The tangents and the chords of the Poincaré circle

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Abstract. After defining the tangent and chord concepts, we determine the chord length and tangent lines of Poincaré circle and the relationship between the tangent lines and the chords of a Poincaré circle.

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1 Introduction

The Poincaré upper half plane geometry has been introduced by Henri Poincaré. The Poincaré upper half plane $\mathbb{H}$ is the upper half plane of the Euclidean analytical plane $\mathbb{R}^2$. Although the points in *Poincaré upper half plane* are same as the points in the upper half plane of the Euclidean analytical plane $\mathbb{R}^2$, the lines and the distance function between any two points are different. The lines in the Poincaré upper halfplane are defined by

half lines : ${}_aL = \left\{(x,y) \in \mathbb{R}^2 \mid x = a,\ y > 0,\ a \in \mathbb{R},\ a \text{ is a constant}\right\}$
half circles : ${}_cL_r = \left\{(x,y) \in \mathbb{R}^2 \mid (x-c)^2 + y^2 = r^2,\ y > 0\ ,\ c, r \in \mathbb{R}, c, r \text{ are constants}\right\}$.

If $A = (x_1, y_1)$ and $B = (x_2, y_2)$ are any two points in H then the *Poincar*'e *distance* between these points is given by

$$d_H(A,B) = \begin{cases} \left|\ln\left(\frac{y_2}{y_1}\right)\right|, x_1 = x_2 \\ \\ \left|\ln\left(\frac{y_2(x_1-c+r)}{y_1(x_2-c+r)}\right)\right|, x_1 \neq x_2, \end{cases}$$

where

$$c = \frac{y_2^2 - y_1^2 + x_2^2 - x_1^2}{2(x_2 - x_1)}$$
$$r = \sqrt{(x_1 - c)^2 + y_1^2} = \sqrt{(x_2 - c)^2 + y_2^2}$$

The Poincaré upper half plane geometry is a non-Euclidean, since it fails to satisfy the parallel postulate within the thirteen axioms [2, 3, 5] of the Euclidean plane geometry. In Poincaré upper half plane geometry, the lines and the function of distance

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are different, therefore, it seems interesting to study the Poincaré analogues of the topics that include the concept of distance in the Euclidean geometry. A few of such topics have been studied by some authors, see [1, 3, 4, 5, 6, 7, 8]. One of these is a study in [5, pp. 80–81] that is about every Poincaré circle in the upper half plane is an Euclidean circle. In this work, I defined tangent and chord concepts. Then, the chord length and tangent lines of Poincaré circle and the relationship between the tangent lines and the chords of a Poincaré circle have been determined.

In [5], Stahl showed the following results concerning Poincaré circle and Euclidean circle.

Proposition 1.1. *If a circle Euclidean center (h, k) and a Euclidean radius r, then it has the hyperbolic center (H, K), and the hyperbolic radius R, where*

$$H = h, \quad K = \sqrt{k^2 - r^2}, \quad R = \frac{1}{2}\left|\ln\left(\frac{k+r}{k-r}\right)\right|$$

and

$$h = H, \quad k = K\cosh(R), \quad r = K\sinh(R)$$

Theorem 1. *Every Euclidean circle in the upper half-plane is also a* hyperbolic circle. *Since the equation of a circle with Euclidean center (h, k) and an Euclidean radius r is $(x-h)^2 + (y-k)^2 = r^2$ the equation of a circle in the Poincaré upper half plane* H *is*

$$(x - H)^2 + (y - K\cosh(R))^2 = K^2\sinh^2(R)$$

Proposition 1.2. *Let q is a circle with Euclidean center (h, k) and Euclidean radius r. The hyperbolic length of q is*

$$Ç = \frac{2\pi r}{\sqrt{k^2 - r^2}} = \frac{2\pi r}{K}$$

2 The Chords of Poincaré Circle

Definition 2.1. *A line segment that joins two points on the Poincaré circle is called as* chord.

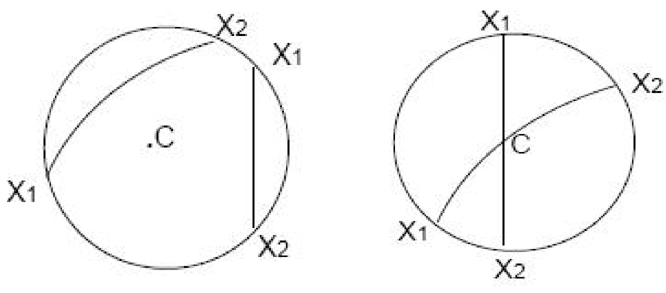


Fig.1. Some chords Fig.2. Some diameters

If X_1 and X_2 are points on the Poincaré circle $Q\,(X_1 \neq X_2)$ then the length of the chord X_1X_2 is equal to the length of Poincaré line which is joined the points X_1 and X_2 on the Poincaré circle Q.

Let $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$ $(y_1, y_2 > 0)$ are points on the Poincaré circle Q with hyperbolic center (H, K) and hyperbolic radius R. Since the equation of Poincaré circle is

$$(x - H)^2 + (y - K\cosh(R))^2 = K^2 \sinh^2(R)$$

(see Theorem 1) the coordinates of the points X_1 and X_2 are defined

$$X_1 = \left(x_1, K\cosh(R) \pm \sqrt{K^2\sinh^2(R) - (x_1 - H)^2}\right)$$
$$X_2 = \left(x_2, K\cosh(R) \pm \sqrt{K^2\sinh^2(R) - (x_2 - H)^2}\right)$$

(**A**) If $x_1 \neq x_2$.

(**i**) Let $y_1 \neq y_2$. Then

$$y_1 = K\cosh(R) + \sqrt{K^2\sinh^2(R) - (x_1 - H)^2}$$

or

$$y_1 = K\cosh(R) - \sqrt{K^2\sinh^2(R) - (x_1 - H)^2}$$

and

$$y_2 = K\cosh(R) + \sqrt{K^2\sinh^2(R) - (x_2 - H)^2}$$

or

$$y_2 = K\cosh(R) - \sqrt{K^2\sinh^2(R) - (x_2 - H)^2}$$

and

$$d_H(X_1, X_2) = \left|\ln\left(\frac{y_2(x_1 - d + s)}{y_1(x_2 - d + s)}\right)\right|$$
$$d = \frac{y_2^2 - y_1^2 + x_2^2 - x_1^2}{2(x_2 - x_1)}$$
$$s = \sqrt{(x_1 - d)^2 + y_1^2} = \sqrt{(x_2 - d)^2 + y_2^2}.$$

(**ii**) Let $y_1 = y_2$

$$d_H(X_1, X_2) = \left|\ln\left(\frac{x_1 - d + s}{x_2 - d + s}\right)\right|$$
$$d = \frac{x_1 + x_2}{2}$$
$$s = \sqrt{(x_1 - d)^2 + y_1^2}$$

(**B**) If $x_1 = x_2$.

$$X_1 = \left(x_1, K\cosh(R) - \sqrt{K^2\sinh^2(R) - (x_1 - H)^2}\right),$$
$$X_2 = \left(x_2, K\cosh(R) + \sqrt{K^2\sinh^2(R) - (x_2 - H)^2}\right)$$

$$d_H(X_1, X_2) = \left|\ln\left(\frac{y_2}{y_1}\right)\right| = \left|\ln\left(\frac{K\cosh(R) + \sqrt{K^2\sinh^2(R) - (x_1 - H)^2}}{K\cosh(R) - \sqrt{K^2\sinh^2(R) - (x_1 - H)^2}}\right)\right|$$

Also, since the hyperbolic length of a circle is $\frac{2\pi r}{\sqrt{k^2-r^2}}$ the length of Poincaré arc X_1X_2 is

$$d_H(X_1, X_2) = \frac{\alpha}{360}\frac{2\pi r}{\sqrt{k^2 - r^2}}$$

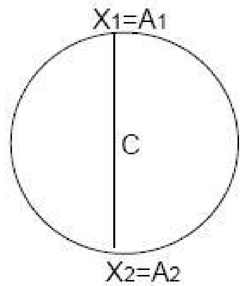
(see Proposition 1.2).

Definition 2.2. *If the length of a chord is equal to the length of a diameter then it is called as* long chord*.*

(**C**) If $X_1 = A_1$ and $X_2 = A_2$ then the length of a long chord is

$$d_H(X_1, X_2) = \left|\ln\left(\frac{y_2}{y_1}\right)\right| = \left|\ln\left(\frac{K\cosh(R) + \sqrt{K^2\sinh^2(R)}}{K\cosh(R) - \sqrt{K^2\sinh^2(R)}}\right)\right| = \left| ln\frac{k+r}{k-r}\right| = 2R$$

(see the following figure).



(**D**) If $X_1 = B_1$ and $X_2 = B_2$ then the length of a long chord is as follows.

(**i**) Let $d = H$ ($x_1 \neq x_2$ and $y_1 = y_2$). Then

$$d_H(X_1, X_2) = \left|\ln\left(\frac{(x_1 - d + s)}{(x_2 - d + s)}\right)\right| = \left|\ln\left(\frac{(x_1 - H + s)}{(x_2 - H + s)}\right)\right|$$
$$s = \sqrt{(x_1 - d)^2 + y_1^2} = \sqrt{(x_1 - H)^2 + y_1^2}$$

(ii) Let $d \neq H$ ($x_1 \neq x_2$ and $y_1 \neq y_2$). Then

$$d_H(X_1, X_2) = \left| \ln \left(\frac{y_2(x_1 - d + s)}{y_1(x_2 - d + s)} \right) \right|$$
$$d = \frac{y_2^2 - y_1^2 + x_2^2 - x_1^2}{2(x_2 - x_1)},$$
$$s = \sqrt{(x_1 - d)^2 + y_1^2} = \sqrt{(x_2 - d)^2 + y_2^2}$$

(see the following figures).

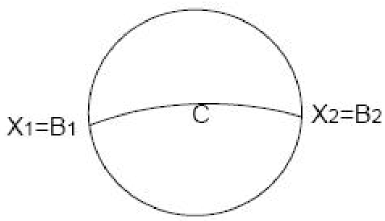


Fig.4. $d = H$

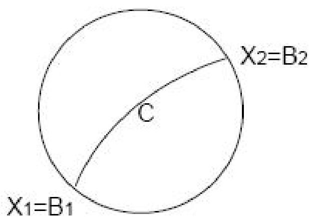


Fig.2. $d \neq H$

In two cases, since we can find to the hyperbolic reflection that transforms respectively the points B_1 and B_2 on to the points A_1 and A_2 [5], the distance between the points is

$$d_H(X_1, X_2) = \left| \ln \left(\frac{k + r}{k - r} \right) \right| = 2R$$

3 The Tangents of Poincaré Circle

In the Poincaré upper half plane, the tangent line is Euclidean line that is a line through a point on a circle [3]. We know that the tangent line that is a line through every point on a circle is perpendicular to diameter. Now, we can examine this condition.

Let Q is a circle with hyperbolic center $C = (H, K)$ and hyperbolic radius R.

(1) $x = H + R$ is the tangent line that is a line through a point $A_1 = (x_1, y_1) = (H + R, K)$ and is perpendicular to diameter. The chord that is perpendicular to this tangent is the line ${}_{H+R}L_P$ that is center $M = (H + R, 0)$ and radius P [4]. If $C = (H, K)$ is replaced in equation of line then

$$(x - (H + R))^2 + y^2 = P^2$$
$$\sqrt{P^2 - (H - (H + R))^2} = \sqrt{P^2 - R^2} \neq K$$

(2) $x = H - R$ is the tangent line that is a line through a point $A_2 = (H - R, K)$ and is perpendicular to diameter. The chord that is perpendicular to this tangent is the line ${}_{H-R}L_S$ that is center $M = (H - R, 0)$ and radius S . If $C = (H, K)$ is replaced in equation of line then

$$(x - (H - R))^2 + y^2 = S^2$$
$$\sqrt{S^2 - (H - (H - R))^2} = \sqrt{S^2 - R^2} \neq K$$

In two cases, center isn't on the chord and also the chord isn't diameter. But the chord that is perpendicular to tangent is long chord and $d_H(X, A_1) = 2R = d_H(X, A_2)\, X \in Q$ and $d_H(C, A_1) = R = d_H(C, A_2)$.

(**3**) $y = K + R$ is the tangent line that is a line through a point $A_3 = (H, K + R)$ and is perpendicular to diameter. As the same as $y = K - R$ is the tangent line that is a line through a point $A_4 = (H, K - R)$ and is perpendicular to diameter. In two cases the chord that is perpendicular to this tangents is the line ${}_HL$ and for the point $C = (H, K)$ since $x = H$ the chord is diameter. In this condition $d_H(C, A_3) = R = d_H(C, A_4)$.

Corollary 3.1. *The tangents $x = H + R$, $x = H - R$, $y = K + R$ and $y = K - R$ are perpendicular to the Poincaré diameter .*

References

[1] J.W. Anderson, *Hyperbolic Geometry*, Springer-Verlag London Berlin Heidelberg 1999.

[2] E.F. Krause, *Taxicab Geometry*, Addison-Wesley, Menlo Park, California 1975.

[3] R.S. Millman, and G.D. Parker, *Geometry. A Metric Approach with Models*, Springer-Verlag New York Inc. 1991.

[4] N. Sönmez, *On The Geometry of Poincare Half Plane*, Ph.D. Thesis, Osmangazi Üniversity Institute of Science and Tecnology, 2006.

[5] Stahl, S.,*The Poincare Half Plane - A Gateway to Modern Geometry*, Jones and Barlett Publishers, Boston 1993.

[6] J. Stilwell, *Sources of Hyperbolic Geometry*, American Mathematical Society 1996.

[7] H.H. Uğurlu, M. Kazaz, A. Özdemir, *Stewart, Ceva and Menelaus Theorems for geodesic triangles on the dual hyperbolic unit sphere H_0^2 of the dual Lorentzian 3-space D_1^3*, III. Geometry Symposium, Eskisehir, 2005.

[8] Yakut, A.T., *Vertex Angles of Simplices in Hyperbolıc Space*, Ph. D. Thesis, Gazi Üniversity Institute of Science and Tecnology 2004.

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