Distinguished torsion, curvature and deflection tensors in the multi-time Hamilton Geometry

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Abstract. The purpose of this paper is to present the main geometrical objects on the dual 1-jet vector bundle $J^{1*}(\mathcal{T}, M)$ that characterize our approach of multi-time Hamilton geometry. In this direction, we firstly introduce the geometrical concept of a nonlinear connection N on the dual 1-jet space $J^{1*}(\mathcal{T}, M)$. Then, starting with a given N-linear connection D on $J^{1*}(\mathcal{T}, M)$, we describe the adapted components of the torsion, curvature and deflection distinguished tensors attached to the N-linear connection D.

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1 Introduction

It is well known that the 1-jet spaces are the basic mathematical objects used in the study of classical and quantum field theories. For this reason, the differential geometry of 1-jet bundles was intensively studied by a lot of authors like, for example, (in chronological order) Saunders [24], Asanov [4], Neagu and Udrişte [23], [20], [22].

In the last decades, numerous physicists and geometers were preoccupied with the development of a branch of mathematical-physics, which is situated at the junction of the Theoretical Physics with the Differential Geometry and the Theory of PDEs. This branch of mathematical-physics is characterized, on the one hand (from the point of view of physicists), by the geometrical quantization of the covariant Hamiltonian field theories, and, on the other hand (from the point of view of geometers), by the geometrization of ordinary Lagrangians and Hamiltonians from Analytical Mechanics or of multi-time Lagrangians and Hamiltonians used in Theoretical Physics.

In order to reach their aim, the physicists use in their studies that so-called the *covariant Hamiltonian geometry of physical fields*, which is the *multi-parameter*, or *multi-time*, extension of the classical Hamiltonian formulation from Mechanics (please see Abraham and Marsden [1]).

It is important to point out that the covariant Hamiltonian geometry of physical fields appears in the literature of specialty in three distinct variants (the *multisymplectic* geometry, the *polysymplectic* geometry and the *De Donder-Weyl covariant*

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Hamiltonian geometry), which differ by the *phase space* used in study. All these three different alternative extensions of the Hamiltonian formulation to field theory (these extensions originate from the calculus of variations of multiple integrals) reduce to the classical Hamiltonian formalism from Mechanics if the number of space-time dimensions equals to one.

Using the technics of the symplectic geometry, the *multisymplectic covariant geometry of physical fields* is developed by Gotay, Isenberg, Marsden, Montgomery and their co-workers [11], [12] on a finite-dimensional multisymplectic phase space.

In order to quantize the covariant Hamiltonian field theory (this is the final purpose in the framework of quantum field theory), Giachetta, Mangiarotti and Sardanashvily [9], [10] develop the *polysymplectic Hamiltonian geometry*, which studies the relations between the equations of first order Lagrangian field theory on fiber bundles and the covariant Hamilton equations on a finite-dimensional polysymplectic phase space.

Another convenient approach for quantization of the Hamiltonian field theory is the *De Donder-Weyl Hamiltonian canonical formulation of field theory* (this is known from about 80 years), which is intensively studied by Kanatchikov (please see [13], [14], [15] and references therein) as opposed to the conventional field-theoretical Hamiltonian formalism, which requires the space + time decomposition and leads to the picture of a field as a mechanical system with infinitely many degrees of freedom. The De Donder-Weyl Hamiltonian approach is achieved by assigning the canonical momentum like variables to the whole set of space-time derivatives of a field, that is $\partial_b x^j \to p^b_j$, where x^i denote field variables $(i = 1, ..., n)$, t^a are space-time variables $(a = 1, ..., m)$, $\partial_b x^j$ are space-time derivatives (or first jets) of field variables and p^b_j denote *polymomenta*.

In this direction, in the De Donder-Weyl polymomentum canonical theory, given a Lagrangian function $L = L(t^a, x^i, \partial_b x^j)$, the polymomenta are introduced by the formula $p^b_j := \partial L/\partial(\partial_b x^j)$ and the corresponding *De Donder-Weyl Hamiltonian function* is given by $H := \left(\partial_a x^i\right) p^a_i - L$, where it is obvious that H is a function of variables $z^I := (t^a, x^i, p^a_i)$. In these variables the Euler-Lagrange field equations can be equivalent rewritten in the form of *De Donder-Weyl Hamiltonian field equations* (please see [13], [14] or [8, for that so-called the *multi-time Hamilton equations*])

$$\frac{\partial x^i}{\partial t^a} = \frac{\partial H}{\partial p^a_i}, \quad \frac{\partial p^a_i}{\partial t^a} = -\frac{\partial H}{\partial x^i},$$

which, for $m = 1$, reduce to the standard Hamilton-Jacobi equations from Mechanics, and for $m > 1$, provide a multi-time covariant generalization of the Hamiltonian formalism.

From the perspective of geometers, we point out that, following the geometrical ideas initially stated by Asanov in the paper [4], a *multi-time Lagrange contravariant geometry on 1-jet spaces* (in the sense of distinguished linear connections, torsions and curvatures) was recently developed by Neagu and Udrişte [20], [22], [23]. This geometrical theory is a natural multi-time extension on 1-jet spaces of the already classical *Lagrange geometrical theory of the tangent bundle* elaborated by Miron, Anastasiei and Bucătaru [17], [7]. Remark that recent new geometrical developments, which relies on the multi-time Lagrange contravariant geometrical ideas from [20], are given by Udrişte and his co-workers in the paper [25]. On the other hand, suggested by the field theoretical extension of the basic structures of classical Analytical Mechanics within the framework of the De Donder-Weyl covariant Hamiltonian formulation, the studies of Miron [16], Atanasiu [5], [6] and others led to the development of the *Hamilton geometry of the cotangent bundle* exposed in the monograph [18].

In such a physical and geometrical context, the aim of this paper is to expose the basic geometrical objects (such as the distinguished linear connections, torsions and curvatures) on the dual 1-jet vector bundle $J^{1*}(\mathcal{T},M)$ (the polymomentum phase space), coordinated by (t^a,x^i,p_i^a), where $a=\overline{1,m}$ and $i=\overline{1,n}$. This geometrical theory (which finally represents a geometrization for multi-time Hamiltonian functions) is called by us the *multi-time covariant Hamilton geometry.* Note that the multi-time covariant Hamilton geometry is a natural multi-time generalization of the Hamilton geometry of the cotangent bundle [18]. We sincerely hope that the geometrical results exposed in this paper to have a physical meaning for physicists, physical meaning which we do not know yet.

Finally, we would like to point out that the *multi-time Legendre jet duality* between this multi-time covariant Hamilton geometry and the already constructed multi-time contravariant Lagrange geometry [20] (this is a geometrization for jet multi-time Lagrangian functions, which is different of the geometrization from the paper [19]) is a part of our work in progress and represents a general direction for our future studies.

It is important to also note that, when the multi-time manifold $\mathcal{T}$ coincides with the usual single-time represented by the set of real numbers $\mathbb{R}$, we recover geometrical approaches of the *time dependent Lagrange geometry* developed in distinct ways by Anastasiei and Kawaguchi [2], [3], on the one hand, or Neagu [21], on the other hand.

2 The dual 1-jet bundle $J^{1*}(\mathcal{T},M)$

We start our geometrical study considering two smooth real manifolds $\mathcal{T}^m$ and M^n having the dimensions m, respectively n, and which are coordinated by $(t^a)_{a=\overline{1,m}}$, respectively $(x^i)_{i=\overline{1,n}}$.

Remark 2.1. We point out that, throughout this paper, the indices a,b,c,d,f,g run over the set $\{1,2,\ldots,m\}$ and the indices i,j,k,l,r,s run over the set $\{1,2,\ldots,n\}$.

Let us consider the 1-jet space $E \overset{not}{=} J^1(\mathcal{T}\times M)\to\mathcal{T}\times M$, coordinated by (t^a,x^i,x_a^i), where x_a^i behave as *partial derivatives.* It is well known that the transformations of coordinates on the 1-jet vector bundle $J^1(\mathcal{T},M)$ are given by

$$\left\{\begin{array}{l}\tilde{t}^a=\tilde{t}^a(t^b)\\[2mm] \tilde{x}^i=\tilde{x}^i(x^j)\\[2mm] \tilde{x}_a^i=\dfrac{\partial\tilde{x}^i}{\partial x^j}\dfrac{\partial t^b}{\partial\tilde{t}^a}x_b^j,\end{array}\right. \tag{2.1}$$

where $\det(\partial\tilde{t}^a/\partial t^b)\neq 0$ and $\det(\partial\tilde{x}^i/\partial x^j)\neq 0$.

Now, using the general theory of vector bundles (please see [17], for example), let us consider the *dual 1-jet vector bundle* $E^* \overset{not}{=} J^{1*}(\mathcal{T},M)\to\mathcal{T}\times M$, whose local coordinates are denoted by (t^a,x^i,p_i^a).

Remark 2.2. According to the Kanatchikov's physical terminology [14], which generalizes the Hamiltonian terminology from Analytical Mechanics, the coordinates p_i^a are called *polymomenta* and the dual 1-jet space E^* is called the *polymomentum phase space*.

It is easy to see that the transformations of coordinates on the dual 1-jet space E^* have the expressions

$$\begin{cases} \tilde{t}^a = \tilde{t}^a(t^b) \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{p}_i^a = \dfrac{\partial x^j}{\partial \tilde{x}^i}\dfrac{\partial \tilde{t}^a}{\partial t^b}p_j^b, \end{cases} \tag{2.2}$$

where $\det(\partial\tilde{t}^a/\partial t^b) \neq 0$ and $\det(\partial\tilde{x}^i/\partial x^j) \neq 0$. In the sequel, doing a transformation of coordinates (2.2) on E^*, we obtain

Proposition 2.1. *The elements of the local natural basis* $\left\{\dfrac{\partial}{\partial t^a}, \dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial p_i^a}\right\}$ *of the Lie algebra of vector fields* $\mathcal{X}(E^*)$ *transform by the rules*

$$\begin{aligned} \frac{\partial}{\partial t^a} &= \frac{\partial \tilde{t}^b}{\partial t^a}\frac{\partial}{\partial \tilde{t}^b} + \frac{\partial \tilde{p}_j^b}{\partial t^a}\frac{\partial}{\partial \tilde{p}_j^b}, \\ \frac{\partial}{\partial x^i} &= \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{p}_j^b}{\partial x^i}\frac{\partial}{\partial \tilde{p}_j^b}, \\ \frac{\partial}{\partial p_i^a} &= \frac{\partial x^i}{\partial \tilde{x}^j}\frac{\partial \tilde{t}^b}{\partial t^a}\frac{\partial}{\partial \tilde{p}_j^b}. \end{aligned} \tag{2.3}$$

Proposition 2.2. *The elements of the local natural cobasis* $\{dt^a, dx^i, dp_i^a\}$ *of the Lie algebra of covector fields* $\mathcal{X}^*(E^*)$ *transform by the rules*

$$\begin{aligned} dt^a &= \frac{\partial t^a}{\partial \tilde{t}^b}d\tilde{t}^b, \\ dx^i &= \frac{\partial x^i}{\partial \tilde{x}^j}d\tilde{x}^j, \\ dp_i^a &= \frac{\partial p_i^a}{\partial \tilde{t}^b}d\tilde{t}^b + \frac{\partial p_i^a}{\partial \tilde{x}^j}d\tilde{x}^j + \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial t^a}{\partial \tilde{t}^b}d\tilde{p}_j^b. \end{aligned} \tag{2.4}$$

It is well known the importance of tensors in the development of a fertile geometry on a vector bundle. Following the geometrical ideas developed in the books [17] and [18], in our study upon the geometry of the dual 1-jet bundle E^* a central role is played by the *distinguished tensors* or, briefly, *d-tensors*.

Definition 2.1. A geometrical object $T = \left(T^{ai(k)(d)\ldots}_{bj(c)(l)\ldots}\right)$ on the dual 1-jet vector bundle E^*, whose local components, with respect to a transformation of coordinates (2.2) on E^*, transform by the rules

$$T^{ai(k)(d)\ldots}_{bj(c)(l)\ldots} = \tilde{T}^{ep(r)(h)\ldots}_{fq(g)(s)\ldots}\frac{\partial t^a}{\partial \tilde{t}^e}\frac{\partial x^i}{\partial \tilde{x}^p}\left(\frac{\partial x^k}{\partial \tilde{x}^r}\frac{\partial \tilde{t}^g}{\partial t^c}\right)\frac{\partial \tilde{t}^f}{\partial t^b}\frac{\partial \tilde{x}^q}{\partial x^j}\left(\frac{\partial \tilde{x}^s}{\partial x^l}\frac{\partial t^d}{\partial \tilde{t}^h}\right)\ldots, \tag{2.5}$$

is called a d-tensor or a distinguished tensor field on the dual 1-jet space E^*.

Example 2.1. If $H : E^* \to \mathbb{R}$ is a Hamiltonian function depending on the polymomenta p_i^a, then the local components

$$G_{(a)(b)}^{(i)(j)} = \frac{1}{2}\frac{\partial^2 H}{\partial p_i^a \partial p_j^b}$$

represent a d-tensor field $\mathbb{G} = \left(G_{(a)(b)}^{(i)(j)}\right)$ on the dual 1-jet space E^*, which is called the *fundamental vertical metrical d-tensor associated to the Hamiltonian function of polymomenta H*.

Example 2.2. Let us consider the d-tensor $\mathbb{C}^* = \left(\mathbb{C}_{(i)}^{(a)}\right)$, where $\mathbb{C}_{(i)}^{(a)} = p_i^a$. The distinguished tensor $\mathbb{C}^*$ is called the *Liouville-Hamilton d-tensor field of polymomenta* on the dual 1-jet space E^*.

Example 2.3. Let $h_{ab}(t)$ be a semi-Riemannian metric on the temporal manifold $\mathcal{T}$. The geometrical object $\mathbb{L} = \left(L_{(j)ab}^{(c)}\right)$, where $L_{(j)ab}^{(c)} = h_{ab}p_j^c$, is a d-tensor field on E^*, which is called the *polymomentum Liouville-Hamilton d-tensor field associated to the metric $h_{ab}(t)$*.

Example 2.4. Using the preceding metric $h_{ab}(t)$, we can construct the d-tensor field $\mathbb{J} = \left(J_{(a)bj}^{(i)}\right)$, where $J_{(a)bj}^{(i)} = h_{ab}\delta_j^i$. The distinguished tensor $\mathbb{J}$ is called the *d-tensor of h-normalization on the dual 1-jet vector bundle E^**.

Definition 2.2. A pair of local functions $N = \left(\underset{1}{N}{}_{(k)a}^{(c)}, \underset{2}{N}{}_{(k)i}^{(c)}\right)$ on E^*, which transform by the rules

$$\begin{aligned} \underset{1}{\widetilde{N}}{}_{(j)d}^{(b)} &= \underset{1}{N}{}_{(k)a}^{(c)} \frac{\partial \tilde{t}^b}{\partial t^c}\frac{\partial x^k}{\partial \tilde{x}^j}\frac{\partial t^a}{\partial \tilde{t}^d} - \frac{\partial t^a}{\partial \tilde{t}^d}\frac{\partial \tilde{p}_j^b}{\partial t^a}, \\ \underset{2}{\widetilde{N}}{}_{(j)r}^{(b)} &= \underset{2}{N}{}_{(k)i}^{(c)} \frac{\partial \tilde{t}^b}{\partial t^c}\frac{\partial x^k}{\partial \tilde{x}^j}\frac{\partial x^i}{\partial \tilde{x}^r} - \frac{\partial x^i}{\partial \tilde{x}^r}\frac{\partial \tilde{p}_j^b}{\partial x^i}, \end{aligned} \tag{2.6}$$

is called a nonlinear connection on the dual 1-jet bundle E^*.

Remark 2.3. The nonlinear connections are very important in the study of the differential geometry of the dual 1-jet space E^* because they produce the *adapted distinguished 1-forms*

$$\delta p_i^a = dp_i^a + \underset{1}{N}{}_{(i)b}^{(a)} dt^b + \underset{2}{N}{}_{(i)j}^{(a)} dx^j, \tag{2.7}$$

together with their *dual adapted vector fields*

$$\begin{aligned} \frac{\delta}{\delta t^a} &= \frac{\partial}{\partial t^a} - \underset{1}{N}{}_{(j)a}^{(b)} \frac{\partial}{\partial p_j^b}, \\ \frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - \underset{2}{N}{}_{(j)i}^{(b)} \frac{\partial}{\partial p_j^b}, \end{aligned} \tag{2.8}$$

which behave like d-tensors, that is they transform as in the tensorial rules (2.5).

3 N-linear connections

A linear connection on $E^* = J^{1*}(\mathcal{T}, M)$ is an application

$$D : \chi(E^*) \times \chi(E^*) \to \chi(E^*), \quad (X, Y) \to D_X Y,$$

having, for all X, X_1, X_2, Y_1, Y_2, $Y \in \chi(E^*)$, $f \in \mathcal{F}(E^*)$, the properties:

1. $D_{X_1+X_2} Y = D_{X_1} Y + D_{X_2} Y$,
2. $D_{fX} Y = f D_X Y$,
3. $D_X (Y_1 + Y_2) = D_X Y_1 + D_X Y_2$,
4. $D_X (fY) = X(f)Y + f D_X Y$.

Obviously, with respect to the adapted basis

$$\left\{\frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_i^a}\right\} \subset \chi(E^*), \tag{3.1}$$

the linear connection D on E^* can be uniquely determined by *27* local adapted coefficients, which are written in the adapted basis (3.1) in the form:

$$\begin{aligned} D_{\frac{\delta}{\delta t^c}} \frac{\delta}{\delta t^b} &= A^a_{bc} \frac{\delta}{\delta t^a} + A^i_{bc} \frac{\delta}{\delta x^i} + A^{(a)}_{(i)bc} \frac{\partial}{\partial p^a_i}, \\ D_{\frac{\delta}{\delta t^c}} \frac{\delta}{\delta x^j} &= A^a_{jc} \frac{\delta}{\delta t^a} + A^i_{jc} \frac{\delta}{\delta x^i} + A^{(a)}_{(i)jc} \frac{\partial}{\partial p^a_i}, \\ -D_{\frac{\delta}{\delta t^c}} \frac{\partial}{\partial p^b_j} &= A^{a(j)}_{\ \ (b)c} \frac{\delta}{\delta t^a} + A^{i(j)}_{\ \ (b)c} \frac{\delta}{\delta x^i} + A^{(a)(j)}_{(i)(b)c} \frac{\partial}{\partial p^a_i}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta t^b} &= H^a_{bk} \frac{\delta}{\delta t^a} + H^i_{bk} \frac{\delta}{\delta x^i} + H^{(a)}_{(i)bk} \frac{\partial}{\partial p^a_i}, \\ D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} &= H^a_{jk} \frac{\delta}{\delta t^a} + H^i_{jk} \frac{\delta}{\delta x^i} + H^{(a)}_{(i)jk} \frac{\partial}{\partial p^a_i}, \\ -D_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial p^b_j} &= H^{a(j)}_{\ \ (b)k} \frac{\delta}{\delta t^a} + H^{i(j)}_{\ \ (b)k} \frac{\delta}{\delta x^i} + H^{(a)(j)}_{(i)(b)k} \frac{\partial}{\partial p^a_i}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} D_{\frac{\partial}{\partial p^c_k}} \frac{\delta}{\delta t^b} &= C^{a(k)}_{b(c)} \frac{\delta}{\delta t^a} + C^{i(k)}_{b(c)} \frac{\delta}{\delta x^i} + C^{(a)\ (k)}_{(i)b(c)} \frac{\partial}{\partial p^a_i}, \\ D_{\frac{\partial}{\partial p^c_k}} \frac{\delta}{\delta x^j} &= C^{a(k)}_{j(c)} \frac{\delta}{\delta t^a} + C^{i(k)}_{j(c)} \frac{\delta}{\delta x^i} + C^{(a)\ (k)}_{(i)j(c)} \frac{\partial}{\partial p^a_i}, \\ -D_{\frac{\partial}{\partial p^c_k}} \frac{\partial}{\partial p^b_j} &= C^{a(j)(k)}_{\ \ (b)(c)} \frac{\delta}{\delta t^a} + C^{i(j)(k)}_{\ \ (b)(c)} \frac{\delta}{\delta x^i} + C^{(a)(j)(k)}_{(i)(b)(c)} \frac{\partial}{\partial p^a_i}. \end{aligned} \tag{3.4}$$

To work with these 27 adapted coefficients is not imposible, but is laborious. We construct in the sequel linear connections whose coefficients are much easy to be used. In this direction, let us consider a nonlinear connection N on E^*.

Definition 3.1. A linear connection D on E^* is called an N-linear connection if, for any vector field $X \in \chi(E^*)$, we have

$$D_X h_{\mathcal{T}} = 0,\ D_X h_M = 0,\ D_X w = 0, \tag{3.5}$$

where $h_{\mathcal{T}}$, h_M and w are the $\mathcal{T}$- horizontal, M-horizontal and vertical, respectively, canonical projections associated to the canonical decomposition of vector fields

$$\chi(E^*) = \chi(\mathcal{H}_{\mathcal{T}}) \oplus \chi(\mathcal{H}_M) \oplus \chi(\mathcal{W}), \tag{3.6}$$

where

$$\chi(\mathcal{H}_{\mathcal{T}}) = Span\left\{\frac{\delta}{\delta t^a}\right\},\ \chi(\mathcal{H}_M) = Span\left\{\frac{\delta}{\delta x^i}\right\},\ \chi(\mathcal{W}) = Span\left\{\frac{\partial}{\partial p_i^a}\right\}.$$

In other words, a linear connection is an N-linear connection if and only if, for any vector field $X \in \chi(E^*)$, D_X carries the $\mathcal{T}$-horizontal vector fields into $\mathcal{T}$-horizontal vector fields, the M-horizontal vector fields into M-horizontal vector fields and the vertical vector fields into vertical vector fields. Thus, we have $D_X Y^{\mathcal{H}_{\mathcal{T}}} \in \chi(\mathcal{H}_{\mathcal{T}})$, $D_X Y^{\mathcal{H}_M} \in \chi(\mathcal{H}_M)$ and $D_X Y^{\mathcal{W}} \in \chi(\mathcal{W})$, which can be written in the form

$$D_X(h_{\mathcal{T}}Y) = h_{\mathcal{T}} D_X Y,\ D_X(h_M Y) = h_M D_X Y,\ D_X(wY) = wD_X Y. \tag{3.7}$$

Consequently, using the adapted basis of vector fields on E^*, given by (3.1), and the above results, we prove without difficulties

Theorem 3.1. *An N-linear connection can be uniquely written in the adapted basis of vector fields on E^* with 9 adapted coefficients given by the relations:*

$$\begin{aligned}
D_{\frac{\delta}{\delta t^c}}\frac{\delta}{\delta t^b} &= A^a_{bc}\frac{\delta}{\delta t^a},\ D_{\frac{\delta}{\delta t^c}}\frac{\delta}{\delta x^j} = A^i_{jc}\frac{\delta}{\delta x^i},\ D_{\frac{\delta}{\delta t^c}}\frac{\partial}{\partial p^b_j} = -A^{(a)(j)}_{(i)(b)c}\frac{\partial}{\partial p^a_i},\\
D_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta t^b} &= H^a_{bk}\frac{\delta}{\delta t^a},\ D_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^j} = H^i_{jk}\frac{\delta}{\delta x^i},\ D_{\frac{\delta}{\delta x^k}}\frac{\partial}{\partial p^b_j} = -H^{(a)(j)}_{(i)(b)k}\frac{\partial}{\partial p^a_i},\\
D_{\frac{\partial}{\partial p^c_k}}\frac{\delta}{\delta t^b} &= C^{a(k)}_{b(c)}\frac{\delta}{\delta t^a},\ D_{\frac{\partial}{\partial p^c_k}}\frac{\delta}{\delta x^j} = C^{i(k)}_{j(c)}\frac{\delta}{\delta x^i},\ D_{\frac{\partial}{\partial p^c_k}}\frac{\partial}{\partial p^b_j} = -C^{(a)(j)(k)}_{(i)(b)(c)}\frac{\partial}{\partial p^a_i}.
\end{aligned}$$

Definition 3.2. The local functions

$$D\Gamma(N) = \left(A^a_{bc}, A^i_{jc}, A^{(a)(j)}_{(i)(b)c}, H^a_{bk}, H^i_{jk}, H^{(a)(j)}_{(i)(b)k}, C^{a(k)}_{b(c)}, C^{i(k)}_{j(c)}, C^{(a)(j)(k)}_{(i)(b)(c)}\right) \tag{3.8}$$

are called the adapted coefficients of the N-linear connection D on E^*.

Taking into acount the tensorial transformation laws of the d-vector fields of the adapted basis (3.1), by a straightforward calculation, we obtain

Theorem 3.2. *(i) With respect to the coordinate transformations (2.2) on E^* (that imply the tensorial rules of the adapted basis of vector fields), the adapted coefficients of the N-linear connection $D\Gamma(N)$ obey the following rules of transformations:*

$$
(h_{\mathcal{T}})\ \begin{cases}
A^d_{bc} = \tilde{A}^a_{fg}\dfrac{\partial \tilde{t}^f}{\partial t^b}\dfrac{\partial \tilde{t}^g}{\partial t^c}\dfrac{\partial t^d}{\partial \tilde{t}^a} + \dfrac{\partial t^d}{\partial \tilde{t}^a}\dfrac{\partial^2 \tilde{t}^a}{\partial t^b \partial t^c},\\[2mm]
A^i_{jc} = \tilde{A}^k_{ld}\dfrac{\partial x^i}{\partial \tilde{x}^k}\dfrac{\partial \tilde{x}^l}{\partial x^j}\dfrac{\partial \tilde{t}^d}{\partial t^c},\\[2mm]
A^{(a)(j)}_{(i)(b)c} = \tilde{A}^{(d)(l)}_{(k)(f)g}\dfrac{\partial t^a}{\partial \tilde{t}^d}\dfrac{\partial \tilde{x}^k}{\partial x^i}\dfrac{\partial x^j}{\partial \tilde{x}^l}\dfrac{\partial \tilde{t}^f}{\partial t^b}\dfrac{\partial \tilde{t}^g}{\partial t^c} - \delta^j_i\dfrac{\partial t^a}{\partial \tilde{t}^d}\dfrac{\partial^2 \tilde{t}^d}{\partial t^b \partial t^c},
\end{cases}
$$

$$
(h_M)\ \begin{cases}
H^a_{bk} = \tilde{H}^c_{dl}\dfrac{\partial t^a}{\partial \tilde{t}^c}\dfrac{\partial \tilde{t}^d}{\partial t^b}\dfrac{\partial \tilde{x}^l}{\partial x^k},\\[2mm]
H^l_{jk} = \tilde{H}^i_{rs}\dfrac{\partial \tilde{x}^r}{\partial x^j}\dfrac{\partial \tilde{x}^s}{\partial x^k}\dfrac{\partial x^l}{\partial \tilde{x}^i} + \dfrac{\partial x^l}{\partial \tilde{x}^i}\dfrac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k},\\[2mm]
H^{(a)(j)}_{(i)(b)k} = \tilde{H}^{(f)(l)}_{(r)(g)s}\dfrac{\partial t^a}{\partial \tilde{t}^f}\dfrac{\partial \tilde{x}^r}{\partial x^i}\dfrac{\partial x^j}{\partial \tilde{x}^l}\dfrac{\partial \tilde{t}^g}{\partial t^b}\dfrac{\partial \tilde{x}^s}{\partial x^k} - \delta^a_b\dfrac{\partial \tilde{x}^r}{\partial x^i}\dfrac{\partial \tilde{x}^s}{\partial x^k}\dfrac{\partial^2 x^j}{\partial \tilde{x}^r \partial \tilde{x}^s},
\end{cases}
$$

$$
(\ w\ )\ \begin{cases}
C^{a(k)}_{b(c)} = \tilde{C}^{d(r)}_{f(g)}\dfrac{\partial t^a}{\partial \tilde{t}^d}\dfrac{\partial \tilde{t}^f}{\partial t^b}\dfrac{\partial x^k}{\partial \tilde{x}^r}\dfrac{\partial \tilde{t}^g}{\partial t^c},\\[2mm]
C^{i(k)}_{j(c)} = \tilde{C}^{r(s)}_{l(f)}\dfrac{\partial x^i}{\partial \tilde{x}^r}\dfrac{\partial \tilde{x}^l}{\partial x^j}\dfrac{\partial \tilde{t}^f}{\partial t^c}\dfrac{\partial x^k}{\partial \tilde{x}^s},\\[2mm]
C^{(a)(j)(k)}_{(i)(b)(c)} = \tilde{C}^{(d)(r)(s)}_{(l)(f)(g)}\dfrac{\partial t^a}{\partial \tilde{t}^d}\dfrac{\partial \tilde{x}^l}{\partial x^i}\dfrac{\partial x^j}{\partial \tilde{x}^r}\dfrac{\partial \tilde{t}^f}{\partial t^b}\dfrac{\partial x^k}{\partial \tilde{x}^s}\dfrac{\partial \tilde{t}^g}{\partial t^c}.
\end{cases}
$$

(ii) Conversely, to give an N-linear connection D on E^* is equivalent to give a set of nine local coefficients $D\Gamma(N)$ as in (3.8), which transform by the rules described in (i).

The following result proves the existence of the N-linear connections on E^*.

Theorem 3.3. *If the manifolds $\mathcal{T}$ and M are paracompacts, then there exists an N-linear connection on E^*.*

Proof. Because the manifold $\mathcal{T}$ (resp. M) is paracompact, there exists a linear connection on $\mathcal{T}$ (resp. M), whose local coefficients we denote by $\chi^a_{bc}(t)$ (resp. $\Gamma^i_{jk}(x)$). Let us consider the local coefficients

$$
\overset{B}{\underset{1}{N}}{}^{(a)}_{(i)b} = \chi^a_{cb}p^c_i,\ \overset{B}{\underset{2}{N}}{}^{(b)}_{(j)k} = -\Gamma^i_{jk}p^b_i, \tag{3.9}
$$

which define a nonlinear connection $\overset{B}{N}$ on E^*. We set

$$
B\Gamma\left(\overset{B}{N}\right) = \left(\chi^a_{bc}, 0, A^{(a)(j)}_{(i)(b)c}, 0, \Gamma^i_{jk}, H^{(a)(j)}_{(i)(b)k}, 0, 0, 0\right) \tag{3.10}
$$

where

$$
A^{(a)(j)}_{(i)(b)c} = -\delta^j_i\chi^a_{bc},\ H^{(a)(j)}_{(i)(b)k} = \delta^a_b\Gamma^j_{ik}. \tag{3.11}
$$

Then, $B\Gamma\begin{pmatrix} B \\ N \end{pmatrix}$ defines an N-linear connection on E^*. □

Definition 3.3. The N-linear connection $B\Gamma\begin{pmatrix} B \\ N \end{pmatrix}$ on E^*, which is given by the relations (3.9), (3.10) and (3.11), is called the canonical Berwald connection attached to the linear connections $\chi^a_{bc}(t)$ and $\Gamma^i_{jk}(x)$.

Now, let us consider that D is a fixed N-linear connection on E^*, defined by the adapted local coefficients (3.8). The linear connection $D\Gamma(N)$ naturally induces derivations on the set of the d-tensor fields on the dual 1-jet bundle E^*, in the following way. Starting from a d-vector field $X \in \chi(E^*)$ and a d-tensor field T on E^*, locally expressed by

$$\begin{aligned} X &= X^a \frac{\delta}{\delta t^a} + X^i \frac{\delta}{\delta x^i} + X^{(a)}_{(i)} \frac{\partial}{\partial p^a_i}, \\ T &= T^{ai(k)(d)\ldots}_{cj(b)(l)\ldots} \frac{\delta}{\delta t^a} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial p^d_l} \otimes dt^c \otimes dx^j \otimes \delta p^b_k \otimes \ldots, \end{aligned}$$

we obtain

$$\begin{aligned} & D_X T = X^g D_{\frac{\delta}{\delta t^g}} T + X^s D_{\frac{\delta}{\delta x^s}} T + X^{(g)}_{(s)} D_{\frac{\partial}{\partial p^g_s}} T = \\ & = \Big\{ X^g T^{ai(k)(d)\ldots}_{cj(b)(l)\ldots/g} + X^s T^{ai(k)(d)\ldots}_{cj(b)(l)\ldots|s} + \\ & + X^{(g)}_{(s)} T^{ai(k)(d)\ldots}_{cj(b)(l)\ldots}\, |^{(s)}_{(g)} \Big\} \frac{\delta}{\delta t^a} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial p^d_l} \otimes dt^c \otimes dx^j \otimes \delta p^b_k \otimes \ldots, \end{aligned}$$

where

$$(h_{\mathcal{T}}) \begin{cases} T^{ai(k)(d)\ldots}_{cj(b)(l)\ldots/g} = \dfrac{\delta T^{ai(k)(d)\ldots}_{cj(b)(l)\ldots}}{\delta t^g} + T^{fi(k)(d)\ldots}_{cj(b)(l)\ldots} A^a_{fg} + \\ + T^{ar(k)(d)\ldots}_{cj(b)(l)\ldots} A^i_{rg} + T^{ai(r)(d)\ldots}_{cj(f)(l)\ldots} A^{(f)(k)}_{(r)(b)g} + \ldots - \\ - T^{ai(k)(d)\ldots}_{fj(b)(l)\ldots} A^f_{cg} - T^{ai(k)(d)\ldots}_{cr(b)(l)\ldots} A^r_{jg} - T^{ai(k)(f)\ldots}_{cj(b)(r)\ldots} A^{(d)(r)}_{(l)(f)g} - \ldots, \end{cases}$$

$$(h_M) \begin{cases} T^{ai(k)(d)\ldots}_{cj(b)(l)\ldots|s} = \dfrac{\delta T^{ai(k)(d)\ldots}_{cj(b)(l)\ldots}}{\delta x^s} + T^{fi(k)(d)\ldots}_{cj(b)(l)\ldots} H^a_{fs} + \\ + T^{ar(k)(d)\ldots}_{cj(b)(l)\ldots} H^i_{rs} + T^{ai(r)(d)\ldots}_{cj(f)(l)\ldots} H^{(f)(k)}_{(r)(b)s} + \ldots - \\ - T^{ai(k)(d)\ldots}_{fj(b)(l)\ldots} H^f_{cs} - T^{ai(k)(d)\ldots}_{cr(b)(l)\ldots} H^r_{js} - T^{ai(k)(f)\ldots}_{cj(b)(r)\ldots} H^{(d)(r)}_{(l)(f)s} - \ldots, \end{cases}$$

$$(\ w\ ) \begin{cases} T^{ai(k)(d)\ldots}_{cj(b)(l)\ldots} \Big|^{(s)}_{(g)} = \dfrac{\partial T^{ai(k)(d)\ldots}_{cj(b)(l)\ldots}}{\partial p^g_s} + T^{fi(k)(d)\ldots}_{cj(b)(l)\ldots} C^{a(s)}_{f(g)} + \\ + T^{ar(k)(d)\ldots}_{cj(b)(l)\ldots} C^{i(s)}_{r(g)} + T^{ai(r)(d)\ldots}_{cj(f)(l)\ldots} C^{(f)(k)(s)}_{(r)(b)(g)} + \ldots - \\ - T^{ai(k)(d)\ldots}_{fj(b)(l)\ldots} C^{f(s)}_{c(g)} - T^{ai(k)(d)\ldots}_{cr(b)(l)\ldots} C^{r(s)}_{j(g)} - T^{ai(k)(f)\ldots}_{cj(b)(r)\ldots} C^{(d)(r)(s)}_{(l)(f)(g)} - \ldots. \end{cases}$$

Definition 3.4. The local derivative operators "$_{/a}$", "$_{|i}$" and "$|_{(a)}^{(i)}$" are called the $\mathcal{T}$-horizontal covariant derivative, the M-horizontal covariant derivative and the vertical covariant derivative attached to the N-linear connection $D\Gamma(N)$. They are applied to the local components of an arbitrary distinguished tensor field T on the dual 1-jet space E^*.

By a direct calculation, we obtain

Proposition 3.4. *The operators "$_{/a}$", "$_{|i}$" and "$|_{(a)}^{(i)}$" have the properties:*

(i) They are distributive with respect to the addition of the d-tensor fields of the same type.

(ii) They commute with the operation of contraction.

(iii) They verify the Leibniz rule with respect to the tensor product.

Remark 3.1. (i) If $T = f$ is a function on E^*, then the following expressions of the local covariant derivatives hold good:

$$f_{/b} = \frac{\delta f}{\delta t^b} = \frac{\partial f}{\partial t^b} - \underset{1}{N}{}_{(i)b}^{(a)} \frac{\partial f}{\partial p_i^a}, \quad f_{|j} = \frac{\delta f}{\delta x^j} = \frac{\partial f}{\partial x^j} - \underset{(2)}{N}{}_{(i)j}^{(a)} \frac{\partial f}{\partial p_i^a},$$

$$f|_{(a)}^{(i)} = \frac{\partial f}{\partial p_i^a}.$$

(ii) If $T = Y$ is a d-vector field on E^*, locally expressed by

$$Y = Y^a \frac{\delta}{\delta t^a} + Y^i \frac{\delta}{\delta x^i} + Y_{(i)}^{(a)} \frac{\partial}{\partial p_i^a},$$

then the following expressions of the local covariant derivatives hold good:

$$(h_{\mathcal{T}}) \begin{cases} Y^a{}_{/c} = \dfrac{\delta Y^a}{\delta t^c} + Y^b A_{bc}^a, \\[2ex] Y^i{}_{/c} = \dfrac{\delta Y^i}{\delta t^c} + Y^j A_{jc}^i, \\[2ex] Y_{(i)/c}^{(a)} = \dfrac{\delta Y_{(i)}^{(a)}}{\delta t^c} - Y_{(j)}^{(b)} A_{(i)(b)c}^{(a)(j)}, \end{cases} \qquad (h_M) \begin{cases} Y^a{}_{|k} = \dfrac{\delta Y^a}{\delta x^k} + Y^b H_{bk}^a, \\[2ex] Y^i{}_{|k} = \dfrac{\delta Y^i}{\delta x^k} + Y^j H_{jk}^i, \\[2ex] Y_{(i)|k}^{(a)} = \dfrac{\delta Y_{(i)}^{(a)}}{\delta x^k} - Y_{(j)}^{(b)} H_{(i)(b)k}^{(a)(j)}, \end{cases}$$

$$(w) \begin{cases} Y^i\, |_{(c)}^{(k)} = \dfrac{\partial Y^a}{\partial p_k^c} + Y^b C_{b(c)}^{a(k)}, \\[2ex] Y^i\, |_{(c)}^{(k)} = \dfrac{\partial Y^i}{\partial p_k^c} + Y^j C_{j(c)}^{i(k)}, \\[2ex] Y_{(i)}^{(a)}\, |_{(c)}^{(k)} = \dfrac{\partial Y_{(i)}^{(a)}}{\partial p_k^c} - Y_{(j)}^{(b)} C_{(i)(b)(c)}^{(a)(j)(k)}. \end{cases}$$

(iii) If $T = \omega$ is a d-covector field on E^*, locally expressed by

$$\omega = \omega_a dx^a + \omega_i dx^i + \omega_{(a)}^{(i)} \delta p_i^a,$$

then the following expressions of the local covariant derivatives hold good:

$$(h_{\mathcal{T}})\begin{cases} \omega_{a/c}=\dfrac{\delta\omega_a}{\delta t^c}-A^b_{ac}\omega_b,\\[2mm] \omega_{i/c}=\dfrac{\delta\omega_i}{\delta t^c}-A^j_{ic}\omega_j,\\[2mm] \omega^{(i)}_{(a)/c}=\dfrac{\delta\omega^{(i)}_{(a)}}{\delta t^c}+A^{(b)(i)}_{(j)(a)c}\omega^{(j)}_{(b)},\end{cases}\qquad (h_M)\begin{cases} \omega_{a|k}=\dfrac{\delta\omega_a}{\delta x^k}-H^b_{ak}\omega_b,\\[2mm] \omega_{i|k}=\dfrac{\delta\omega_i}{\delta x^k}-H^j_{ik}\omega_j,\\[2mm] \omega^{(i)}_{(a)|k}=\dfrac{\delta\omega^{(i)}_{(a)}}{\delta x^k}+H^{(b)(i)}_{(j)(a)k}\omega^{(j)}_{(b)},\end{cases}$$

$$(w)\begin{cases} \omega_a\mid^{(k)}_{(c)}=\dfrac{\partial\omega_a}{\partial p^c_k}-C^{b(k)}_{a(c)}\omega_b,\\[2mm] \omega_i\mid^{(k)}_{(c)}=\dfrac{\partial\omega_i}{\partial p^c_k}-C^{j(k)}_{i(c)}\omega_j,\\[2mm] \omega^{(i)}_{(a)}\mid^{(k)}_{(c)}=\dfrac{\partial\omega^{(i)}_{(a)}}{\partial p^c_k}+C^{(b)(i)(k)}_{(j)(a)(c)}\omega^{(j)}_{(b)}.\end{cases}$$

(iv) Taking into account that, for any 1-form $\omega\in\chi^*(E^*)$, we have, by definition,

$$(D_X\omega)(Y)=X\omega(Y)-\omega(D_XY),\ \forall\ X,Y\in\chi(E^*),$$

then, it follows that we have:

$$D_{\frac{\delta}{\delta t^c}}dt^a=-A^a_{bc}dt^b,\quad D_{\frac{\delta}{\delta t^c}}dx^i=-A^i_{jc}dx^j,\quad D_{\frac{\delta}{\delta t^c}}\delta p^a_i=A^{(a)(j)}_{(i)(b)c}\delta p^b_j,$$

$$D_{\frac{\delta}{\delta x^k}}dt^a=-H^a_{bk}dt^b,\quad D_{\frac{\delta}{\delta x^k}}dx^i=-H^i_{jk}dx^j,\quad D_{\frac{\delta}{\delta x^k}}\delta p^a_i=H^{(a)(j)}_{(i)(b)k}\delta p^b_j,$$

$$D_{\frac{\partial}{\partial p^c_k}}dt^a=-C^{a(k)}_{b(c)}dt^b,\ D_{\frac{\partial}{\partial p^c_k}}dx^i=-C^{i(k)}_{j(c)}dx^j,\ D_{\frac{\partial}{\partial p^c_k}}\delta p^a_i=C^{(a)(j)(k)}_{(i)(b)(c)}\delta p^b_j.$$

(v) In the particular case of the canonical Berwald $\overset{B}{N}$-linear connection $B\Gamma\left(\overset{B}{N}\right)$, defined by the relations (3.9), (3.10) and (3.11), the local covariant derivatives are denoted by "$_{//a}$", "$_{||i}$" and " $\|^{(i)}_{(a)}$ ".

Now, we shall give an application of this paragraph. In this direction, let us consider the *canonical Liouville-Hamilton d-tensor field of polymomenta* on E^*, given by

$$\mathbb{C}^*=\mathbb{C}^{(a)}_{(i)}\frac{\partial}{\partial p^a_i},$$

where $\mathbb{C}^{(a)}_{(i)}=p^a_i$.

Definition 3.5. The d-tensor fields, defined by

$$\Delta^{(a)}_{(i)b}=\mathbb{C}^{(a)}_{(i)/b},\quad \Delta^{(a)}_{(i)j}=\mathbb{C}^{(a)}_{(i)|j},\quad \vartheta^{(a)(j)}_{(i)(b)}=\mathbb{C}^{(a)}_{(i)}|^{(j)}_{(b)},\tag{3.12}$$

are called the polymomentum deflection d-tensor fields attached to the N-linear connection $D\Gamma(N)$.

By a direct calculation, we find

Proposition 3.5. *The polymomentum deflection d-tensor fields on E^*, attached to the N-linear connection $D\Gamma(N)$, have the expressions*

$$\begin{gathered}\Delta^{(a)}_{(i)b} = -\underset{1}{N}{}^{(a)}_{(i)b} - A^{(a)(k)}_{(i)(c)b} p^c_k, \quad \Delta^{(a)}_{(i)j} = -\underset{2}{N}{}^{(a)}_{(i)j} - H^{(a)(k)}_{(i)(c)j} p^c_k, \\ \vartheta^{(a)(j)}_{(i)(b)} = \delta^a_b \delta^j_i - C^{(a)(k)(j)}_{(i)(c)(b)} p^c_k.\end{gathered} \tag{3.13}$$

Remark 3.2. The polymomentum deflection d-tensor fields (3.13) will be used in a future paper for the construction of a *generalized polymomentum electromagnetic geometrical theory* (governed by some *generalized Maxwell equations*), which is derived starting from a given *Kronecker h-regular polymomentum Hamiltonian function.*

4 The torsion of an N-linear connection

Let D be an N-linear connection on E^*. The torsion $\mathbb{T}$ of D is given by

$$\mathbb{T}(X,Y) = D_X Y - D_Y X - [X,Y], \ \forall\ X, Y \ \in \chi(E^*). \tag{4.1}$$

It is obvious that the torsion $\mathbb{T}$ can be evaluated by the pairs of d-vector fields $\left(X^{\mathcal{W}_\beta}, Y^{\mathcal{W}_\gamma}\right)$, $\left(X^{\mathcal{W}_\beta}, Y^{\mathcal{W}}\right)$, $\left(X^{\mathcal{W}}, Y^{\mathcal{W}}\right)$, where $\beta, \gamma = 1, 2$ and $\beta \leq \gamma$, $\mathcal{W}_1 = \mathcal{H}_{\mathcal{T}}$, $\mathcal{W}_2 = \mathcal{H}_M$, and then we obtain the vector fields:

$$\mathbb{T}\left(X^{\mathcal{W}_\beta}, Y^{\mathcal{W}_\gamma}\right), \quad \mathbb{T}\left(X^{\mathcal{W}_\beta}, Y^{\mathcal{W}}\right), \quad \mathbb{T}\left(X^{\mathcal{W}}, Y^{\mathcal{W}}\right).$$

Because the N-linear connection D preserves by parallelism the distributions $\mathcal{H}_{\mathcal{T}}$, $\mathcal{H}_M$ and $\mathcal{W}$, and the vertical distribution $\mathcal{W}$ is integrable, we find

Proposition 4.1. *The following properties of the torsion $\mathbb{T}$ hold good:*

$$\begin{gathered}h_M \mathbb{T}\left(X^{\mathcal{H}_{\mathcal{T}}}, Y^{\mathcal{H}_{\mathcal{T}}}\right) = 0, \quad h_M \mathbb{T}\left(X^{\mathcal{H}_{\mathcal{T}}}, Y^{\mathcal{W}}\right) = 0, \quad h_{\mathcal{T}} \mathbb{T}\left(X^{\mathcal{H}_M}, Y^{\mathcal{H}_M}\right) = 0, \\ h_{\mathcal{T}} \mathbb{T}\left(X^{\mathcal{H}_M}, Y^{\mathcal{W}}\right) = 0, \quad h_{\mathcal{T}} \mathbb{T}\left(X^{\mathcal{W}}, Y^{\mathcal{W}}\right) = 0, \quad h_M \mathbb{T}\left(X^{\mathcal{W}}, Y^{\mathcal{W}}\right) = 0.\end{gathered}$$

From the preceding statement, we deduce

Proposition 4.2. *The torsion tensor field $\mathbb{T}$ of an N-linear connection D is uniquely determined by the following components:*

$$\left\{\begin{aligned} &\mathbb{T}\left(X^{\mathcal{H}_{\mathcal{T}}}, Y^{\mathcal{H}_{\mathcal{T}}}\right) = h_{\mathcal{T}} \mathbb{T}\left(X^{\mathcal{H}_{\mathcal{T}}}, Y^{\mathcal{H}_{\mathcal{T}}}\right) + w \mathbb{T}\left(X^{\mathcal{H}_{\mathcal{T}}}, Y^{\mathcal{H}_{\mathcal{T}}}\right), \\ &\mathbb{T}\left(X^{\mathcal{H}_{\mathcal{T}}}, Y^{\mathcal{H}_M}\right) = h_{\mathcal{T}} \mathbb{T}\left(X^{\mathcal{H}_{\mathcal{T}}}, Y^{\mathcal{H}_M}\right) + h_M \mathbb{T}\left(X^{\mathcal{H}_{\mathcal{T}}}, Y^{\mathcal{H}_M}\right) + \\ &\qquad\qquad + w \mathbb{T}\left(X^{\mathcal{H}_{\mathcal{T}}}, Y^{\mathcal{H}_M}\right), \\ &\mathbb{T}\left(X^{\mathcal{H}_{\mathcal{T}}}, Y^{\mathcal{W}}\right) = h_{\mathcal{T}} \mathbb{T}\left(X^{\mathcal{H}_{\mathcal{T}}}, Y^{\mathcal{W}}\right) + w \mathbb{T}\left(X^{\mathcal{H}_{\mathcal{T}}}, Y^{\mathcal{W}}\right), \end{aligned}\right. \tag{4.2}$$

$$\left\{\begin{aligned} &\mathbb{T}\left(X^{\mathcal{H}_M}, Y^{\mathcal{H}_M}\right) = h_M \mathbb{T}\left(X^{\mathcal{H}_M}, Y^{\mathcal{H}_M}\right) + w \mathbb{T}\left(X^{\mathcal{H}_M}, Y^{\mathcal{H}_M}\right), \\ &\mathbb{T}\left(X^{\mathcal{H}_M}, Y^{\mathcal{W}}\right) = h_M \mathbb{T}\left(X^{\mathcal{H}_M}, Y^{\mathcal{W}}\right) + w \mathbb{T}\left(X^{\mathcal{H}_M}, Y^{\mathcal{W}}\right), \end{aligned}\right. \tag{4.3}$$

$$\mathbb{T}\left(X^{\mathcal{W}}, Y^{\mathcal{W}}\right) = w\mathbb{T}\left(X^{\mathcal{W}}, Y^{\mathcal{W}}\right), \tag{4.4}$$

where in the right part of each equality we have d-tensor fields on E^*.

Definition 4.1. The terms from (4.2), (4.3) and (4.4) are called the d-tensors of torsion of the N-linear connection D. More exactly, $h_{\mathcal{T}}\mathbb{T}\left(X^{\mathcal{H}_{\mathcal{T}}}, Y^{\mathcal{H}_{\mathcal{T}}}\right)$ is called the $h_{\mathcal{T}}$ $(h_{\mathcal{T}}h_{\mathcal{T}})$-tensor of torsion of D, $h_M\mathbb{T}(X^{\mathcal{H}_{\mathcal{T}}}, Y^{\mathcal{H}_M})$ is called the h_M $(h_{\mathcal{T}}h_M)$-tensor of torsion of D and so on.

Now, let us suppose that the N-linear connection D is given in the adapted basis (3.1) by the coefficients $D\Gamma(N)$ from (3.8). In such a context, we have

Theorem 4.3. *The torsion d-tensors of the N-linear connection D on E^* have the expressions:*

$$h_{\mathcal{T}}\mathbb{T}\left(\frac{\delta}{\delta t^b}, \frac{\delta}{\delta t^a}\right) = T^c_{ab}\frac{\delta}{\delta t^c}, \quad h_M\mathbb{T}\left(\frac{\delta}{\delta t^b}, \frac{\delta}{\delta t^a}\right) = T^k_{ab}\frac{\delta}{\delta x^k},$$

$$w\mathbb{T}\left(\frac{\delta}{\delta t^b}, \frac{\delta}{\delta t^a}\right) = T^{(f)}_{(r)ab}\frac{\partial}{\partial p^f_r},$$

$$h_{\mathcal{T}}\mathbb{T}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta t^a}\right) = T^c_{aj}\frac{\delta}{\delta t^c}, \quad h_M\mathbb{T}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta t^a}\right) = T^k_{aj}\frac{\delta}{\delta x^k},$$

$$w\mathbb{T}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta t^a}\right) = T^{(f)}_{(r)aj}\frac{\partial}{\partial p^f_r},$$

$$h_{\mathcal{T}}\mathbb{T}\left(\frac{\partial}{\partial p^b_j}, \frac{\delta}{\delta t^a}\right) = P^{c(j)}_{a(b)}\frac{\delta}{\delta t^c}, \quad h_M\mathbb{T}\left(\frac{\partial}{\partial p^b_j}, \frac{\delta}{\delta t^a}\right) = P^{k(j)}_{a(b)}\frac{\delta}{\delta x^k},$$

$$w\mathbb{T}\left(\frac{\partial}{\partial p^b_j}, \frac{\delta}{\delta t^a}\right) = P^{(f)\ (j)}_{(r)a(b)}\frac{\partial}{\partial p^f_r},$$

$$h_{\mathcal{T}}\mathbb{T}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) = T^c_{ij}\frac{\delta}{\delta t^c}, \quad h_M\mathbb{T}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) = T^k_{ij}\frac{\delta}{\delta x^k},$$

$$w\mathbb{T}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) = T^{(f)}_{(r)ij}\frac{\partial}{\partial p^f_r},$$

$$h_{\mathcal{T}}\mathbb{T}\left(\frac{\partial}{\partial p^b_j}, \frac{\delta}{\delta x^i}\right) = P^{c(j)}_{i(b)}\frac{\delta}{\delta t^c}, \quad h_M\mathbb{T}\left(\frac{\partial}{\partial p^b_j}, \frac{\delta}{\delta x^i}\right) = P^{k(j)}_{i(b)}\frac{\delta}{\delta x^k},$$

$$w\mathbb{T}\left(\frac{\partial}{\partial p^b_j}, \frac{\delta}{\delta x^i}\right) = P^{(f)\ (j)}_{(r)i(b)}\frac{\partial}{\partial p^f_r},$$

$$h_{\mathcal{T}}\mathbb{T}\left(\frac{\partial}{\partial p^b_j}, \frac{\partial}{\partial p^a_i}\right) = S^{c(i)(j)}_{\ \ (a)(b)}\frac{\delta}{\delta t^c}, \quad h_M\mathbb{T}\left(\frac{\partial}{\partial p^b_j}, \frac{\partial}{\partial p^a_i}\right) = S^{k(i)(j)}_{\ \ (a)(b)}\frac{\delta}{\delta x^k},$$

$$w\mathbb{T}\left(\frac{\partial}{\partial p_j^b},\frac{\partial}{\partial p_i^a}\right)=S^{(f)(i)(j)}_{(r)(a)(b)}\frac{\partial}{\partial p_r^f},$$

where

$$(4.5)\qquad \begin{cases} T^c_{ab}=A^c_{ab}-A^c_{ba}, \quad T^k_{ab}=0, \quad T^{(f)}_{(r)ab}=R^{(f)}_{(r)ab},\\ T^c_{aj}=H^c_{aj}, \quad T^k_{aj}=-A^k_{ja}, \quad T^{(f)}_{(r)aj}=R^{(f)}_{(r)aj},\\ P^{c(j)}_{a(b)}=C^{c(j)}_{a(b)}, \quad P^{k(j)}_{a(b)}=0, \quad P^{(f)\ (j)}_{(r)a(b)}=B^{(f)\ (j)}_{(r)a(b)}+A^{(f)(j)}_{(r)(b)a}, \end{cases}$$

$$(4.6)\qquad \begin{cases} T^c_{ij}=0, \quad T^k_{ij}=H^k_{ij}-H^k_{ji}, \quad T^{(f)}_{(r)ij}=R^{(f)}_{(r)ij},\\ T^{c(j)}_{i(b)}=0, \quad P^{k(j)}_{i(b)}=C^{k(j)}_{i(b)}, \quad P^{(f)\ (j)}_{(r)i(b)}=B^{(f)\ (j)}_{(r)i(b)}+H^{(f)(j)}_{(r)(b)i}, \end{cases}$$

$$(4.7)\qquad S^{c(i)(j)}_{(a)(b)}=0, \quad S^{k(i)(j)}_{(a)(b)}=0, \quad S^{(f)(i)(j)}_{(r)(a)(b)}=-\left(C^{(f)(i)(j)}_{(r)(a)(b)}-C^{(f)(j)(i)}_{(r)(b)(a)}\right)$$

and the distinguished tensors

$$R^{(f)}_{(r)ab},\ R^{(f)}_{(r)aj},\ R^{(f)}_{(r)ij},\ B^{(f)\ (j)}_{(r)a(b)} \text{ and } B^{(f)\ (j)}_{(r)i(b)}$$

are given by the formulas:

$$(4.8)\qquad \begin{aligned} R^{(a)}_{(i)bc}&=\frac{\delta \underset{1}{N}{}^{(a)}_{(i)b}}{\delta t^c}-\frac{\delta \underset{1}{N}{}^{(a)}_{(i)c}}{\delta t^b},\\ R^{(a)}_{(i)bk}&=\frac{\delta \underset{1}{N}{}^{(a)}_{(i)b}}{\delta x^k}-\frac{\delta \underset{2}{N}{}^{(a)}_{(i)k}}{\delta t^b},\\ R^{(a)}_{(i)jk}&=\frac{\delta \underset{2}{N}{}^{(a)}_{(i)j}}{\delta x^k}-\frac{\delta \underset{2}{N}{}^{(a)}_{(i)k}}{\delta x^j},\\ B^{(a)\ (k)}_{(i)b(c)}&=\frac{\partial \underset{1}{N}{}^{(a)}_{(i)b}}{\partial p_k^c}, \quad B^{(a)\ (k)}_{(i)j(c)}=\frac{\partial \underset{2}{N}{}^{(a)}_{(i)j}}{\partial p_k^c}. \end{aligned}$$

Proof. Taking into account the expression of the Poisson brackets of the adapted vector fields, together with the description in the adapted basis (3.1) of the N-linear connection $D\Gamma(N)$ given by the local coefficients (3.8), we successively obtain

$$\begin{aligned} h_{\mathcal{T}}\mathbb{T}\left(\frac{\delta}{\delta t^b},\frac{\delta}{\delta t^a}\right) &= h_{\mathcal{T}}D_{\frac{\delta}{\delta t^b}}\frac{\delta}{\delta t^a}-h_{\mathcal{T}}D_{\frac{\delta}{\delta t^a}}\frac{\delta}{\delta t^b}-h_{\mathcal{T}}\left[\frac{\delta}{\delta t^b},\frac{\delta}{\delta t^a}\right]=\\ &= \left(A^c_{ab}-A^c_{ba}\right)\frac{\delta}{\delta t^c}. \end{aligned}$$

Consequently, the first equality from (4.5) is true. In the sequel, we have

$$\begin{aligned} h_M\mathbb{T}\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta t^a}\right) &= h_M D_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta t^a} - h_M D_{\frac{\delta}{\delta t^a}}\frac{\delta}{\delta x^j} - h_M\left[\frac{\delta}{\delta x^j},\frac{\delta}{\delta t^a}\right] = \\ &= -A^k_{ja}\frac{\delta}{\delta x^k} \end{aligned}$$

and the fifth equality from (4.5) is correct. Then, for example, we have

$$\begin{aligned} w\mathbb{T}\left(\frac{\partial}{\partial p^b_j},\frac{\delta}{\delta t^a}\right) &= wD_{\frac{\partial}{\partial p^b_j}}\frac{\delta}{\delta t^a} - wD_{\frac{\delta}{\delta t^a}}\frac{\partial}{\partial p^b_j} - w\left[\frac{\partial}{\partial p^b_j},\frac{\delta}{\delta t^a}\right] = \\ &= \left(A^{(f)(j)}_{(r)(b)a} + B^{(f)\ (j)}_{(r)a(b)}\right)\frac{\partial}{\partial p^f_r} \end{aligned}$$

and the ninth equality from (4.5) is true. In the same manner, we obtain the other equalities. □

Corollary 4.4. *The torsion $\mathbb{T}$ of an arbitrary N-linear connection D on E^* is determined by* 12 *effective local d-tensors, arranged in the following table:*

	$h_{\mathcal{T}}$	h_M	w
$h_{\mathcal{T}}h_{\mathcal{T}}$	T^c_{ab}	0	$R^{(f)}_{(r)ab}$
$h_M h_{\mathcal{T}}$	T^c_{aj}	T^k_{aj}	$R^{(f)}_{(r)aj}$
$wh_{\mathcal{T}}$	$P^{c(j)}_{a(b)}$	0	$P^{(f)\ (j)}_{(r)a(b)}$
$h_M h_M$	0	T^k_{ij}	$R^{(f)}_{(r)ij}$
wh_M	0	$P^{k(j)}_{i(b)}$	$P^{(f)\ (j)}_{(r)i(b)}$
ww	0	0	$S^{(f)(i)(j)}_{(r)(a)(b)}$

5 The curvature of an N-linear connection

Let D be an N-linear connection on E^*. The curvature $\mathbb{R}$ of D is given by

(5.1) $$\mathbb{R}(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z,\ \forall\ X,Y,Z\in\chi(E^*).$$

We will express $\mathbb{R}$ by his adapted components, taking into account the decomposition (3.6) of the vector fields on E^*. In this direction, we firstly prove

Theorem 5.1. *The curvature tensor field $\mathbb{R}$ of the N-linear connection D on E^* has the properties:*

(5.2) $$\begin{array}{ll} h_{\mathcal{T}}\mathbb{R}(X,Y)Z^{\mathcal{H}_M} = 0, & h_{\mathcal{T}}\mathbb{R}(X,Y)Z^{\mathcal{W}} = 0, \\ h_M\mathbb{R}(X,Y)Z^{\mathcal{H}_{\mathcal{T}}} = 0, & h_M\mathbb{R}(X,Y)Z^{\mathcal{W}} = 0, \\ w\mathbb{R}(X,Y)Z^{\mathcal{H}_{\mathcal{T}}} = 0, & w\mathbb{R}(X,Y)Z^{\mathcal{H}_M} = 0, \end{array}$$

$$\mathbb{R}(X,Y)Z = h_{\mathcal{T}}\mathbb{R}(X,Y)Z^{\mathcal{H}_{\mathcal{T}}} + h_M\mathbb{R}(X,Y)Z^{\mathcal{H}_M} + w\mathbb{R}(X,Y)Z^{\mathcal{W}}.$$

Proof. Because the N-linear connection D preserves by parallelism the $\mathcal{H}_{\mathcal{T}}$-horizontal, $\mathcal{H}_M$-horizontal and vertical distributions, via the formula (5.1), the operator $\mathbb{R}\,(X,Y)$ carries $h_{\mathcal{T}}$-horizontal (resp. h_M-horizontal) vector fields into $h_{\mathcal{T}}$-horizontal (resp. h_M-horizontal) vector fields and the vertical vector fields into vertical vector fields. Thus, the first six equations from (5.2) hold good. The next one is an easy consequence of the first six. □

Taking into account the preceding geometrical result, by straightforward calculus, we obtain

Theorem 5.2. *The curvature tensor* $\mathbb{R}$ *of the* N*-linear connection* D *is completely determined by* 18 *local* d-tensors of curvature:

$$\mathbb{R}\left(\frac{\delta}{\delta t^c},\frac{\delta}{\delta t^b}\right)\frac{\delta}{\delta t^a}=R^d_{abc}\frac{\delta}{\delta t^d},\quad \mathbb{R}\left(\frac{\delta}{\delta t^c},\frac{\delta}{\delta t^b}\right)\frac{\delta}{\delta x^i}=R^l_{ibc}\frac{\delta}{\delta x^l},$$

$$\mathbb{R}\left(\frac{\delta}{\delta t^c},\frac{\delta}{\delta t^b}\right)\frac{\partial}{\partial p^a_i}=-R^{(d)(i)}_{(l)(a)bc}\frac{\partial}{\partial p^d_l},$$

$$\mathbb{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta t^b}\right)\frac{\delta}{\delta t^a}=R^d_{abk}\frac{\delta}{\delta t^d},\quad \mathbb{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta t^b}\right)\frac{\delta}{\delta x^i}=R^l_{ibk}\frac{\delta}{\delta x^l},$$

$$\mathbb{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta t^b}\right)\frac{\partial}{\partial p^a_i}=-R^{(d)(i)}_{(l)(a)bk}\frac{\partial}{\partial p^d_l},$$

$$\mathbb{R}\left(\frac{\partial}{\partial p^c_k},\frac{\delta}{\delta t^b}\right)\frac{\delta}{\delta t^a}=P^{d\ (k)}_{ab(c)}\frac{\delta}{\delta t^d},\quad \mathbb{R}\left(\frac{\partial}{\partial p^c_k},\frac{\delta}{\delta t^b}\right)\frac{\delta}{\delta x^i}=P^{l\ (k)}_{ib(c)}\frac{\delta}{\delta x^l},$$

$$\mathbb{R}\left(\frac{\partial}{\partial p^c_k},\frac{\delta}{\delta t^b}\right)\frac{\partial}{\partial p^a_i}=-P^{(d)(i)\ (k)}_{(l)(a)b(c)}\frac{\partial}{\partial p^d_l},$$

$$\mathbb{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta t^a}=R^d_{ajk}\frac{\delta}{\delta t^d},\quad \mathbb{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^i}=R^l_{ijk}\frac{\delta}{\delta x^l},$$

$$\mathbb{R}\left(\frac{\delta}{\delta x^k},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial p^a_i}=-R^{(d)(i)}_{(l)(a)jk}\frac{\partial}{\partial p^d_l},$$

$$\mathbb{R}\left(\frac{\partial}{\partial p^c_k},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta t^a}=P^{d\ (k)}_{aj(c)}\frac{\delta}{\delta t^d},\quad \mathbb{R}\left(\frac{\partial}{\partial p^c_k},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^i}=P^{l\ (k)}_{ij(c)}\frac{\delta}{\delta x^l},$$

$$\mathbb{R}\left(\frac{\partial}{\partial p^c_k},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial p^a_i}=-P^{(d)(i)\ (k)}_{(l)(a)j(c)}\frac{\partial}{\partial p^d_l},$$

$$\mathbb{R}\left(\frac{\partial}{\partial p^c_k},\frac{\partial}{\partial p^b_j}\right)\frac{\delta}{\delta t^a}=S^{d(j)(k)}_{a(b)(c)}\frac{\delta}{\delta t^d},\quad \mathbb{R}\left(\frac{\partial}{\partial p^c_k},\frac{\partial}{\partial p^b_j}\right)\frac{\delta}{\delta x^i}=S^{l(j)(k)}_{i(b)(c)}\frac{\delta}{\delta x^l},$$

$$\mathbb{R}\left(\frac{\partial}{\partial p^c_k},\frac{\partial}{\partial p^b_j}\right)\frac{\partial}{\partial p^a_i}=-S^{(d)(i)(j)(k)}_{(l)(a)(b)(c)}\frac{\partial}{\partial p^d_l},$$

which we can arrange in the following table:

(5.3)

	$h_{\mathcal{T}}$	h_M	w
$h_{\mathcal{T}}h_{\mathcal{T}}$	R^d_{abc}	R^l_{ibc}	$R^{(d)(i)}_{(l)(a)bc}$
$h_M h_{\mathcal{T}}$	R^d_{abk}	R^l_{ibk}	$R^{(d)(i)}_{(l)(a)bk}$
$wh_{\mathcal{T}}$	$P^{d\ (k)}_{ab(c)}$	$P^{l\ (k)}_{ib(c)}$	$P^{(d)(i)\ (k)}_{(l)(a)b(c)}$
$h_M h_M$	R^d_{ajk}	R^l_{ijk}	$R^{(d)(i)}_{(l)(a)jk}$
wh_M	$P^{d\ (k)}_{aj(c)}$	$P^{l\ (k)}_{ij(c)}$	$P^{(d)(i)\ (k)}_{(l)(a)j(c)}$
ww	$S^{d(j)(k)}_{a(b)(c)}$	$S^{l(j)(k)}_{i(b)(c)}$	$S^{(d)(i)(j)(k)}_{(l)(a)(b)(c)}$

Theorem 5.3. *The eighteen local curvature d-tensors (5.3) are given by the following formulas:*

$$\left\{\begin{array}{lll}
1. & R^d_{abc} & = \dfrac{\delta A^d_{ab}}{\delta t^c} - \dfrac{\delta A^d_{ac}}{\delta t^b} + A^f_{ab}A^d_{fc} - A^f_{ac}A^d_{fb} + C^{d(r)}_{a(f)}R^{(f)}_{(r)bc}, \\
2. & R^d_{abk} & = \dfrac{\delta A^d_{ab}}{\delta x^k} - \dfrac{\delta H^d_{ak}}{\delta t^b} + A^f_{ab}H^d_{fk} - H^f_{ak}A^d_{fb} + C^{d(r)}_{a(f)}R^{(f)}_{(r)bk}, \\
3. & P^{d\ (k)}_{ab(c)} & = \dfrac{\partial A^d_{ab}}{\partial p^c_k} - C^{d(k)}_{a(c)/b} + C^{d(r)}_{a(f)}P^{(f)\ (k)}_{(r)b(c)}, \\
4. & R^d_{ajk} & = \dfrac{\delta H^d_{aj}}{\delta x^k} - \dfrac{\delta H^d_{ak}}{\delta x^j} + H^f_{aj}H^d_{fk} - H^f_{ak}H^d_{fj} + C^{d(r)}_{a(f)}R^{(f)}_{(r)jk}, \\
5. & P^{d\ (k)}_{aj(c)} & = \dfrac{\partial H^d_{aj}}{\partial p^c_k} - C^{d(k)}_{a(c)|j} + C^{d(r)}_{a(f)}P^{(f)\ (k)}_{(r)j(c)}, \\
6. & S^{d(j)(k)}_{a(b)(c)} & = \dfrac{\partial C^{d(j)}_{a(b)}}{\partial p^c_k} - \dfrac{\partial C^{d(k)}_{a(c)}}{\partial p^b_j} + C^{f(j)}_{a(b)}C^{d(k)}_{f(c)} - C^{f(k)}_{a(c)}C^{d(j)}_{f(b)},
\end{array}\right.$$

$$\left\{\begin{array}{lll}
7. & R^l_{ibc} & = \dfrac{\delta A^l_{ib}}{\delta t^c} - \dfrac{\delta A^l_{ic}}{\delta t^b} + A^r_{ib}A^l_{rc} - A^r_{ic}A^l_{rb} + C^{l(r)}_{i(f)}R^{(f)}_{(r)bc}, \\
8. & R^l_{ibk} & = \dfrac{\delta A^l_{ib}}{\delta x^k} - \dfrac{\delta H^l_{ik}}{\delta t^b} + A^r_{ib}H^l_{rk} - H^r_{ik}A^l_{rb} + C^{l(r)}_{i(f)}R^{(f)}_{(r)bk}, \\
9. & P^{l\ (k)}_{ib(c)} & = \dfrac{\partial A^l_{ib}}{\partial p^c_k} - C^{l(k)}_{i(c)/b} + C^{l(r)}_{i(f)}P^{(f)\ (k)}_{(r)b(c)}, \\
10. & R^l_{ijk} & = \dfrac{\delta H^l_{ij}}{\delta x^k} - \dfrac{\delta H^l_{ik}}{\delta x^j} + H^r_{ij}H^l_{rk} - H^r_{ik}H^l_{rj} + C^{l(r)}_{i(f)}R^{(f)}_{(r)jk}, \\
11. & P^{l\ (k)}_{ij(c)} & = \dfrac{\partial H^l_{ij}}{\partial p^c_k} - C^{l(k)}_{i(c)|j} + C^{l(f)}_{i(r)}P^{(r)\ (k)}_{(f)j(c)}, \\
12. & S^{l(j)(k)}_{i(b)(c)} & = \dfrac{\partial C^{l(j)}_{i(b)}}{\partial p^c_k} - \dfrac{\partial C^{l(k)}_{i(c)}}{\partial p^b_j} + C^{r(j)}_{i(b)}C^{l(k)}_{r(c)} - C^{r(k)}_{i(c)}C^{l(j)}_{r(b)},
\end{array}\right.$$

$$\left\{\begin{array}{lll}
13.\quad R^{(d)(i)}_{(l)(a)bc} & = & \dfrac{\delta A^{(d)(i)}_{(l)(a)b}}{\delta t^c}-\dfrac{\delta A^{(d)(i)}_{(l)(a)c}}{\delta t^b}+A^{(d)(r)}_{(l)(f)b}A^{(f)(i)}_{(r)(a)c}- \\
 & & -A^{(d)(r)}_{(l)(f)c}A^{(f)(i)}_{(r)(a)b}+C^{(d)(i)(r)}_{(l)(a)(f)}R^{(f)}_{(r)bc}, \\
14.\quad R^{(d)(i)}_{(l)(a)bk} & = & \dfrac{\delta A^{(d)(i)}_{(l)(a)b}}{\delta x^k}-\dfrac{\delta H^{(d)(i)}_{(l)(a)k}}{\delta t^b}+A^{(d)(r)}_{(l)(f)b}H^{(f)(i)}_{(r)(a)k}- \\
 & & -H^{(d)(r)}_{(l)(f)k}A^{(f)(i)}_{(r)(a)b}+C^{(d)(i)(r)}_{(l)(a)(f)}R^{(f)}_{(r)bk}, \\
15.\quad P^{(d)(i)\ (k)}_{(l)(a)b(c)} & = & \dfrac{\partial A^{(d)(i)}_{(l)(a)b}}{\partial p^c_k}-C^{(d)(i)(k)}_{(l)(a)(c)/b}+C^{(d)(i)(r)}_{(l)(a)(f)}P^{(f)\ (k)}_{(r)b(c)}, \\
16.\quad R^{(d)(i)}_{(l)(a)jk} & = & \dfrac{\delta H^{(d)(i)}_{(l)(a)j}}{\delta x^k}-\dfrac{\delta H^{(d)(i)}_{(l)(a)k}}{\delta x^j}+H^{(d)(r)}_{(l)(f)j}H^{(f)(i)}_{(r)(a)k}- \\
 & & -H^{(d)(r)}_{(l)(f)k}H^{(f)(i)}_{(r)(a)j}+C^{(d)(i)(r)}_{(l)(a)(f)}R^{(f)}_{(r)jk}, \\
17.\quad P^{(d)(i)\ (k)}_{(l)(a)j(c)} & = & \dfrac{\partial H^{(d)(i)}_{(l)(a)j}}{\partial p^c_k}-C^{(d)(i)(k)}_{(l)(a)(c)|j}+C^{(d)(i)(r)}_{(l)(a)(f)}P^{(f)\ (k)}_{(r)j(c)}, \\
18.\quad S^{(d)(i)(j)(k)}_{(l)(a)(b)(c)} & = & \dfrac{\partial C^{(d)(i)(j)}_{(l)(a)(b)}}{\partial p^c_k}-\dfrac{\partial C^{(d)(i)(k)}_{(l)(a)(c)}}{\partial p^b_j}+C^{(d)(r)(j)}_{(l)(f)(b)}C^{(f)(i)(k)}_{(r)(a)(c)}- \\
 & & -C^{(d)(r)(k)}_{(l)(f)(c)}C^{(f)(i)(j)}_{(r)(a)(b)}.
\end{array}\right.$$

Proof. Taking into account the description in the adapted basis (3.1) of the N-linear connection $D\Gamma(N)$ given by the local coefficients (3.8), together with the Poisson brackets formulas, we obtain, for example,

$$\mathbb{R}\left(\frac{\partial}{\partial p^c_k},\frac{\delta}{\delta t^b}\right)\frac{\partial}{\partial p^a_i}=-P^{(d)(i)\ (k)}_{(l)(a)b(c)}\frac{\partial}{\partial p^d_l}=$$

$$=D_{\frac{\partial}{\partial p^c_k}}D_{\frac{\delta}{\delta t^b}}\frac{\partial}{\partial p^a_i}-D_{\frac{\delta}{\delta t^b}}D_{\frac{\partial}{\partial p^c_k}}\frac{\partial}{\partial p^a_i}-D_{\left[\frac{\partial}{\partial p^c_k},\frac{\delta}{\delta t^b}\right]}\frac{\partial}{\partial p^a_i}=$$

$$=-D_{\frac{\partial}{\partial p^c_k}}\left(A^{(f)(i)}_{(r)(a)b}\frac{\partial}{\partial p^f_r}\right)+D_{\frac{\delta}{\delta t^b}}\left(C^{(f)(i)(k)}_{(r)(a)(c)}\frac{\partial}{\partial p^f_r}\right)+B^{(f)\ (k)}_{(r)b(c)}D_{\frac{\partial}{\partial p^f_r}}\frac{\partial}{\partial p^a_i}=$$

$$\begin{aligned}
= \quad & -\frac{\partial A^{(d)(i)}_{(l)(a)b}}{\partial p^c_k}\frac{\partial}{\partial p^d_l}+A^{(f)(i)}_{(r)(a)b}C^{(d)(r)(k)}_{(l)(f)(c)}\frac{\partial}{\partial p^d_l}+\\
& +\frac{\delta C^{(d)(i)(k)}_{(l)(a)(c)}}{\delta t^b}\frac{\partial}{\partial p^d_l}-C^{(f)(i)(k)}_{(r)(a)(c)}A^{(d)(r)}_{(l)(f)b}\frac{\partial}{\partial p^d_l}-B^{(f)\ (k)}_{(r)b(c)}C^{(d)(i)(r)}_{(l)(a)(f)}\frac{\partial}{\partial p^d_l}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}P^{(d)(i)\ (k)}_{(l)(a)b(c)} &= \frac{\partial A^{(d)(i)}_{(l)(a)b}}{\partial p^c_k} - \underline{\underline{A^{(f)(i)}_{(r)(a)b} C^{(d)(r)(k)}_{(l)(f)(c)}}} - \\ &- \underline{\underline{\frac{\delta C^{(d)(i)(k)}_{(l)(a)(c)}}{\delta t^b} + C^{(f)(i)(k)}_{(r)(a)(c)} A^{(d)(r)}_{(l)(f)b}}} + B^{(f)\ (k)}_{(r)b(c)} C^{(d)(i)(r)}_{(l)(a)(f)}.\end{aligned}$$

Now, using the formula of the $\mathcal{T}$-horizontal covariant derivative, we get

$$\begin{aligned}C^{(d)(i)(k)}_{(l)(a)(c)/b} &= \underline{\underline{\frac{\delta C^{(d)(i)(k)}_{(l)(a)(c)}}{\delta t^b} - C^{(f)(i)(k)}_{(r)(a)(c)} A^{(d)(r)}_{(l)(f)b} + C^{(d)(r)(k)}_{(l)(f)(c)} A^{(f)(i)}_{(r)(a)b}}} + \\ &+ C^{(d)(i)(r)}_{(l)(a)(f)} A^{(f)(k)}_{(r)(c)b},\end{aligned}$$

and, consequently, interchanging the underlined terms, it follows

$$P^{(d)(i)\ (k)}_{(l)(a)b(c)} = \frac{\partial A^{(d)(i)}_{(l)(a)b}}{\partial p^c_k} - C^{(d)(i)(k)}_{(l)(a)(c)/b} + C^{(d)(i)(r)}_{(l)(a)(f)} P^{(f)\ (k)}_{(r)b(c)},$$

where we also used the last formula from (4.5). Obviously, this is the $15^{\text{-}th}$ relation of the above lot.

The other equalities are given in the same manner. □

Remark 5.1. Finally, let us consider the particular nonlinear connection $\overset{0}{N}$, whose local coefficients are given by

$$\underset{1}{\overset{0}{N}}{}^{(a)}_{(i)b} = \varkappa^a_{cb} p^c_i, \ \underset{2}{\overset{0}{N}}{}^{(b)}_{(j)k} = -\gamma^i_{jk} p^b_i,$$

where $\varkappa^a_{cb}(t)$ and $\gamma^i_{jk}(x)$ are the Christoffel symbols of a given pair of semi-Riemannian metrics $h_{ab}(t)$ and $\varphi_{ij}(x)$. Now, let us consider the Berwald $\overset{0}{N}$-linear connection

$$B\Gamma\left(\overset{0}{N}\right) = \left(\varkappa^a_{bc}, 0, A^{(a)(j)}_{(i)(b)c}, 0, \gamma^i_{jk}, H^{(a)(j)}_{(i)(b)k}, 0, 0, 0\right),$$

where

$$A^{(a)(j)}_{(i)(b)c} = -\delta^j_i \varkappa^a_{bc}, \qquad H^{(a)(j)}_{(i)(b)k} = \delta^a_b \gamma^j_{ik}.$$

Then, by direct local computations, we deduce that all d-tensors of torsion of $B\Gamma\left(\overset{0}{N}\right)$ vanish, except

$$R^{(f)}_{(r)ab} = \varkappa^f_{gab} p^g_r, \quad R^{(f)}_{(r)ij} = -\mathbf{r}^s_{rij} p^f_s,$$

where $\varkappa^f_{gab}(t)$ (resp. $\mathbf{r}^s_{rij}(x)$) are the local curvature tensors of the semi-Riemannian metric $h_{ab}(t)$ (resp. $\varphi_{ij}(x)$). At the same time, by direct local computations, we find that all local curvature d-tensors of $B\Gamma\left(\overset{0}{N}\right)$ vanish, except

$$R^d_{abc} = \varkappa^d_{abc}, \quad R^{(d)(i)}_{(l)(a)bc} = -\delta^i_l \varkappa^d_{abc}, \quad R^l_{ijk} = \mathbf{r}^l_{ijk}, \quad R^{(d)(i)}_{(l)(a)jk} = \delta^d_a \mathbf{r}^i_{ljk}.$$

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