On concircular φ-recurrent LP-Sasakian manifolds

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Abstract. Concircular φ-recurrent LP-Sasakian manifolds are studied. It is shown that any concircular φ-recurrent LP-Sasakian manifold is an Einstein manifold and that any concircular φ-recurrent LP-Sasakian manifold having a non-zero constant sectional curvature is locally concircular φ-symmetric.

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Key words: LP-Sasakian manifold, concircular φ-symmetric manifold, concircular φ-recurrent manifold, Einstein manifold.

1 Introduction

In 1989, K. Matsumoto [4] introduced the notion of LP-Sasakian manifold. Then I. Mihai and R. Rosca [6] introduced the same notion independently and they obtained several results on this manifold. LP-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [5] , U.C.De and et al., [2].

T. Takahashi in his paper [7] introduced the notion of locally φ-symmetric Sasakian manifold and obtained few interesting properties. Many authors like U.C. De and G. Pathak [3], Venkatesh and C.S. Bagewadi [8], A.A. Shaikh and U.C. De [1] have extended this notion to 3-dimensional Kenmotsu, trans-Sasakian and LP-Sasakian manifolds respectively. In this paper we study a concircular φ-recurrent LP-Sasakian manifold which generalizes the notion of locally concircular φ-symmetric LP-Sasakian manifold and obtain some interesting results. Here we show that a concircular φ-recurrent LP-Sasakian manifold is an Einstein manifold and concircular φ-recurrent LP-Sasakian manifold having a non-zero constant sectional curvature is locally concircular φ-symmetric.

2 Preliminaries

An 2n+1 dimensional differentiable manifold M^{2n+1} is called an LP-Sasakian manifold [4],[5] if it admits a (1, 1) tensor field φ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy:

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$$\varphi^2 = I + \eta \otimes \xi, \tag{2.1}$$
$$\eta(\xi) = -1, \tag{2.2}$$
$$g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$
$$(a)\ \nabla_X \xi = \varphi X, \quad (b)\ g(X,\xi) = \eta(X), \tag{2.4}$$
$$(\nabla_X \varphi)Y = g(X,Y)\xi + 2\eta(X)\eta(Y)\xi \tag{2.5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

It can be easily seen that in an LP-Sasakian manifold, the following relations hold:

$$\varphi\xi = 0, \quad \eta(\varphi X) = 0. \tag{2.6}$$

Again if we put

$$\Omega(X,Y) = g(X, \varphi Y), \tag{2.7}$$

for any vector fields X and Y, then the tensor field $\Omega(X,Y)$ is a symmetric (0, 2) tensor field [4].
Also since the vector field η is closed in an LP-Sasakian manifold, we have [4], [2],

$$(\nabla_X \eta)(Y) = \Omega(X,Y), \ \Omega(X,\xi) = 0, \tag{2.8}$$

for any vector fields X and Y.

Also in an LP-Sasakian manifold, the following relations hold[5],[2]:

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.9}$$
$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.10}$$
$$S(X,\xi) = 2n\ \eta(X), \tag{2.11}$$
$$S(\varphi X, \varphi Y) = S(X,Y) + 2n\ \eta(X)\eta(Y), \tag{2.12}$$

for any vector fields X, Y, Z, where R(X, Y)Z is the curvature tensor, and S is the Ricci tensor.

Definition 1. *[1] An LP-Sasakian manifold is said to be locally φ-symmetric if*

$$\varphi^2((\nabla_W R)(X,Y)Z) = 0, \tag{2.13}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 2. *An LP-Sasakian manifold is said to be locally concircular φ-symmetric if*

$$\varphi^2((\nabla_W \widetilde{C})(X,Y)Z) = 0, \tag{2.14}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 3. *An LP-Sasakian manifold is said to be concircular φ-recurrent manifold if there exists a non-zero 1-form A such that*

$$\varphi^2((\nabla_W \widetilde{C})(X,Y)Z) = A(W)\widetilde{C}(X,Y)Z, \tag{2.15}$$

for arbitrary vector fields X, Y, Z, W, where $\widetilde{C}$ is a concircular curvature tensor given by

$$\widetilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}\left[g(Y,Z)X - g(X,Z)Y\right], \tag{2.16}$$

where R is the curvature tensor, and r is the scalar curvature.

If the 1-form A vanishes, then the manifold reduces to a locally concircular φ-symmetric manifold.

3 Concircular φ-recurrent LP-Sasakian manifold

Let us consider a concircular φ-recurrent LP-Sasakian manifold. Then by virtue of (2.1) and (2.13) we have,

$$(\nabla_W \widetilde{C})(X,Y)Z + \eta((\nabla_W \widetilde{C})(X,Y)Z)\xi = A(W)\widetilde{C}(X,Y)Z, \tag{3.1}$$

from which it follows that

$$\begin{aligned} g((\nabla_W \widetilde{C})(X,Y)Z,U) + \eta((\nabla_W \widetilde{C})(X,Y)Z)\eta(U) \\ = A(W)g(\widetilde{C}(X,Y)Z,U). \end{aligned} \tag{3.2}$$

Let $\{e_i\}, i = 1,2,...,2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.2) and taking summation over $i, 1 \leq i \leq 2n+1$, we get

$$\begin{aligned} (\nabla_W S)(Y,Z) &= \frac{dr(W)}{2n+1}g(Y,Z) - \frac{dr(W)}{2n(2n+1)}\left[g(Y,Z) + \eta(Y)\eta(Z)\right] \\ &\quad +A(W)\left[S(Y,Z) - \frac{r}{(2n+1)}g(Y,Z)\right]. \end{aligned} \tag{3.3}$$

Replacing Z by ξ in (3.3)and using (2.4), we have

$$(\nabla_W S)(Y,\xi) = \frac{dr(W)}{2n+1}\eta(Y). \tag{3.4}$$

Now we have

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi)$$

Using (2.4) and (2.11) in the above relation, it follows that,

$$(\nabla_W S)(Y,\xi) = 2ng(\varphi Y,W) - S(\varphi Y,W). \tag{3.5}$$

In view of (3.4) and (3.5) we have

$$S(Y,\varphi W)=2ng(Y,\varphi W)-\frac{dr(W)}{2n+1}\eta(Y). \tag{3.6}$$

Replacing Y by φY in (3.6) and using (2.3),(2.6) and (2.12), we obtain

$$S(Y,W)=2ng(Y,W). \tag{3.7}$$

This leads to the following theorem:

Theorem 3.1. *A Concircular φ-recurrent LP-Sasakian manifold* (M^{2n+1},g) *is an Einstein manifold.*

Now from (3.1) we have

$$(\nabla_W\widetilde{C})(X,Y)Z=-\eta((\nabla_W\widetilde{C})(X,Y)Z)\xi+A(W)\widetilde{C}(X,Y)Z. \tag{3.8}$$

This implies

$$\begin{aligned}(\nabla_W R)(X,Y)Z \;=\;& -\eta((\nabla_W R)(X,Y)Z)\xi+A(W)R(X,Y)Z\\ &+\frac{dr(W)}{2n(2n+1)}[g(Y,Z)X-g(X,Z)Y\\ &+g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi]\\ &-\frac{r}{2n(2n+1)}A(W)\left[g(Y,Z)X-g(X,Z)Y\right]\end{aligned} \tag{3.9}$$

From (3.9) and the Bianchi identity we get

$$\begin{aligned}&A(W)\eta(R(X,Y)Z)+A(X)\eta(R(Y,W)Z)+A(Y)\eta(R(W,X)Z)\\ =\;&\frac{r}{2n(2n+1)}A(W)\left[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\right]\\ &+\frac{r}{2n(2n+1)}A(X)\left[g(W,Z)\eta(Y)-g(Y,Z)\eta(W)\right]\\ &+\frac{r}{2n(2n+1)}A(Y)\left[g(X,Z)\eta(W)-g(W,Z)\eta(X)\right]\end{aligned} \tag{3.10}$$

By virtue of (2.9) we obtain from (3.10) that

$$\begin{aligned}&A(W)\left[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\right]+A(X)[g(W,Z)\eta(Y)\\ &-g(Y,Z)\eta(W)]+A(Y)\left[g(X,Z)\eta(W)-g(W,Z)\eta(X)\right]\\ =\;&\frac{r}{2n(2n+1)}A(W)\left[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\right]\\ &+\frac{r}{2n(2n+1)}A(X)\left[g(W,Z)\eta(Y)-g(Y,Z)\eta(W)\right]\\ &+\frac{r}{2n(2n+1)}A(Y)\left[g(X,Z)\eta(W)-g(W,Z)\eta(X)\right].\end{aligned} \tag{3.11}$$

Putting $Y = Z = e_i$ in (3.11) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$A(W)\eta(X) = A(X)\eta(W), \tag{3.12}$$

for all vector fields X, W. Replacing X by ξ in (3.12), we get

$$A(W) = -\eta(W)\eta(\rho), \tag{3.13}$$

for any vector field W, where $A(\xi) = g(\xi, \rho) = \eta(\rho)$, ρ being the vector field associated to the 1-form A i.e.,

$$g(X, \rho) = A(X).$$

From (3.12) and (3.13) we can state the following;

Theorem 3.2. *In a Concircular φ-recurrent LP-Sasakian manifold (M^{2n+1}, g), $n \geq 1$, the characteristic vector field ξ and the vector field ρ associated to the 1-form A are co-directional and the 1-form A is given by (3.13).*

From (2.16) it follows that

$$(\nabla_W \widetilde{C})(X,Y)\xi = (\nabla_W R)(X,Y)\xi - \frac{dr(W)}{2n(2n+1)}\left[\eta(Y)X - \eta(X)Y\right]. \tag{3.14}$$

In view of (2.4) and (2.10) it can be easily seen that in a LP-Sasakian manifold the following relation holds:

$$(\nabla_W R)(X,Y)\xi = g(\varphi Y, W)X - g(\varphi X, W)Y - R(X,Y)\varphi W. \tag{3.15}$$

Substituting (3.15) in (3.14), we obtain

$$\begin{aligned}(\nabla_W \widetilde{C})(X,Y)\xi &= g(\varphi Y, W)X - g(\varphi X, W)Y - R(X,Y)\varphi W \\ &\quad - \frac{dr(W)}{2n(2n+1)}\left[\eta(Y)X - \eta(X)Y\right].\end{aligned} \tag{3.16}$$

By virtue of (2.9) it follows from (3.16) that

$$\eta\left((\nabla_W \widetilde{C})(X,Y)\xi\right) = 0. \tag{3.17}$$

Also in a LP-Sasakian manifold the following relation holds:

$$\begin{aligned}R(X,Y)\varphi W &= g(Y,W)\varphi X + g(\varphi X, W)Y + 2g(\varphi X, W)\eta(Y)\xi \\ &\quad + 2\eta(Y)\eta(W)\varphi X - g(X,W)\varphi Y - g(\varphi Y, W)X \\ &\quad - 2g(\varphi Y, W)\eta(X)\xi - 2\eta(X)\eta(W)\varphi Y \\ &\quad - 2\eta(X)\eta(W)\varphi Y + \varphi R(X,Y)W,\end{aligned} \tag{3.18}$$

for any X, Y, W $\in T_p$ M. Using (3.18) in (3.16) we get

$$
\begin{aligned}
(\nabla_W \widetilde{C})(X,Y)\xi \;=\;& 2g(\varphi Y,W)X - 2g(\varphi X,W) - 2g(\varphi X,W)\eta(Y)\xi \\
&+2g(\varphi Y,W)\eta(X))\xi - g(Y,W)\varphi X \\
&+g(X,W)\varphi Y + 2\eta(X)\eta(W)\varphi Y - 2\eta(Y)\eta(W)\varphi X \\
&-\varphi R(X,Y)W - \frac{dr(W)}{2n(2n+1)}\left[\eta(Y)X - \eta(X)Y\right]. \qquad (3.19)
\end{aligned}
$$

We now suppose that a LP-Sasakian manifold $(M^{2n+1}, g), (n > 1)$, is concircular φ-recurrent. Then from (3.8) and (3.19), it follows that

$$
\begin{aligned}
(\nabla_W \widetilde{C})(X,Y)Z \;=\;& [2g(\varphi Y,W)g(X,Z) - 2g(\varphi X,W)g(Y,Z) \\
&+2g(\varphi Y,W)\eta(X)\eta(Z) - 2g(\varphi X,W)\eta(Y)\eta(Z) \\
&-g(Y,W)g(\varphi X,Z) + g(X,W)g(\varphi Y,Z) \\
&+2g(\varphi Y,Z)\eta(X)\eta(W) + g(X,W)g(\varphi Y,Z) \\
&-2g(\varphi X,Z)\eta(Y)\eta(W) - g(\varphi R(X,Y)W,Z) \\
&-\frac{dr(W)}{2n(2n+1)}\left[\eta(Y)X - \eta(X)Y\right] - \frac{dr(W)}{2n(2n+1)} \\
&\times\left[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\right] + A(W)\widetilde{C}(X,Y)Z. \qquad (3.20)
\end{aligned}
$$

Next, we suppose that in a concircular φ-recurrent LP-Sasakian manifold, the sectional curvature of a plane $\pi \subset T_pM$ defined by

$$K_p(\pi) = g(R(X,Y)Y,X), \qquad (3.21)$$

is a non-zero constant k, where $\{X,Y\}$ is any orthonormal basis of π. Then we have

$$g((\nabla_Z R)(X,Y)Y,X) = 0. \qquad (3.22)$$

Again from (2.16) we have

$$
\begin{aligned}
(\nabla_Z \widetilde{C})(X,Y)Y \;=\;& (\nabla_Z R)(X,Y)Y \\
&-\frac{dr(Z)}{2n(2n+1)}\left[g(Y,Y)X - g(X,Y)Y\right]. \qquad (3.23)
\end{aligned}
$$

In view of (3.21) it follows from (3.22) that

$$g\left((\nabla_Z \widetilde{C})(X,Y)Y,X\right) = 0. \qquad (3.24)$$

By virtue of (3.23) and (3.1) we obtain

$$g\left((\nabla_Z \widetilde{C})(X,Y)Y,\xi\right)\eta(X) = A(Z)g(\widetilde{C}(X,Y)Y,X). \qquad (3.25)$$

Since in a concircular φ-recurrent LP-Sasakian manifold, the relation (3.20) holds good, using (3.20) in (3.24) we get

$$
\begin{aligned}
kA(Z) &= A(Z)\eta(X)[g(Y,Y)\eta(X) - g(X,Y)\eta(Y) \\
&- \frac{r}{2n(2n+1)}[g(Y,Y)\eta(X) - g(X,Y)\eta(Y)]\ ] \\
&- \eta(X)[2g(\varphi Y,Z)g(X,Y) - 2g(\varphi X,Z)g(Y,Y) \\
&- 2g(\varphi X,Z)\eta(Y)\eta(Y) + 2g(\varphi Y,Z)\eta(X)\eta(Y) \\
&+ g(Y,Z)g(\varphi X,Y) + 2g(\varphi Y,Y)\eta(X)\eta(Z) \\
&- 2g(\varphi X,Y)\eta(Y)\eta(Z) - g(\varphi R(X,Y)Z,Y) \\
&- \frac{dr(Z)}{2n(2n+1)}[g(X,Y)\eta(Y) - g(Y,Y)\eta(X)]\ ].
\end{aligned}
\tag{3.26}
$$

Putting $Y = Z = \xi$ in (3.25) and simplifying we get $\eta(\rho) = 0$. Hence by (3.13), we obtain from (2.15) that

$$\varphi^2((\nabla_W \widetilde{C})(X,Y)Z) = 0.$$

This leads to the following theorem:

Theorem 3.3. *If a concircular φ-recurrent LP-Sasakian manifold $(M^{2n+1}, g), (n > 1)$, has a non-zero constant sectional curvature, then it reduces to a locally concircular φ-symmetric manifold.*

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