14}$$

with ξ is the associated vector field. i.e., $g(X, \xi) = \eta(X)$.

The relation between the semi-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection ∇ of (M, g) has been obtained by K. Yano [17], which is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \tag{2.15}$$

where $\eta(Y) = g(Y, \xi)$.

Further, a relation between the curvature tensor R and $\widetilde{R}$ of type $(1,3)$ of connections ∇ and $\widetilde{\nabla}$ respectively is given by [17]:

$$\widetilde{R}(X,Y)Z = R(X,Y)Z - K(Y,Z)X + K(X,Z)Y - g(Y,Z)FX + g(X,Z)FY. \tag{2.16}$$

where K is a tensor field of type $(0,2)$ defined by

$$\begin{aligned} K(Y,Z) = g(FY,Z) &= (\nabla_Y \eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}\eta(\xi)g(Y,Z) \\ &= (\widetilde{\nabla}_Y \eta)(Z) - \frac{1}{2}g(Y,Z), \end{aligned} \tag{2.17}$$

for any vector fields X and Y.

From (2.16), it follows that

$$\widetilde{S}(Y,Z) = S(Y,Z) - (n-2)K(Y,Z) - A.g(Y,Z) \tag{2.18}$$

where $\widetilde{S}$ denotes the Ricci tensor with respect to $\widetilde{\nabla}$, $A = Tr \cdot K$. Differentiating (2.18) covariantly with respect to X, we obtain

$$\begin{aligned} (\widetilde{\nabla}_X \widetilde{S})(Y,Z) = &(\nabla_X S)(Y,Z) - (n-2)(\nabla_X K)(Y,Z) \\ &-\eta(Y)S(X,Z) - \eta(Z)S(X,Y) \\ &+(n-2)\eta(Y)K(X,Z) + (n-2)\eta(Z)K(Y,X) \\ &+g(X,Y)S(\xi,Z) + g(X,Z)S(Y,\xi) \\ &-(n-2)g(X,Z)K(Y,\xi) - (n-2)g(X,Y)K(Z,\xi). \end{aligned} \tag{2.19}$$

Now let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold for $i = 1, 2, ..., n$. Putting $Y = Z = e_i$ in (2.19) and then taking summation over the index i, we get

$$\widetilde{\nabla}_X \widetilde{r} = \nabla_X r - (n-2)(\nabla_X A) \tag{2.20}$$

3 Basic results

Theorem 3.1. *For a trans-Sasakian manifold M under the condition (2.10), we have*

$$\begin{aligned} & [(\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi,Z)] \\ = & \ \beta S(Y,Z) - (n-1)(\alpha^2-\beta^2)\beta g(Y,Z) \\ & +\alpha S(Y,\phi Z) - (n-1)(\alpha^2-\beta^2)\alpha g(Y,\phi Z). \end{aligned} \tag{3.1}$$

Proof. For a symmetric endomorphism Q of the tangent space at a point of M, we express the Ricci tensor S as

$$S(X,Y) = g(QX,Y). \tag{3.2}$$

Further, it is known that[8]

$$(L_\xi g)(X,Y) = 2(g - \eta \otimes \eta), \tag{3.3}$$

where L is the Lie derivation.
Using (3.2) and (3.3), for a trans-Sasakian manifold we have

$$(L_\xi S)(X,Y) = 2\beta S(X,Y) - 2\beta(n-1)(\alpha^2-\beta^2)\eta(X)\eta(Y). \tag{3.4}$$

But

$$\begin{aligned} & (\nabla_\xi S)(Y,Z) \\ = & \ \xi S(Y,Z) - S(\nabla_\xi Y,Z) - S(Y,\nabla_\xi Z) \\ = & \ \xi S(Y,Z) - S([\xi,Y]+\nabla_Y\xi,Z) - S(Y,[\xi,Z]+\nabla_Z\xi) \\ = & \ \xi S(Y,Z) - S([\xi,Y],Z) - S(\nabla_Y\xi,Z) - S(Y,[\xi,Z]) - S(Y,\nabla_Z\xi) \\ = & \ (L_\xi S)(Y,Z) - S(\nabla_Y\xi,Z) - S(Y,\nabla_Z\xi). \end{aligned}$$

Using (2.5), (2.12) and (3.4) in the above equation, we obtain

$$(\nabla_\xi S)(Y,Z) = 0. \tag{3.5}$$

Which implies

$$\nabla_\xi r = 0. \tag{3.6}$$

Again

$$(\nabla_Y S)(\xi,Z) = YS(\xi,Z) - S(\nabla_Y\xi,Z) - S(\xi,\nabla_Y\xi).$$

Using (2.5) and (2.12) in the above equation, we get

$$\begin{aligned} (\nabla_Y S)(\xi,Z) = & \ (n-1)(\alpha^2-\beta^2)\nabla_Y.\eta(Z) - S(-\alpha\phi Y + \beta(Y-\eta(Y)\xi,Z)) \\ & -(n-1)(\alpha^2-\beta^2)\eta(\nabla_Y Z). \end{aligned}$$

Further simplification gives

$$\begin{aligned} (\nabla_Y S)(\xi,Z) = & \ (n-1)(\alpha^2-\beta^2)\beta g(Y,Z) - \beta S(Y,Z) \\ & +(n-1)(\alpha^2-\beta^2)\alpha g(Y,\phi Z) - \alpha S(Y,\phi Z). \end{aligned} \tag{3.7}$$

Putting (3.5) and (3.7) in left hand side of (3.1), result follows. □

Theorem 3.2. *For a trans-Sasakian manifold M under the condition (2.10), the following results are true:*

$$\begin{aligned}
&i) \quad K(Y,Z) = \alpha g(Y,\phi Z) + \left(\beta + \frac{1}{2}\right) g(Y,Z) - (\beta+1)\eta(Y)\eta(Z)\\
&ii) \quad K(Y,\xi) = K(\xi,Y) = -\frac{1}{2}\eta(Y)\\
&iii) \quad K(\nabla_Y \xi, Z) = -\alpha^2[g(Y,Z) - \eta(Y)\eta(Z)] - 2\alpha\beta g(\phi Y, Z)\\
&\qquad\qquad -\frac{\alpha}{2} g(\phi Y, Z) + \beta\left(\beta + \frac{1}{2}\right)[g(Y,Z) - \eta(Y)\eta(Z)]\\
&iv) \quad K(Y, \nabla_Z \xi) = \alpha^2[g(Y,Z) - \eta(Y)\eta(Z)] + \frac{\alpha}{2} g(\phi Y, Z)\\
&\qquad\qquad +\beta\left(\beta + \frac{1}{2}\right)[g(Y,Z) - \eta(Y)\eta(Z)].
\end{aligned} \tag{3.8}$$

Proof. From (2.17), we have

$$K(Y,Z) = g(FY,Z) = (\nabla_Y \eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}g(Y,Z). \tag{3.9}$$

Using (2.6) in (3.9), we get (3.8 (i)).

Taking $Z = \xi$ in (3.8 (i)) and then using (2.1) and (2.3), we get (3.8 (ii)).

Next, by considering $Y = \nabla_Y \xi$ in (3.8 (i)) and then using (2.1) and (2.5), (3.8(iii)) follows.

From the result (3.8(iii)), proof of (3.8 (iv)) is obvious. □

Theorem 3.3. *For a trans-Sasakian manifold M under the condition (2.10), we have*

$$\begin{aligned}
&[(\nabla_\xi K)(Y,Z) - (\nabla_Y K)(\xi, Z)]\\
= \; &-\alpha g(\phi Y, Z) - 2\alpha\beta g(\phi Y, Z)\\
&-[\alpha^2 - \beta(\beta+1)](g(Y,Z) - \eta(Y)\eta(Z)).
\end{aligned} \tag{3.10}$$

Proof. From (3.3) and $K(X,Y) = g(FX,Y)$, we have

$$(L_\xi K)(Y,Z) = 2\beta K(Y,Z) + \beta\eta(Y)\eta(Z). \tag{3.11}$$

Also

$$\begin{aligned}
&(\nabla_\xi K)(Y,Z)\\
= \; &\xi K(Y,Z) - K(\nabla_\xi Y, Z) - K(Y, \nabla_\xi Z)\\
= \; &\xi K(Y,Z) - K([\xi,Y] + \nabla_Y \xi, Z) - K(Y, [\xi, Z] + \nabla_Z \xi)\\
= \; &\xi K(Y,Z) - K([\xi,Y], Z) - K(\nabla_Y \xi, Z) - K(Y, [\xi, Z]) - K(Y, \nabla_Z \xi)\\
= \; &(L_\xi K)(Y,Z) - K(\nabla_Y \xi, Z) - K(Y, \nabla_Z \xi).
\end{aligned}$$

Using (2.5), (3.8(ii),(iii)& (iv)) and (3.11) in the above equation, we obtain

$$(\nabla_\xi K)(Y,Z) = 0. \tag{3.12}$$

Which implies

$$\nabla_\xi A = 0. \tag{3.13}$$

And

$$(\nabla_Y K)(\xi, Z) = YK(\xi, Z) - K(\nabla_Y \xi, Z) - K(\xi, \nabla_Y \xi). \tag{3.14}$$

Using (2.5), (3.8(ii),(iii)& (iv)) in (3.14), we get

$$\begin{aligned}(\nabla_Y K)(\xi, Z) &= [\alpha^2 - \beta(\beta+1)](g(Y, Z) - \eta(Y)\eta(Z)) \\ &\quad +\alpha g(\phi Y, Z) + 2\alpha\beta g(\phi Y, Z).\end{aligned} \tag{3.15}$$

By putting (3.12) and (3.15) in left hand side of (3.10), the result follows. □

4 Trans-Sasakian manifolds admitting semi-symmetric metric connection with $Div\ \tilde{C} = 0$

In this section we prove the following theorem.

Theorem 4.1. *If in a trans-Sasakian manifold M $(n > 3)$ under the condition (2.10) admitting a semi-symmetric metric connection whose conformal curvature tensor with respect to this connection is conservative, then the manifold M is an Einstein manifold with respect to Levi-Civita connection and the scalar curvature tensor of such a manifold is $n(n-1)(\alpha^2 - \beta^2)$.*

Proof. A conformal curvature tensor on trans-Sasakian manifold with respect to semi-symmetric metric connection is given by [4]

$$\begin{aligned}\widetilde{C}(X, Y)Z &= R(X, Y)Z - \frac{1}{n-2}[g(Y, Z)QX \\ &\quad -g(X, Z)QY + S(Y, Z)X - S(X, Z)Y] \\ &\quad +\frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y].\end{aligned} \tag{4.1}$$

Differentiating (4.1) covariantly and contracting we obtain

$$\begin{aligned}DivC &= DivR - \frac{1}{(n-2)}[(\widetilde{\nabla}_X S)(Y, Z) - (\widetilde{\nabla}_Y S)(X, Z)] \\ &\quad +\frac{1}{n-1}[g(Y, Z)(\widetilde{\nabla}_X r) - g(X, Z)(\widetilde{\nabla}_Y r)].\end{aligned} \tag{4.2}$$

Let us suppose that in a trans-Sasakian manifold

$$Div \cdot \widetilde{C} = 0. \tag{4.3}$$

where Div denotes the divergence.
From (4.2) and (4.3) it follows that

$$\begin{aligned}&DivR - \tfrac{1}{(n-2)}[(\widetilde{\nabla}_X S)(Y, Z) - (\widetilde{\nabla}_Y S)(X, Z)] \\ &\quad = -\tfrac{1}{n-1}[g(Y, Z)(\widetilde{\nabla}_X r) - g(X, Z)(\widetilde{\nabla}_Y r)].\end{aligned} \tag{4.4}$$

From (2.19) and (2.20) in (4.4), we get

$$
\begin{aligned}
(4.5)\qquad & \frac{n-3}{n-2}[(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)]\\
= & \frac{n-1}{n-2}[S(Y,Z)\eta(X)-S(X,Z)\eta(Y)]-(n-1)\eta(R(X,Y)Z)\\
& -\frac{n-3}{n-2}K(X,Z)\eta(Y)+(n-A-1)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\\
& +S(X,Y)\eta(Z)-\frac{1}{n-2}[g(Y,Z)S(X,\xi)-g(X,Z)S(Y,\xi)]\\
& +\frac{1}{n-1}[g(Y,Z)\{(\nabla_X r)+(\nabla_X A)\}-g(X,Z)\{(\nabla_Y r)-(\nabla_Y A)\}].
\end{aligned}
$$

By taking account of (3.8(i)) and (3.8(ii)) in (4.5), we have

$$
\begin{aligned}
(4.6)\qquad & \frac{n-3}{n-2}[(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)]-S(X,Y)\eta(Z)\\
= & \frac{n-1}{n-2}[S(Y,Z)\eta(X)-S(X,Z)\eta(Y)]+(n-1)\eta(R(X,Y)Z)\\
& -\frac{1}{(n-2)}[g(Y,Z)S(X,\xi)-g(X,Z)S(Y,\xi)]-\left[\frac{1}{2}-(n-A-1)\right]g(Y,Z)\eta(X)\\
& -\frac{(n-3)}{(n-2)}g(X,\phi Z)\eta(Y)-\left[\frac{(n-3)}{2(n-2)}-(n-A-1)+\frac{1}{2}\right]g(X,Z)\eta(Y)\\
& +\frac{(n-3)}{(n-2)}\eta(X)\eta(Y)\eta(Z)+\frac{1}{n-1}[g(Y,Z)dA(X)-g(X,Z)dA(Y)]\\
& +\frac{1}{n-1}[g(Y,Z)dr(X)-g(X,Z)dr(Y)].
\end{aligned}
$$

Now putting $X=\xi$ in (4.6) and using (2.1), (2.3), (2.11) and (2.12), we get

$$
\begin{aligned}
(4.7)\qquad & \frac{n-3}{n-2}[(\nabla_\xi S)(Y,Z)-(\nabla_Y S)(\xi,Z)]\\
= & \frac{(n-1)}{(n-2)}S(Y,Z)-\left[\frac{(n-1)}{(n-2)}+\frac{1}{2}-(n-1)(\alpha^2-\beta^2)-(n-A-1)\right]g(Y,Z)\\
& -\left[\frac{(n-3)}{2(n-2)}-A-\frac{1}{2}\right]\eta(Y)\eta(Z)-\frac{1}{(n-1)}\eta(Z)[dA(Y)+dr(Y)].
\end{aligned}
$$

Using (3.5), (3.6), (3.12) and (3.13) in (4.7), we obtain

$$
\begin{aligned}
(4.8)\qquad & \left[\frac{(n-3)}{(n-2)}\beta-\frac{(n-1)}{(n-2)}\right]S(Y,Z)-\frac{(n-3)}{(n-2)}\alpha S(\phi Y,Z)\\
= & \left[\frac{(n-3)}{(n-2)}(n-1)(\alpha^2-\beta^2)\beta-\frac{(n-1)}{(n-2)}-\frac{1}{2}+(n-1)(\alpha^2-\beta^2)\right.\\
& \left.+(n-A-1)\right]g(Y,Z)-\frac{(n-3)}{(n-2)}(n-1)(\alpha^2-\beta^2)\alpha g(\phi Y,Z)\\
& -\left[\frac{(n-3)}{2(n-2)}-A-\frac{1}{2}\right]\eta(Y)\eta(Z)-\frac{1}{(n-1)}\eta(Z)[dA(Y)+dr(Y)].
\end{aligned}
$$

Next, by replacing Z by ϕZ in (4.8) and then using (2.1), we obtain

$$\begin{aligned}(4.9)\qquad & \left[\frac{(n-3)}{(n-2)}\beta-\frac{(n-1)}{(n-2)}\right]S(Y,\phi Z)-\frac{(n-3)}{(n-2)}\alpha S(\phi Y,\phi Z)\\ = & \left[\frac{(n-3)}{(n-2)}(n-1)(\alpha^2-\beta^2)\beta-\frac{(n-1)}{(n-2)}-\frac{1}{2}+(n-1)(\alpha^2-\beta^2)\right.\\ & \left.+(n-A-1)\right]g(Y,\phi Z)-\frac{(n-3)}{(n-2)}(n-1)(\alpha^2-\beta^2)\alpha g(\phi Y,\phi Z).\end{aligned}$$

Interchanging Y and Z in the above equation, we have

$$\begin{aligned}(4.10)\qquad & \left[\frac{(n-3)}{(n-2)}\beta-\frac{(n-1)}{(n-2)}\right]S(\phi Y,Z)-\frac{(n-3)}{(n-2)}\alpha S(\phi Y,\phi Z)\\ = & \left[\frac{(n-3)}{(n-2)}(n-1)(\alpha^2-\beta^2)\beta-\frac{(n-1)}{(n-2)}-\frac{1}{2}+(n-1)(\alpha^2-\beta^2)\right.\\ & \left.+(n-A-1)\right]g(\phi Y,Z)-\frac{(n-3)}{(n-2)}(n-1)(\alpha^2-\beta^2)\alpha g(\phi Y,\phi Z).\end{aligned}$$

Adding (4.9) with (4.10), and using the skew-symmetric property of ϕ, one can get

$$(4.11)\qquad S(Y,Z)=(n-1)(\alpha^2-\beta^2)g(Y,Z).$$

Thus the manifold is an Einstein manifold. From (4.11), it follows that

$$r=n(n-1)(\alpha^2-\beta^2).$$

□

5 Trans-Sasakian manifolds admitting a semi-symmetric metric connection with $Div\,\tilde{W}=0$

In this section we prove the following theorem.

Theorem 5.1. *If in a trans-Sasakian manifold $M(n>2)$ under the condition (2.10) admitting a semi-symmetric metric connection whose quasi-conformal curvature tensor with respect to this connection is conservative, then the manifold M is an η-Einstein manifold with respect to Levi-Civita connection; moreover the scalar curvature of the manifold is constant if and only if $\beta=-1$.*

Proof. A quasi-conformal curvature tensor on trans-Sasakian manifold with respect to semi-symmetric metric connection is given by [1]

$$\begin{aligned}(5.1)\qquad & \widetilde{W}(X,Y)Z=aR(X,Y)Z\\ & +b[g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X-S(X,Z)Y]\\ & \frac{r}{n(n-1)}[a+(n-2)b][g(Y,Z)X-g(X,Z)Y].\end{aligned}$$

Differentiating (5.1) covariantly and contracting we obtain

$$\begin{aligned} Div\cdot\widetilde{W}(X,Y)Z &= aDivR+b[(\widetilde{\nabla}_X S)(Y,Z)-(\widetilde{\nabla}_Y S)(X,Z)] \\ &\quad -\frac{[a-b(n-1)(n-2)]}{n(n-1)}[g(Y,Z)(\widetilde{\nabla}_X\widetilde{r})-g(X,Z)(\widetilde{\nabla}_Y\widetilde{r})]. \end{aligned} \tag{5.2}$$

Let us suppose that in a trans-Sasakian manifold

$$Div\cdot\widetilde{W}=0. \tag{5.3}$$

where Div denotes the divergence.
From (5.2) and (5.3) it follows that

$$\begin{aligned} &aDivR+b[(\widetilde{\nabla}_X S)(Y,Z)-(\widetilde{\nabla}_Y S)(X,Z)] \\ =\ &\frac{[a-b(n-1)(n-2)]}{n(n-1)}[g(Y,Z)(\widetilde{\nabla}_X\widetilde{r})-g(X,Z)(\widetilde{\nabla}_Y\widetilde{r})]. \end{aligned} \tag{5.4}$$

From (2.19) and (2.20) in (5.4), we have

$$\begin{aligned} &(a+b)[(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)]+\alpha(a+b(n-2))g(\phi Y,Z) \\ &-(a+b(n-2))[(\nabla_X K)(Y,Z)-(\nabla_Y K)(X,Z)] \\ =\ &(a-b)[S(X,Z)\eta(Y)-S(Y,Z)\eta(X)]+aS(X,Y)\eta(Z) \\ &-(a+b(n-2))[K(X,Z)\eta(Y)-K(Y,Z)\eta(X) \\ &+K(Y,X)\eta(Z)-K(X,Y)\eta(Z)]-g(X,Z)(n-2)(\nabla_Y A) \\ &-aA[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]+a[g(Y,Z)(\nabla_X A) \\ &-g(X,Z)(\nabla_Y A)]-b[g(X,Z)S(Y,\xi)-g(Y,Z)S(X,\xi)] \\ &+b(n-2)[g(X,Z)K(Y,\xi)-g(Y,Z)K(X,\xi)]-g(X,Z)(\nabla_Y r) \\ &+\left[\frac{a}{n(n-1)}-\frac{b(n-2)}{n}\right][g(Y,Z)(\nabla_X r)-g(Y,Z)(n-2)(\nabla_X A)]. \end{aligned} \tag{5.5}$$

By taking account of (3.8(i)) and (3.8(ii)) in (5.5), we get

$$\begin{aligned} &(a+b)[(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)]-aS(X,Y)\eta(Z) \\ &-[a+b(n-2)][(\nabla_X K)(Y,Z)-(\nabla_Y K)(X,Z)]-a(n-1)\eta(R(X,Y)Z) \\ =\ &(a-b)[S(X,Z)\eta(Y)-S(Y,Z)\eta(X)]+b[g(Y,Z)S(X,\xi)-g(X,Z)S(Y,\xi)] \\ &-[a+b(n-2)][2\alpha g(\phi X,Y)\eta(Z)+\alpha g(X,\phi Z)\eta(Y)-\alpha g(Y,\phi Z)\eta(X)] \\ &+\left[a\left(\beta+\frac{1}{2}\right)+b(n-2)(\beta+1)+a(n-A-1)\right][g(Y,Z)\eta(X)-g(X,Z)\eta(Y)] \\ &+\left[\frac{a}{n(n-1)}-\frac{b(n-2)}{n}\right][g(Y,Z)dr(X)-g(X,Z)dr(Y)] \\ &+a[g(Y,Z)dA(X)-g(X,Z)dA(Y)]. \end{aligned} \tag{5.6}$$

Now taking $X = \xi$ in (5.6), and using (2.1),(2.3), (2.11) and (2.12) we get

$$\begin{aligned} (5.7)\quad & (a+b)[(\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi,Z)] + (a-b)S(Y,Z) \\ & -[a+b(n-2)][(\nabla_\xi K)(Y,Z) - (\nabla_Y K)(\xi,Z)] - [a+b(n-2)]\alpha g(Y,\phi Z) \\ = & \left[a\left(\beta+\frac{1}{2}\right) + b(n-2)(\beta+1) + a(n-A-1)\right. \\ & \left. -(a-b)(n-1)(\alpha^2-\beta^2)\right] g(Y,Z) + \left[2(a-b)(n-1)(\alpha^2-\beta^2) + a\left(\beta+\frac{1}{2}\right)\right. \\ & \left. + b(n-2)(\beta+1) + a(n-A-1)\right] \eta(Y)\eta(Z) \\ & + \left[\frac{a}{n(n-1)} - \frac{b(n-2)}{n}\right] [g(Y,Z)dr(\xi) - (n-2)g(Y,Z)dA(\xi) - \eta(Z)dr(Y) \\ & +(n-2)\eta(Z)dA(Y)] + a[g(Y,Z)dA(\xi) - \eta(Y)dA(Y)]. \end{aligned}$$

Putting (3.5), (3.6), (3.12) and (3.13)) in (5.7), we get

$$\begin{aligned} (5.8)\quad & [(a+b)\beta + (a-b)]S(Y,Z) - (a+b)\alpha S(\phi Y,Z) \\ = & \left[\alpha(n-1)(\alpha^2-\beta^2)(a+b) + 2\alpha(1+\beta)\{a+b(n-2)\}\right] g(Y,\phi Z) \\ & + \left[a\left(\beta+\frac{1}{2}\right) + b(n-2)(\beta+1) - \{a+b(n-2)\}\{\alpha^2-\beta(\beta+1)\}\right. \\ & \left. + a(n-A-1) - (n-1)(\alpha^2-\beta^2)\{(a-b)-\beta(a+b)\}\right] g(Y,Z) \\ & + \left[\{a+b(n-2)\}\{\alpha^2-\beta(\beta+1)\} + 2(a-b)(n-1)(\alpha^2-\beta^2)\right. \\ & \left. + a\left(\beta+\frac{1}{2}\right) - b(n-2)(\beta+1) + a(n-A-1)\right] \eta(Y)\eta(Z) \\ & + \left[\frac{a}{n(n-1)} - \frac{b(n-2)}{n}\right] [\eta(Z)\{(n-2)dA(Y) - dr(Y)\}] - a\eta(Z)dA(Y). \end{aligned}$$

Next, by replacing Z by ϕZ in (5.8) and then using (2.1), we obtain

$$\begin{aligned} (5.9)\quad & [(a+b)\beta + (a-b)]S(Y,\phi Z) - (a+b)\alpha S(\phi Y,\phi Z) \\ = & \left[\alpha(1-n)(\alpha^2-\beta^2)(a+b) - 2\alpha(1+\beta)\{a+b(n-2)\}\right] g(\phi Y,\phi Z) \\ & + \left[a\left(\beta+\frac{1}{2}\right) + b(n-2)(\beta+1) - \{a+b(n-2)\}\{\alpha^2-\beta(\beta+1)\}\right. \\ & \left. + a(n-A-1) - (n-1)(\alpha^2-\beta^2)\{(a-b)-\beta(a+b)\}\right] g(Y,\phi Z). \end{aligned}$$

Interchanging Y and Z in above equation, we have

$$\begin{aligned} (5.10)\quad & [(a+b)\beta + (a-b)]S(\phi Y,Z) - (a+b)\alpha S(\phi Y,\phi Z) \\ = & \left[\alpha(1-n)(\alpha^2-\beta^2)(a+b) - 2\alpha(1+\beta)\{a+b(n-2)\}\right] g(\phi Y,\phi Z) \\ & + \left[a\left(\beta+\frac{1}{2}\right) + b(n-2)(\beta+1) - \{a+b(n-2)\}\{\alpha^2-\beta(\beta+1)\}\right. \\ & \left. + a(n-\widetilde{A}-1) - (n-1)(\alpha^2-\beta^2)\{(a-b)-\beta(a+b)\}\right] g(\phi Y,Z). \end{aligned}$$

Adding (5.9) with (5.10), and using (2.1) and skew-symmetric property of ϕ, one can get

$$S(Y,Z) = \left[(n-1)(\alpha^2-\beta^2)+\frac{(\beta+1)}{(a+b)}2\{a+b(n-2)\}\right]g(Y,Z) - \left[\frac{(\beta+1)}{(a+b)}2\{a+b(n-2)\}\right]\eta(Y)\eta(Z). \tag{5.11}$$

Thus the manifold is an η-Einstein manifold.
Differentiating (5.11) covariantly with respect to X, and then using (2.6), we have

$$(\nabla_X S)(Y,Z) = \frac{\beta+1}{(a+b)}2\{a+b(n-2)\}[\alpha(g(\phi Y,X)\eta(Z)+g(\phi Z,X)\eta(Y)) +\beta(g(X,Y)\eta(Z)g(X,Z)\eta(Y))-2\beta\eta(X)\eta(Y)\eta(Z)]. \tag{5.12}$$

Taking an orthonormal frame field and contracting (5.12) over X and Z, we obtain

$$dr(Y) = \frac{2(1+\beta)}{(a+b)}\{a+b(n-2)\}[\alpha\psi+(n-1)\beta]\eta(Y) \tag{5.13}$$

where $\psi = Tr.\phi$. From (5.13), it follows that

$$dr(Y) = 0 \quad if \quad and \quad only \quad if \quad \beta = -1.$$

□

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