

Geodesic curves of the minimal homothetical hypersurfaces in the semi-Euclidean space

Derya Sağlam and Arif Sabuncuoğlu

Abstract. In this paper, we calculate the equations of the geodesic curves of the minimal homothetical hypersurfaces in \mathbb{R}_1^3 , \mathbb{R}_1^4 and \mathbb{R}_2^4 , and obtain some important results.

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1 Introduction

In the semi-Euclidean space, the minimal homothetical hypersurfaces are studied by I. Van de Woestyne [2]. Here, we obtain the equations of these geodesic curves of this hypersurfaces in \mathbb{R}_1^3 , \mathbb{R}_1^4 and \mathbb{R}_2^4 . Finally, we notice that the geodesic curves of each of the homothetical hypersurfaces determined by

$$(1) \quad \varphi = (x_1, x_2, x_3, x_4), \quad x_4 = (x_1 + p_1) \tan(p_2 x_2 + q)$$

$$(2) \quad \varphi = (x_1, x_2, x_3, x_4), \quad x_4 = (x_1 + p_1) \tanh(p_2 x_2 + q)$$

$$(3) \quad \varphi = (x_1, x_2, x_3, x_4), \quad x_4 = (x_1 + p_1) \coth(p_2 x_2 + q)$$

are the same in \mathbb{R}_1^4 and \mathbb{R}_2^4 , respectively.

2 Minimal homothetical hypersurfaces in \mathbb{R}_ν^{n+1}

Theorem 1. *Let M be a non-degenerate minimal homothetical hypersurface of \mathbb{R}_s^3 . Then, M is either a plane or is one of the following hypersurfaces:*

$$(1) \quad f = (x_i + p_i) \tan(p_j x_j + q), \quad \varepsilon_i \cdot \varepsilon_k = 1.$$

$$(2) \quad f = (x_i + p_i) \tanh(p_j x_j + q), \quad \varepsilon_i \cdot \varepsilon_k = -1.$$

$$(3) \quad f = (x_i + p_i) \coth(p_j x_j + q), \quad \varepsilon_i \cdot \varepsilon_k = -1.$$

$$(4) \quad f = C e^{\beta(x_i \mp x_j)}, \quad \varepsilon_i \cdot \varepsilon_j = -1.$$

Here (ijk) is a permutation of 1, 2 and 3. Furthermore, C, p_i, p_j, q and β are constants [2].

Theorem 2. Let M be a non-degenerate minimal homothetical hypersurface of \mathbb{R}_s^4 . Then, M is either a hyperplane or is one of the following hypersurfaces:

- (1) $f = Ce^{L_i x_i + L_j x_j + L_k x_k}, \varepsilon_i L_i^2 + \varepsilon_j L_j^2 + \varepsilon_k L_k^2 = 0.$
- (2) $f = (x_i + p_i) \tan(p_j x_j + q), \varepsilon_i \cdot \varepsilon_l = 1.$
- (3) $f = (x_i + p_i) \tanh(p_j x_j + q), \varepsilon_i \cdot \varepsilon_l = -1.$
- (4) $f = (x_i + p_i) \coth(p_j x_j + q), \varepsilon_i \cdot \varepsilon_l = -1.$
- (5) $f = (Cx_i + p_i) e^{\beta(x_j \mp x_k)}, \varepsilon_j \cdot \varepsilon_k = -1.$
- (6) $f = \mp \frac{(x_i + p_i)(x_j + p_j)}{x_k + p_k}, \varepsilon_i \cdot \varepsilon_j \cdot \varepsilon_k \cdot \varepsilon_l = 1.$

Here $(ijkl)$ is a permutation of 1, 2, 3 and 4. $C, L_i, L_j, L_k, p_i, p_j, p_k, q$ and β are constants [2].

The following theorem gives the non-degenerate minimal homothetical hypersurfaces of \mathbb{R}_ν^{n+1} in general.

Theorem 3. Let M be a non-degenerate minimal homothetical hypersurface of \mathbb{R}_ν^{n+1} . Then, M is either a hyperplane or is one of the following hypersurfaces:

- (1) $f = Ce^{\sum_{j=1}^n L_j x_{p(j)}}, \sum_{j=1}^n \varepsilon_{p(j)} L_j^2 = 0,$
- (2) $f = (x_{p(1)} + a_1) \tan(a_2 x_{p(2)} + b), \varepsilon_{p(1)} \cdot \varepsilon_{p(n+1)} = 1,$
- (3) $f = (x_{p(1)} + a_1) \tanh(a_2 x_{p(2)} + b), \varepsilon_{p(1)} \cdot \varepsilon_{p(n+1)} = -1,$
- (4) $f = (x_{p(1)} + a_1) \coth(a_2 x_{p(2)} + b), \varepsilon_{p(1)} \cdot \varepsilon_{p(n+1)} = -1,$
- (5) $f = (Cx_{p(1)} + D) e^{\sum_{j=2}^n L_j x_{p(j)}}, \sum_{j=2}^n \varepsilon_{p(j)} L_j^2 = 0,$
- (6) $f = \mp \frac{(x_i + a_i)(x_j + a_j)}{x_k + a_k}, \varepsilon_i \cdot \varepsilon_j \cdot \varepsilon_k \cdot \varepsilon_l = 1,$

where p is a permutation 1, 2, ..., $n + 1$. C, D, a_1, a_2, a_3, b and $L_j (j = 1, 2, \dots, n)$ are constant [2].

3 Geodesic curves

Let M be a n -dimensional surface and (u_1, u_2, \dots, u_n) be a coordinate system of the surface. A curve γ on M is a geodesic curve if and only if

$$(3.1) \quad \frac{d^2(u_k \circ \gamma)}{dt^2} + \sum_{i,j=1}^n \Gamma_{ij}^k(\gamma) \frac{d(u_i \circ \gamma)}{dt} \frac{d(u_j \circ \gamma)}{dt} = 0,$$

where $1 \leq k \leq n$. If the surface M is determined by φ , we know that

$$g_{ij} \circ \varphi = \langle \varphi_i, \varphi_j \rangle, \quad g_{ij} = \langle \partial_i, \partial_j \rangle \quad \text{and} \quad \partial_i \circ \varphi = \varphi_i.$$

We have

$$D_{\partial_i} \partial_j = \sum_{k=1}^n \Gamma_{ij}^k \partial_k$$

and

$$(3.2) \quad \Gamma_{ij}^k = \frac{1}{2} \sum_{h=1}^n \left(\frac{\partial g_{ih}}{\partial u_j} + \frac{\partial g_{jh}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_h} \right) g^{hk},$$

where D , is the Levi-Civita connection on M [1].

4 Geodesic curves of the minimal homothetical hypersurfaces in \mathbb{R}_1^3

Lemma 1. *Let $\varepsilon_1 \cdot \varepsilon_3 = 1$, ($\varepsilon_1 = \varepsilon_3 = 1$, $\varepsilon_2 = -1$) in \mathbb{R}_1^3 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by*

$$(4.1) \quad \varphi = (x_1, x_2, x_3), \quad x_3 = (x_1 + p_1) \tan(p_2 x_2 + q)$$

are

$$\begin{aligned} & \frac{d^2(u_1 \circ \gamma)}{dt^2} + \frac{2p_2 \tan(p_2 u_2 + q)}{1 - p_2^2 (u_1 + p_1)^2 (1 + \tan^2(p_2 u_2 + q))} (\gamma) \\ & \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} + p_2 (u_1 + p_1) \tan(p_2 u_2 + q) (\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0 \end{aligned}$$

and

$$(4.2) \quad \begin{aligned} & \frac{d^2(u_2 \circ \gamma)}{dt^2} - \frac{2p_2^2 (u_1 + p_1) (1 + \tan^2(p_2 u_2 + q))}{1 - p_2^2 (u_1 + p_1)^2 (1 + \tan^2(p_2 u_2 + q))} (\gamma) \\ & \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} + p_2 (u_1 + p_1) \tan(p_2 u_2 + q) (\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0. \end{aligned}$$

Proof. Since $n = 2$, we calculate the functions Γ_{ij}^k in (3.2) and substitute this function in (3.1), for $1 \leq i, j, k \leq 2$. □

Lemma 2. *Let $\varepsilon_1 \cdot \varepsilon_3 = -1$, ($\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = 1$) in \mathbb{R}_1^3 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by*

$$(4.3) \quad \varphi = (x_1, x_2, x_3), \quad x_3 = (x_1 + p_1) \tanh(p_2 x_2 + q)$$

are

$$\begin{aligned} & \frac{d^2(u_1 \circ \gamma)}{dt^2} + \frac{2p_2 \tanh(p_2 u_2 + q)}{-1 - p_2^2 (u_1 + p_1)^2 (1 - \tanh^2(p_2 u_2 + q))} (\gamma) \\ & \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} - p_2 (u_1 + p_1) \tanh(p_2 u_2 + q) (\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0 \end{aligned}$$

and

$$(4.4) \quad \frac{d^2(u_2 \circ \gamma)}{dt^2} - \frac{2p_2^2(u_1 + p_1)(1 - \tanh^2(p_2u_2 + q))}{-1 - p_2^2(u_1 + p_1)^2(1 - \tanh^2(p_2u_2 + q))}(\gamma) \\ \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} - p_2(u_1 + p_1) \tanh(p_2u_2 + q)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0.$$

Lemma 3. Let $\varepsilon_1 \cdot \varepsilon_3 = -1$, ($\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = 1$) in \mathbb{R}_1^3 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by

$$(4.5) \quad \varphi = (x_1, x_2, x_3), \quad x_3 = (x_1 + p_1) \coth(p_2x_2 + q)$$

are

$$\frac{d^2(u_1 \circ \gamma)}{dt^2} + \frac{2p_2 \coth(p_2u_2 + q)}{-1 - p_2^2(u_1 + p_1)^2(1 - \coth^2(p_2u_2 + q))}(\gamma) \\ \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} - p_2(u_1 + p_1) \coth(p_2u_2 + q)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0$$

and

$$(4.6) \quad \frac{d^2(u_2 \circ \gamma)}{dt^2} - \frac{2p_2^2(u_1 + p_1)(1 - \coth^2(p_2u_2 + q))}{-1 - p_2^2(u_1 + p_1)^2(1 - \coth^2(p_2u_2 + q))}(\gamma) \\ \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} - p_2(u_1 + p_1) \coth(p_2u_2 + q)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0.$$

Lemma 4. Let $\varepsilon_1 \cdot \varepsilon_2 = -1$, ($\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = 1$) in \mathbb{R}_1^3 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by

$$\varphi = (x_1, x_2, x_3), \quad x_3 = Ce^{\beta(x_1+x_2)}$$

are

$$\frac{d^2(u_1 \circ \gamma)}{dt^2} - C^2\beta^3 e^{2\beta(u_1+u_2)}(\gamma) \left[\left(\frac{d(u_1 \circ \gamma)}{dt} \right)^2 + 2 \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} + \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0$$

and

$$\frac{d^2(u_2 \circ \gamma)}{dt^2} + C^2\beta^3 e^{2\beta(u_1+u_2)}(\gamma) \left[\left(\frac{d(u_1 \circ \gamma)}{dt} \right)^2 + 2 \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} + \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0.$$

Lemma 5. Let $\varepsilon_1 \cdot \varepsilon_2 = -1$, ($\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = 1$) in \mathbb{R}_1^3 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by

$$\varphi = (x_1, x_2, x_3), \quad x_3 = Ce^{\beta(x_1-x_2)}$$

are

$$\frac{d^2(u_1 \circ \gamma)}{dt^2} - C^2\beta^3 e^{2\beta(u_1-u_2)}(\gamma) \left[\left(\frac{d(u_1 \circ \gamma)}{dt} \right)^2 - 2 \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} + \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0$$

and

$$\frac{d^2(u_2 \circ \gamma)}{dt^2} - C^2 \beta^3 e^{2\beta(u_1 - u_2)}(\gamma) \left[\left(\frac{d(u_1 \circ \gamma)}{dt} \right)^2 - 2 \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} + \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0.$$

Corollary 1. *Each of the minimal homothetical hypersurfaces given by (4.1), (4.3), (4.5) is a plane for $p_2 = 0$ and corresponding geodesic curve is a straight line. The equations for these straight lines are*

$$\begin{aligned} \gamma(t) &= (a_1 t + b_1, a_2 t + b_2, (a_1 t + b_1 + p_1) \tan q), \\ \gamma(t) &= (a_1 t + b_1, a_2 t + b_2, (a_1 t + b_1 + p_1) \tanh q), \\ \gamma(t) &= (a_1 t + b_1, a_2 t + b_2, (a_1 t + b_1 + p_1) \coth q), \end{aligned}$$

respectively.

Corollary 2. *None of the equations of the geodesic curves of the minimal homothetical hypersurfaces given by (4.1), (4.3), (4.5) contains the term $\left(\frac{d(u_1 \circ \gamma)}{dt} \right)^2$.*

5 Geodesic curves of the minimal homothetical hypersurfaces in \mathbb{R}_1^4

Lemma 6. *Let $\varepsilon_1 L_1^2 + \varepsilon_2 L_2^2 + \varepsilon_3 L_3^2 = 0$, ($\varepsilon_1 = -1, \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$) in \mathbb{R}_1^4 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by*

$$\varphi = (x_1, x_2, x_3, x_4), \quad x_4 = C e^{L_1 x_1 + L_2 x_2 + L_3 x_3}$$

are

$$\begin{aligned} &\frac{d^2(u_1 \circ \gamma)}{dt^2} - C^2 L_1 e^{2 \sum L_i u_i}(\gamma) \left[L_1^2 \left(\frac{d(u_1 \circ \gamma)}{dt} \right)^2 + 2 L_1 L_2 \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} \right. \\ &\left. + 2 L_1 L_3 \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} + L_2^2 \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 + 2 L_2 L_3 \frac{d(u_2 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} + L_3^2 \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right] = 0, \end{aligned}$$

$$\begin{aligned} &\frac{d^2(u_2 \circ \gamma)}{dt^2} + C^2 L_2 e^{2 \sum L_i u_i}(\gamma) \left[L_1^2 \left(\frac{d(u_1 \circ \gamma)}{dt} \right)^2 + 2 L_1 L_2 \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} \right. \\ &\left. + 2 L_1 L_3 \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} + L_2^2 \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 + 2 L_2 L_3 \frac{d(u_2 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} + L_3^2 \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right] = 0, \end{aligned}$$

and

$$\begin{aligned} &\frac{d^2(u_3 \circ \gamma)}{dt^2} + C^2 L_3 e^{2 \sum L_i u_i}(\gamma) \left[L_1^2 \left(\frac{d(u_1 \circ \gamma)}{dt} \right)^2 + 2 L_1 L_2 \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} \right. \\ &\left. + 2 L_1 L_3 \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} + L_2^2 \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 + 2 L_2 L_3 \frac{d(u_2 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} + L_3^2 \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right] = 0. \end{aligned}$$

Lemma 7. Let $\varepsilon_1 \cdot \varepsilon_4 = 1$, ($\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = 1$, $\varepsilon_2 = -1$) in \mathbb{R}_1^4 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by

$$(5.1) \quad \varphi = (x_1, x_2, x_3, x_4), \quad x_4 = (x_1 + p_1) \tan(p_2 x_2 + q)$$

are

$$\begin{aligned} & \frac{d^2(u_1 \circ \gamma)}{dt^2} + \frac{2p_2 \tan(p_2 u_2 + q)}{1 - p_2^2(u_1 + p_1)^2(1 + \tan^2(p_2 u_2 + q))}(\gamma) \\ & \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} + p_2(u_1 + p_1) \tan(p_2 u_2 + q)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0, \\ & \frac{d^2(u_2 \circ \gamma)}{dt^2} - \frac{2p_2^2(u_1 + p_1)(1 + \tan^2(p_2 u_2 + q))}{1 - p_2^2(u_1 + p_1)^2(1 + \tan^2(p_2 u_2 + q))}(\gamma) \\ & \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} + p_2(u_1 + p_1) \tan(p_2 u_2 + q)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0 \end{aligned}$$

and

$$(5.2) \quad \frac{d^2(u_3 \circ \gamma)}{dt^2} = 0.$$

Lemma 8. Let $\varepsilon_1 \cdot \varepsilon_4 = -1$, ($\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$) in \mathbb{R}_1^4 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by

$$(5.3) \quad \varphi = (x_1, x_2, x_3, x_4), \quad x_4 = (x_1 + p_1) \tanh(p_2 x_2 + q)$$

are

$$\begin{aligned} & \frac{d^2(u_1 \circ \gamma)}{dt^2} + \frac{2p_2 \tanh(p_2 u_2 + q)}{-1 - p_2^2(u_1 + p_1)^2(1 - \tanh^2(p_2 u_2 + q))}(\gamma) \\ & \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} - p_2(u_1 + p_1) \tanh(p_2 u_2 + q)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0, \\ & \frac{d^2(u_2 \circ \gamma)}{dt^2} - \frac{2p_2^2(u_1 + p_1)(1 - \tanh^2(p_2 u_2 + q))}{-1 - p_2^2(u_1 + p_1)^2(1 - \tanh^2(p_2 u_2 + q))}(\gamma), \\ & \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} - p_2(u_1 + p_1) \tanh(p_2 u_2 + q)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0 \end{aligned}$$

and

$$(5.4) \quad \frac{d^2(u_3 \circ \gamma)}{dt^2} = 0.$$

Lemma 9. Let $\varepsilon_1 \cdot \varepsilon_4 = -1$, ($\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$) in \mathbb{R}_1^4 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by

$$(5.5) \quad \varphi = (x_1, x_2, x_3, x_4), \quad x_4 = (x_1 + p_1) \coth(p_2 x_2 + q)$$

are

$$\begin{aligned} & \frac{d^2(u_1 \circ \gamma)}{dt^2} + \frac{2p_2 \coth(p_2 u_2 + q)}{-1 - p_2^2(u_1 + p_1)^2(1 - \coth^2(p_2 u_2 + q))}(\gamma) \\ & \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} - p_2(u_1 + p_1) \coth(p_2 u_2 + q)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0, \\ & \frac{d^2(u_2 \circ \gamma)}{dt^2} - \frac{2p_2^2(u_1 + p_1)(1 - \coth^2(p_2 u_2 + q))}{-1 - p_2^2(u_1 + p_1)^2(1 - \coth^2(p_2 u_2 + q))}(\gamma) \\ & \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} - p_2(u_1 + p_1) \coth(p_2 u_2 + q)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0 \end{aligned}$$

and

$$(5.6) \quad \frac{d^2(u_3 \circ \gamma)}{dt^2} = 0.$$

Lemma 10. Let $\varepsilon_2 \cdot \varepsilon_3 = -1$, $(\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = 1, \varepsilon_2 = -1)$ in \mathbb{R}_1^4 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by

$$(5.7) \quad \varphi = (x_1, x_2, x_3, x_4), \quad x_4 = (Cx_1 + p_1)e^{\beta(x_2 + x_3)}$$

are

$$\begin{aligned} & \frac{d^2(u_1 \circ \gamma)}{dt^2} + \frac{C\beta e^{2\beta(u_2 + u_3)}}{1 + C^2 e^{2\beta(u_2 + u_3)}}(\gamma) \left[2C \frac{d(u_1 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} + \frac{d(u_3 \circ \gamma)}{dt} \right) \right. \\ & \left. + \beta(Cu_1 + p_1)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} + 2\frac{d(u_3 \circ \gamma)}{dt} \right) + \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right) \right] = 0, \\ & \frac{d^2(u_2 \circ \gamma)}{dt^2} - \frac{\beta^2(Cu_1 + p_1)e^{2\beta(u_2 + u_3)}}{1 + C^2 e^{2\beta(u_2 + u_3)}}(\gamma) \left[2C \frac{d(u_1 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} + \frac{d(u_3 \circ \gamma)}{dt} \right) \right. \\ & \left. + \beta(Cu_1 + p_1)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} + 2\frac{d(u_3 \circ \gamma)}{dt} \right) + \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right) \right] = 0 \end{aligned}$$

and

$$\begin{aligned} & \frac{d^2(u_3 \circ \gamma)}{dt^2} + \frac{\beta^2(Cu_1 + p_1)e^{2\beta(u_2 + u_3)}}{1 + C^2 e^{2\beta(u_2 + u_3)}}(\gamma) \left[2C \frac{d(u_1 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} + \frac{d(u_3 \circ \gamma)}{dt} \right) \right. \\ & \left. + \beta(Cu_1 + p_1)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} + 2\frac{d(u_3 \circ \gamma)}{dt} \right) + \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right) \right] = 0. \end{aligned}$$

Lemma 11. Let $\varepsilon_2 \cdot \varepsilon_3 = -1$, $(\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = 1, \varepsilon_2 = -1)$ in \mathbb{R}_1^4 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by

$$(5.8) \quad \varphi = (x_1, x_2, x_3, x_4), \quad x_4 = (Cx_1 + p_1)e^{\beta(x_2 - x_3)}$$

are

$$\begin{aligned} & \frac{d^2(u_1 \circ \gamma)}{dt^2} + \frac{C\beta e^{2\beta(u_2-u_3)}}{1+C^2 e^{2\beta(u_2-u_3)}}(\gamma) \left[2C \frac{d(u_1 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} - \frac{d(u_3 \circ \gamma)}{dt} \right) \right. \\ & \left. + \beta(Cu_1 + p_1)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} - 2\frac{d(u_3 \circ \gamma)}{dt} \right) + \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right) \right] = 0, \\ & \frac{d^2(u_2 \circ \gamma)}{dt^2} - \frac{\beta^2(Cu_1 + p_1)e^{2\beta(u_2-u_3)}}{1+C^2 e^{2\beta(u_2-u_3)}}(\gamma) \left[2C \frac{d(u_1 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} - \frac{d(u_3 \circ \gamma)}{dt} \right) \right. \\ & \left. + \beta(Cu_1 + p_1)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} - 2\frac{d(u_3 \circ \gamma)}{dt} \right) + \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right) \right] = 0 \end{aligned}$$

and

$$\begin{aligned} & \frac{d^2(u_3 \circ \gamma)}{dt^2} - \frac{\beta^2(Cu_1 + p_1)e^{2\beta(u_2-u_3)}}{1+C^2 e^{2\beta(u_2-u_3)}}(\gamma) \left[2C \frac{d(u_1 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} - \frac{d(u_3 \circ \gamma)}{dt} \right) \right. \\ & \left. - \beta(Cu_1 + p_1)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} + 2\frac{d(u_3 \circ \gamma)}{dt} \right) - \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right) \right] = 0. \end{aligned}$$

Corollary 3. *Each of the minimal homothetical hypersurfaces given by (5.1), (5.3), (5.5) are a plane for $p_2 = 0$ and corresponding geodesic curve is a straight line. The equations for these straight lines are*

$$\begin{aligned} \gamma(t) &= (a_1 t + b_1, a_2 t + b_2, a_3 t + b_3, (a_1 t + b_1 + p_1) \tan q), \\ \gamma(t) &= (a_1 t + b_1, a_2 t + b_2, a_3 t + b_3, (a_1 t + b_1 + p_1) \tanh q), \\ \gamma(t) &= (a_1 t + b_1, a_2 t + b_2, a_3 t + b_3, (a_1 t + b_1 + p_1) \coth q), \end{aligned}$$

respectively.

Corollary 4. *None of the equations of the geodesic curves of the minimal homothetical hypersurfaces given by (5.1), (5.3), (5.5), (5.7), (5.8) contains the term $\left(\frac{d(u_1 \circ \gamma)}{dt}\right)^2$.*

6 Geodesic curves of the minimal homothetical hypersurfaces in \mathbb{R}_2^4

Lemma 12. *Let $\varepsilon_1 L_1^2 + \varepsilon_2 L_2^2 + \varepsilon_3 L_3^2 = 0$, ($\varepsilon_1 = \varepsilon_2 = -1$, $\varepsilon_3 = \varepsilon_4 = 1$) in \mathbb{R}_2^4 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by*

$$\varphi = (x_1, x_2, x_3, x_4), \quad x_4 = C e^{L_1 x_1 + L_2 x_2 + L_3 x_3}$$

are

$$\begin{aligned} & \frac{d^2(u_1 \circ \gamma)}{dt^2} - C^2 L_1 e^{2 \sum L_i u_i}(\gamma) [L_1^2 \left(\frac{d(u_1 \circ \gamma)}{dt} \right)^2 + 2L_1 L_2 \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} \\ & + 2L_1 L_3 \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} + L_2^2 \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 + 2L_2 L_3 \frac{d(u_2 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} \\ & + L_3^2 \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2] = 0, \\ & \frac{d^2(u_2 \circ \gamma)}{dt^2} - C^2 L_2 e^{2 \sum L_i u_i}(\gamma) [L_1^2 \left(\frac{d(u_1 \circ \gamma)}{dt} \right)^2 + 2L_1 L_2 \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} \\ & + 2L_1 L_3 \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} + L_2^2 \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 + 2L_2 L_3 \frac{d(u_2 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} \\ & + L_3^2 \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2] = 0, \end{aligned}$$

and

$$\begin{aligned} & \frac{d^2(u_3 \circ \gamma)}{dt^2} + C^2 L_3 e^{2 \sum L_i u_i}(\gamma) [L_1^2 \left(\frac{d(u_1 \circ \gamma)}{dt} \right)^2 + 2L_1 L_2 \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} \\ & + 2L_1 L_3 \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} + L_2^2 \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 + 2L_2 L_3 \frac{d(u_2 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} \\ & + L_3^2 \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2] = 0. \end{aligned}$$

Lemma 13. Let $\varepsilon_1 \cdot \varepsilon_4 = 1$, $(\varepsilon_1 = \varepsilon_4 = 1, \varepsilon_2 = \varepsilon_3 = -1)$ in \mathbb{R}_2^4 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by

$$(6.1) \quad \varphi = (x_1, x_2, x_3, x_4), \quad x_4 = (x_1 + p_1) \tan(p_2 x_2 + q)$$

are

$$\begin{aligned} & \frac{d^2(u_1 \circ \gamma)}{dt^2} + \frac{2 p_2 \tan(p_2 u_2 + q)}{1 - p_2^2 (u_1 + p_1)^2 (1 + \tan^2(p_2 u_2 + q))}(\gamma) \\ & \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} + p_2 (u_1 + p_1) \tan(p_2 u_2 + q)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0, \\ & \frac{d^2(u_2 \circ \gamma)}{dt^2} - \frac{2 p_2^2 (u_1 + p_1) (1 + \tan^2(p_2 u_2 + q))}{1 - p_2^2 (u_1 + p_1)^2 (1 + \tan^2(p_2 u_2 + q))}(\gamma) \\ & \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} + p_2 (u_1 + p_1) \tan(p_2 u_2 + q)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0 \end{aligned}$$

and

$$(6.2) \quad \frac{d^2(u_3 \circ \gamma)}{dt^2} = 0.$$

Lemma 14. Let $\varepsilon_1 \cdot \varepsilon_4 = -1$, $(\varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = \varepsilon_4 = 1)$ in \mathbb{R}_2^4 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by

$$(6.3) \quad \varphi = (x_1, x_2, x_3, x_4), \quad x_4 = (x_1 + p_1) \tanh(p_2 x_2 + q)$$

are

$$\begin{aligned} & \frac{d^2(u_1 \circ \gamma)}{dt^2} - \frac{2p_2 \tanh(p_2 u_2 + q)}{1 - p_2^2(u_1 + p_1)^2(1 - \tanh^2(p_2 u_2 + q))}(\gamma) \\ & \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} - p_2(u_1 + p_1) \tanh(p_2 u_2 + q)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0, \\ & \frac{d^2(u_2 \circ \gamma)}{dt^2} - \frac{2p_2^2(u_1 + p_1)(1 - \tanh^2(p_2 u_2 + q))}{1 - p_2^2(u_1 + p_1)^2(1 - \tanh^2(p_2 u_2 + q))}(\gamma) \\ & \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} - p_2(u_1 + p_1) \tanh(p_2 u_2 + q)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0 \end{aligned}$$

and

$$(6.4) \quad \frac{d^2(u_3 \circ \gamma)}{dt^2} = 0.$$

Lemma 15. Let $\varepsilon_1 \cdot \varepsilon_4 = -1$, $(\varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = \varepsilon_4 = 1)$ in \mathbb{R}_2^4 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by

$$(6.5) \quad \varphi = (x_1, x_2, x_3, x_4), \quad x_4 = (x_1 + p_1) \coth(p_2 x_2 + q)$$

are

$$\begin{aligned} & \frac{d^2(u_1 \circ \gamma)}{dt^2} - \frac{2p_2 \coth(p_2 u_2 + q)}{1 - p_2^2(u_1 + p_1)^2(1 - \coth^2(p_2 u_2 + q))}(\gamma) \\ & \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} - p_2(u_1 + p_1) \coth(p_2 u_2 + q)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0, \\ & \frac{d^2(u_2 \circ \gamma)}{dt^2} - \frac{2p_2^2(u_1 + p_1)(1 - \coth^2(p_2 u_2 + q))}{1 - p_2^2(u_1 + p_1)^2(1 - \coth^2(p_2 u_2 + q))}(\gamma) \\ & \times \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} - p_2(u_1 + p_1) \coth(p_2 u_2 + q)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \right)^2 \right] = 0 \end{aligned}$$

and

$$(6.6) \quad \frac{d^2(u_3 \circ \gamma)}{dt^2} = 0.$$

Lemma 16. Let $\varepsilon_2 \cdot \varepsilon_3 = -1$, $(\varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = \varepsilon_4 = 1)$ in \mathbb{R}_2^4 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by

$$(6.7) \quad \varphi = (x_1, x_2, x_3, x_4), \quad x_4 = (Cx_1 + p_1)e^{\beta(x_2 + x_3)}$$

are

$$\begin{aligned} & \frac{d^2(u_1 \circ \gamma)}{dt^2} - \frac{C\beta e^{2\beta(u_2+u_3)}}{1 - C^2 e^{2\beta(u_2+u_3)}}(\gamma) [2C \frac{d(u_1 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} + \frac{d(u_3 \circ \gamma)}{dt} \right) \\ & + \beta(Cu_1 + p_1)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} + 2\frac{d(u_3 \circ \gamma)}{dt} \right) + \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right)] = 0, \\ & \frac{d^2(u_2 \circ \gamma)}{dt^2} - \frac{\beta^2(Cu_1 + p_1)e^{2\beta(u_2+u_3)}}{1 - C^2 e^{2\beta(u_2+u_3)}}(\gamma) [2C \frac{d(u_1 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} + \frac{d(u_3 \circ \gamma)}{dt} \right) \\ & + \beta(Cu_1 + p_1)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} + 2\frac{d(u_3 \circ \gamma)}{dt} \right) + \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right)] = 0 \end{aligned}$$

and

$$\begin{aligned} & \frac{d^2(u_3 \circ \gamma)}{dt^2} + \frac{\beta^2(Cu_1 + p_1)e^{2\beta(u_2+u_3)}}{1 - C^2 e^{2\beta(u_2+u_3)}}(\gamma) [2C \frac{d(u_1 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} + \frac{d(u_3 \circ \gamma)}{dt} \right) \\ & + \beta(Cu_1 + p_1)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} + 2\frac{d(u_3 \circ \gamma)}{dt} \right) + \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right)] = 0. \end{aligned}$$

Lemma 17. Let $\varepsilon_2 \cdot \varepsilon_3 = -1$, $(\varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = \varepsilon_4 = 1)$ in \mathbb{R}_2^4 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by

$$(6.8) \quad \varphi = (x_1, x_2, x_3, x_4), \quad x_4 = (Cx_1 + p_1)e^{\beta(x_2 - x_3)}$$

are

$$\begin{aligned} & \frac{d^2(u_1 \circ \gamma)}{dt^2} - \frac{C\beta e^{2\beta(u_2-u_3)}}{1 - C^2 e^{2\beta(u_2-u_3)}}(\gamma) [2C \frac{d(u_1 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} - \frac{d(u_3 \circ \gamma)}{dt} \right) \\ & + \beta(Cu_1 + p_1)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} - 2\frac{d(u_3 \circ \gamma)}{dt} \right) + \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right)] = 0, \\ & \frac{d^2(u_2 \circ \gamma)}{dt^2} - \frac{\beta^2(Cu_1 + p_1)e^{2\beta(u_2-u_3)}}{1 - C^2 e^{2\beta(u_2-u_3)}}(\gamma) [2C \frac{d(u_1 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} - \frac{d(u_3 \circ \gamma)}{dt} \right) \\ & + \beta(Cu_1 + p_1)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} - 2\frac{d(u_3 \circ \gamma)}{dt} \right) + \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right)] = 0 \end{aligned}$$

and

$$\begin{aligned} & \frac{d^2(u_3 \circ \gamma)}{dt^2} - \frac{\beta^2(Cu_1 + p_1)e^{2\beta(u_2-u_3)}}{1 - C^2 e^{2\beta(u_2-u_3)}}(\gamma) [2C \frac{d(u_1 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} - \frac{d(u_3 \circ \gamma)}{dt} \right) \\ & + \beta(Cu_1 + p_1)(\gamma) \left(\frac{d(u_2 \circ \gamma)}{dt} \left(\frac{d(u_2 \circ \gamma)}{dt} - 2\frac{d(u_3 \circ \gamma)}{dt} \right) + \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right)] = 0. \end{aligned}$$

Lemma 18. Let $\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 \cdot \varepsilon_4 = 1$, $(\varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = \varepsilon_4 = 1)$ in \mathbb{R}_2^4 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by

$$(6.9) \quad \varphi = (x_1, x_2, x_3, x_4), \quad x_4 = \frac{(x_1 + p_1)(x_2 + p_2)}{x_3 + p_3}$$

are

$$\begin{aligned} & \frac{d^2(u_1 \circ \gamma)}{dt^2} - 2 \frac{(u_2 + p_2)}{(u_3 + p_3)^2 \det G}(\gamma) \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} - \frac{(u_2 + p_2)}{(u_3 + p_3)}(\gamma) \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} \right. \\ & \left. - \frac{(u_1 + p_1)}{(u_3 + p_3)}(\gamma) \frac{d(u_2 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} + \frac{(u_1 + p_1)(u_2 + p_2)}{(u_3 + p_3)^2}(\gamma) \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right] = 0, \\ & \frac{d^2(u_2 \circ \gamma)}{dt^2} - 2 \frac{(u_1 + p_1)}{(u_3 + p_3)^2 \det G}(\gamma) \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} - \frac{(u_2 + p_2)}{(u_3 + p_3)}(\gamma) \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} \right. \\ & \left. - \frac{(u_1 + p_1)}{(u_3 + p_3)}(\gamma) \frac{d(u_2 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} + \frac{(u_1 + p_1)(u_2 + p_2)}{(u_3 + p_3)^2}(\gamma) \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right] = 0 \end{aligned}$$

and

$$\begin{aligned} & \frac{d^2(u_3 \circ \gamma)}{dt^2} - 2 \frac{(u_1 + p_1)(u_2 + p_2)}{(u_3 + p_3)^3 \det G}(\gamma) \left[\frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_2 \circ \gamma)}{dt} - \frac{(u_2 + p_2)}{(u_3 + p_3)}(\gamma) \frac{d(u_1 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} \right. \\ & \left. - \frac{(u_1 + p_1)}{(u_3 + p_3)}(\gamma) \frac{d(u_2 \circ \gamma)}{dt} \frac{d(u_3 \circ \gamma)}{dt} + \frac{(u_1 + p_1)(u_2 + p_2)}{(u_3 + p_3)^2}(\gamma) \left(\frac{d(u_3 \circ \gamma)}{dt} \right)^2 \right] = 0, \end{aligned}$$

where $\det G = \left(\frac{(u_1 + p_1)^2}{(u_3 + p_3)^2} - 1 \right) \left(\frac{(u_2 + p_2)^2}{(u_3 + p_3)^2} - 1 \right)$.

Lemma 19. Let $\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 \cdot \varepsilon_4 = 1$. ($\varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = \varepsilon_4 = 1$) in \mathbb{R}_2^4 . Then the equations of the geodesic curve of the minimal homothetical hypersurface determined by

$$(6.10) \quad \varphi = (x_1, x_2, x_3, x_4), \quad x_4 = -\frac{(x_1 + p_1)(x_2 + p_2)}{x_3 + p_3}$$

are the same as the equations in the Lemma 18.

Corollary 5. Each of the minimal homothetical hypersurfaces given by (6.1), (6.3), (6.5) is a plane for $p_2 = 0$ and corresponding geodesic curve is a straight line. The equations for these straight lines are

$$\begin{aligned} \gamma(t) &= (a_1t + b_1, a_2t + b_2, a_3t + b_3, (a_1t + b_1 + p_1) \tan q), \\ \gamma(t) &= (a_1t + b_1, a_2t + b_2, a_3t + b_3, (a_1t + b_1 + p_1) \tanh q), \\ \gamma(t) &= (a_1t + b_1, a_2t + b_2, a_3t + b_3, (a_1t + b_1 + p_1) \coth q), \end{aligned}$$

respectively.

Corollary 6. None of the equations of the geodesic curves of the minimal homothetical hypersurfaces given by (6.1), (6.3), (6.5), (6.7), (6.8), (6.9), (6.10) contains the term $\left(\frac{d(u_1 \circ \gamma)}{dt} \right)^2$.

Theorem 4. The geodesic curves of the homothetical hypersurfaces determined by

$$\varphi = (x_1, x_2, x_3, x_4), \quad x_4 = (x_1 + p_1) \tan(p_2 x_2 + q)$$

are the same in \mathbb{R}_1^4 and \mathbb{R}_2^4 . Hence, the geodesic curves are independent of index.

Proof. The proof follows from the equations (5.2) and (6.2). \square

Theorem 5. *The geodesic curves of the homothetical hypersurfaces determined by*

$$\varphi = (x_1, x_2, x_3, x_4), \quad x_4 = (x_1 + p_1) \tanh(p_2 x_2 + q)$$

are the same in \mathbb{R}_1^4 and \mathbb{R}_2^4 . Hence, the geodesic curves are independent of index.

Proof. See equations (5.4) and (6.4). \square

Theorem 6. *The geodesic curves of the homothetical hypersurfaces determined by*

$$\varphi = (x_1, x_2, x_3, x_4), \quad x_4 = (x_1 + p_1) \coth(p_2 x_2 + q)$$

are the same in \mathbb{R}_1^4 and \mathbb{R}_2^4 . Hence, the geodesic curves are independent of index.

Proof. See equations (5.6) and (6.6). \square

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Authors' address:

Derya Sağlam and Arif Sabuncuoğlu
 Department of Mathematics, Faculty of Sciences and Arts,
 ANS Campus, Afyon Kocatepe University, 03200, Afyonkarahisar Turkey.
 E-mail addresses: dryilmaz@science.ankara.edu.tr, sabuncu@science.ankara.edu.tr