Proper affine vector fields in Bianchi type V space-times

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Abstract. A study of Bianchi type V space-times according to its proper affine vector fields is given by using holonomy and decomposability, the rank of the 6×6 Riemannian matrix and direct integration techinques. Studying proper affine vector field in the above space-time it is shown that there exist only two cases when the above space-time admits proper affine vector fields.

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1 Introduction

The aim of this paper is to find the existence of proper affine vector fields in Bianchi type V space-times by using holonomy and decomposability, the rank of the 6×6 Rieman matrix and direct integration techinques. Through out M is representing the four dimensional, connected, Hausdorff space-time manifold with Lorentz metric g of signature $(-,+,+,+)$ The curvature tensor associated with g through Levi-Civita connection Γ, is denoted in component form by $R^a{}_{bcd}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol L, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively. The space-time M will be assumed nonflat in the sense that the Riemann tensor does not vanish over any nonempty open subset of M.

A vector field X on M is called an affine vector field if it satisfies

$$X_{a;bc} = R_{abcd}X^d. \tag{1.1}$$

or equivalently,

$$X_{a,bc} - \Gamma^f_{ac}X_{f,b} - \Gamma^f_{bc}X_{a,f} - \Gamma^e_{ab}X_{e,c} + \Gamma^e_{ab}\Gamma^f_{ec}X_f - (\Gamma^e_{ab})_{,c}X_e - \Gamma^f_{ab}\Gamma^e_{cf}X_e$$
$$+\Gamma^e_{fb}\Gamma^f_{ca}X_e + \Gamma^e_{af}\Gamma^f_{bc}X_e = R_{abcd}X^d.$$

Let X be a smooth vector field on M then in any coordinate system on M, one decompose X in the form

$$X_{a;b} = \frac{1}{2}h_{ab} + F_{ab}, \tag{1.2}$$

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where $h_{ab} = L_X g_{ab}$ and $F_{ab} = -F_{ba}$ are symmetric and skew symmetric tensors on M, respectively. It follows from equation (1.1) that

$$(1.3) \qquad (i) \quad h_{ab;c} = 0 \quad (ii) \quad F_{ab;c} = R_{abcd}X^d \quad (iii) \quad F_{ab;c}X^c = 0.$$

Such a vector field X is called affine if the local diffeomorphism ϕ_t (for appropriate t) associated with X map geodesics into geodesics. If $h_{ab} = 2cg_{ab}$, $c \in R$ then the vector field X is called homothetic (and Killing if $c = 0$.) The vector field X is said to be proper affine if it is not homothetic vector field and also X is said to be proper homothetic vector field if it is not Killing vector field on M [1]. Define the subspace S_p of the tangent space T_pM to M at p as those $k \in T_pM$ satisfying

$$(1.4) \qquad R_{abcd}k^d = 0.$$

2 Affine vector fields

In this section we will briefly discuss when the space-times admit proper affine vector fields for further details see [5].

Suppose that M is a simple connected space-time. Then the holonomy group of M is a connected Lie subgroup of the idenity component of the Lorentz group and is thus characterized by its subalgebra in the Lorentz algebra. These have been labeled into fifteen types $R_1 - R_{15}$ [3]. It follows from [5] that the space-times admitting nowhere zero covariantly constant second order symmetric tensor field h_{ab} are the only space-times which could admit proper affine vector fields. It is also known that this forces the holonomy type to be either R_2, R_3, R_4, R_6, R_7, R_8, R_{10}, R_{11} or R_{13}. Here, we will only discuss the space-times which has the holonomy type R_{10} and R_{13}. A study of the affine vector fields of the space-times which has the holonomy type R_2, R_3, R_4, R_6, R_7, R_8, or R_{11} can be found in [1].

First consider the case when M has type R_{13}. Then one can always set up local coordinates (t, x^1, x^2, x^3) on an open set $U = U_1 \times U_2$, where U_1 is a one dimensional timelike submanifold of U coordinatized by t and U_2 is a three dimensional spacelike submanifold of U coordinatized by x^1, x^2, x^3 and where the above product is a metric product and the metric on U is given by [1]

$$(2.1) \qquad ds^2 = -dt^2 + g_{\alpha\beta}dx^\alpha dx^\beta, \quad (\alpha, \beta = 1, 2, 3)$$

where $g_{\alpha\beta}$ depends on x^γ, $(\gamma = 1, 2, 3)$. The above space-time is clearly $1 + 3$ decomposable. The curvature rank of the above space-time is atmost three and there exists a unique nowhere zero vector field $t_a = t_{,a}$ satisfying $t_{a;b} = 0$ and also $t^a t_a = -1$. From the Ricci identity $R^a{}_{bcd}t^d = 0$. It follows from [5] that affine vector fields in this case are

$$(2.2) \qquad X = (c_1 t + c_2)\frac{\partial}{\partial t} + Y$$

where $c_1, c_2 \in R$ and Y is a homothetic vector field in the induced geometry on each of the three dimensional submanifolds of constant t.

Now consider the situation when M has type R_{10}. The situation is similar to that of previous R_{13} case except that now we have local decomposition is $U = U_1 \times U_2$, where U_1 is a one dimensional spacelike submanifold of U and U_2 is a three dimensional timelike submanifold of U, respectively. The space-time metric on U is given by [1]

$$ds^2 = dx^2 + g_{\alpha\beta} dx^\alpha dx^\beta, \quad (\alpha, \beta = 0, 2, 3) \tag{2.3}$$

where $g_{\alpha\beta}$ depends on x^γ, $(\gamma = 0, 2, 3)$. The above space-time is clearly $1+3$ decomposable. The curvature rank of the above space-time is atmost three and there exists a unique nowhere zero vector field $x_a = x_{,a}$ satisfying $x_{a;b} = 0$ and also $x^a x_a = 1$. From the Ricci identity $R^a{}_{bcd} x^d = 0$. It follows from [5] that affine vector fields in this case are

$$X = (c_1 x + c_2)\frac{\partial}{\partial x} + Y \tag{2.4}$$

where $c_1, c_2 \in R$ and Y is a homothetic vector field in the induced geometry on each of the three dimensional submanifolds of constant x.

3 Main results

As mentioned in section 2, the space-times which can admit proper affine vector fields have holonomy type R_2, R_3, R_4, R_6, R_7, R_8, R_{10}, R_{11} or R_{13} and it follows from [4] that the rank of the 6×6 Riemann matrix is atmost three. Here in this paper we will consider the rank of the 6×6 Riemann matrix to study affine vector fields in Bianchi type V space-time. Consider a Bianchi type V space-time in the usual coordinate system (t, x, y, z) (labeled by (x^0, x^1, x^2, x^3), respectively) with line elememt [7]

$$ds^2 = -dt^2 + A^2(t)dx^2 + e^{2qx}(X^2(t)dy^2 + Y^2(t)dz^2), \tag{3.1}$$

where $A(t)$, $X(t)$ and $Y(t)$ are some nowhere zero functions of t only and q is a non zero constant on M (if $q = 0$ then the above space-time become Bianchi type I and their affine vector fields are given in [6]). The above space-time admits three independent Killing vector fields which are $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial x} - qy\frac{\partial}{\partial y} - qz\frac{\partial}{\partial z}$. The non-zero independent components of the Riemann tensor are

$$R_{0101} = -A\ddot{A} \equiv \alpha_1, \quad R_{0202} = -e^{2qx} X\ddot{X} \equiv \alpha_2,$$

$$R_{0303} = -e^{2qx} Y\ddot{Y} \equiv \alpha_3, \quad R_{1212} = e^{2qx} A^2 X^2 (\frac{\dot{A}\dot{X}}{AX} - \frac{q^2}{A^2}) \equiv \alpha_4,$$

$$R_{1313} = e^{2qx} A^2 Y^2 (\frac{\dot{A}\dot{Y}}{AY} - \frac{q^2}{A^2}) \equiv \alpha_5, \quad R_{2323} = e^{4qx} X^2 Y^2 (\frac{\dot{X}\dot{Y}}{XY} - \frac{q^2}{A^2}) \equiv \alpha_6,$$

$$R_{0212} = qe^{2qx} X^2 (\frac{\dot{A}}{A} - \frac{\dot{X}}{X}) \equiv \alpha_7, \quad R_{0313} = qe^{2qx} Y^2 (\frac{\dot{A}}{A} - \frac{\dot{Y}}{Y}) \equiv \alpha_8.$$

Writing the curvature tensor with components R_{abcd} at p as a 6×6 symmetric matrix in a well known way [2]

$$(3.2)\qquad R_{abcd}=\begin{pmatrix}\alpha_1 & 0 & 0 & 0 & 0 & 0\\ 0 & \alpha_2 & 0 & \alpha_7 & 0 & 0\\ 0 & 0 & \alpha_3 & 0 & \alpha_8 & 0\\ 0 & \alpha_7 & 0 & \alpha_4 & 0 & 0\\ 0 & 0 & \alpha_8 & 0 & \alpha_5 & 0\\ 0 & 0 & 0 & 0 & 0 & \alpha_6\end{pmatrix},$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$, α_6, α_7 and α_8 are real functions of t. The possible rank of the 6×6 Riemann matrix is six, five, four or three. Ranks two and one is not possible for the following reason: Suppose the rank of the 6×6 Riemann matrix is two. Then there exist only two non-zero rows or columns in (3.2). If we set any four rows or columns identically zero in (3.2) with judicious choice of A, X and Y, this forces the rank of the 6×6 Riemann matrix to be zero thus giving a contradiction. Similar argument will also apply to the rank one. We are only interested in those cases when the rank of the 6×6 Riemann matrix is less than or equal to three. Thus there exist only two possibilities:
$(A1) Rank = 3, \alpha_1 = \alpha_2 = \alpha_3 = \alpha_7 = \alpha_8 = 0, \alpha_4 \neq 0, \alpha_5 \neq 0$ and $\alpha_6 \neq 0$.
$(A2) Rank = 3, \alpha_1 = \alpha_2 = \alpha_4 = \alpha_7 = 0, \alpha_3 \neq 0, \alpha_5 \neq 0, \alpha_6 \neq 0$ and $\alpha_8 \neq 0$.
We will consider each case in turn.

Case A1

In this case $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_7 = \alpha_8 = 0, \alpha_4 \neq 0, \alpha_5 \neq 0, \alpha_6 \neq 0$, the rank of the 6×6 Riemann matrix is 3 and there exists a unique (up to a multiple) nowhere zero timelike vector field $t_a = t_{,a}$ solution of equation (1.4) and $t_{a;b} \neq 0$. From the above constraints we have $A = a_1t + a_2$, $X = b_1t + b_2$, $Y = c_1t + c_2$, $A = k_1X$, $Y = k_2A$, and $\frac{a_2b_1}{b_2} \neq \pm q$, where $a_1, a_2, b_1, b_2, c_1, c_2, k_1, k_1 \in R - \{0\}$. The line element can, after a rescaling of x, y and z, be written in the form

$$(3.3)\qquad ds^2 = -dt^2 + (b_1t + b_2)^2(dx^2 + e^{2qx}(dy^2 + dz^2)).$$

Substituting the above information into the equations (1.1) and after some calculation one find affine vector fields in this case are

$$(3.4)\qquad Z^0 = td_5, Z^1 = d_1, Z^2 = -qyd_1 - zd_3 + d_2, Z^3 = -qzd_1 + yd_3 + d_4,$$

where $d_1, d_2, d_3, d_4, d_5 \in R$. One can write the above equation (3.4), after subtracting Killing vector fields as

$$(3.5)\qquad Z = (t, 0, 0, 0).$$

Clearly, in this case the above space-time admit proper affine vector field.

Now consider the sub case when $b_1 = 0$ in (3.3) the line element can, after a rescaling of x, y and z, be written in the form

$$(3.6)\qquad ds^2 = -dt^2 + (dx^2 + e^{2qx}(dy^2 + dz^2)),$$

where $q \neq 0, 1$. The space-time is clearly $1 + 3$ decomposable. In this case the rank of the 6×6 Riemann matrix is 3 there exists a unique (up to a multiple) nowhere zero timelike vector field $t_a = t_{,a}$ such that $t_{a;b} = 0$ (and so, from the Ricci idenity $R^a{}_{bcd}t_a = 0$). The affine vector fields in this case [5] are

$$X = (c_7 x + c_8)\frac{\partial}{\partial t} + Z^{'} \tag{3.7}$$

where $c_7, c_8 \in R$ and $Z^{'}$ is a homothetic vector field in the induced geometry on each of the three dimensional submanifolds of constant t. The completion of case A1 necessities finding an homothetic vector fields in the induced geometry of the submanifolds of constant t. The induced metric $g_{\alpha\beta}$ (where $\alpha, \beta = 1, 2, 3$) with nonzero components is given by

$$g_{11} = 1, \quad g_{22} = e^{2qx}, \quad g_{33} = e^{2qx}. \tag{3.8}$$

A vector field $Z^{'}$ is a homothetic vector field if it satisfies $L_{Z'}g_{\alpha\beta} = 2cg_{\alpha\beta}$, where $c \in R$. One can expand by using (3.8) to get

$$\begin{array}{c} Z^1{}_{,1} = c, \quad Z^1{}_{,2} + e^{2qx}Z^2{}_{,1} = 0, \quad Z^1{}_{,3} + e^{2qx}Z^3{}_{,1} = 0 \\ qZ^1 + Z^2{}_{,2} = c, \qquad Z^2{}_{,3} + X^3{}_{,2} = 0, \qquad qZ^1 + Z^3{}_{,3} = c. \end{array} \tag{3.9}$$

Equations (3.9) give

$$Z^1 = cx + A^1(y,z), \quad Z^2 = \frac{1}{2q}e^{-2qx}A^1_y(y,z) + A^2(y,z),$$

$$Z^3 = \frac{1}{2q}e^{-2qx}A^1_z(y,z) + A^3(y,z),$$

where $A^1(y,z)$, $A^2(y,z)$ and $A^3(y,z)$ are functions of integration. If one proceeds further after a strightforward calculation one can find that proper homothetic vector fields exist if $q = 0$, which is not possible. So homothetic vector fields in the induced gemetry are the Killing vector fields which are

$$\begin{aligned} Z^1 &= yc_1 + zc_3 + c_5, \\ Z^2 &= \frac{1}{2q}e^{-2qx}c_1 + \frac{q}{2}(z^2 - y^2)c_1 - qyzc_3 - qyc_5 - zc_6 + c_2, \\ Z^3 &= \frac{1}{2q}e^{-2qx}c_3 + \frac{q}{2}(y^2 - z^2)c_3 - qyzc_1 - qzc_5 + yc_6 + c_4, \end{aligned} \tag{3.10}$$

where $c_1, c_2, c_3, c_4, c_5, c_6 \in R$. Affine vector fields in this case are (from (3.7) and (3.10))

$$\begin{aligned} Z^0 &= tc_7 + c_8, \\ Z^1 &= yc_1 + zc_3 + c_5, \\ Z^2 &= \frac{1}{2q}e^{-2qx}c_1 + \frac{q}{2}(z^2 - y^2)c_1 - qyzc_3 - qyc_5 - zc_6 + c_2, \\ Z^3 &= \frac{1}{2q}e^{-2qx}c_3 + \frac{q}{2}(y^2 - z^2)c_3 - qyzc_1 - qzc_5 + yc_6 + c_4, \end{aligned} \tag{3.11}$$

One can write the above equation (3.11), after subtracting Killing vector fields as

$$Z = (tc_7 + c_8, 0, 0, 0). \tag{3.12}$$

Clearly, the above space-time (3.6) admits proper affine vector fields.

Case A2

In this case $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_7 = 0, \alpha_3 \neq 0, \alpha_5 \neq 0, \alpha_6 \neq 0, \alpha_8 \neq 0$, the rank of the 6×6 Riemann matrix is 3 and there exist no non-trival solutions of equation (1.4). Equations $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_7 = 0 \Rightarrow A = a_1 t + a_2, X = b_1 t + b_2, q = \pm a_1$ and $X = kA$, respectively where $a_1, a_2, b_1, b_2, k \in R (a_1, b_1, k \neq 0)$. The line element can, after a rescaling of y, be written in the form

$$ds^2 = -dt^2 + (a_1 t + a_2)^2 dx^2 + e^{2qx}((a_1 t + a_2)^2 dy^2 + Y^2(t) dz^2). \tag{3.13}$$

Affine vector fields in this case are

$$Z^0 = 0, Z^1 = c_1, Z^2 = -qyc_1 + c_2, Z^3 = -qzc_1 + c_3, \tag{3.14}$$

where $c_1, c_2, c_3 \in R$. Affine vector fields in this case are Killing vector fields.

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