

The dynamical economic model with discrete time and consumer sentiment

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Abstract. The paper describes the Hick Samuelson Keynes dynamical economic model with discrete time and consumer sentiment. We seek to demonstrate that consumer sentiment may create fluctuations in the economical activities. The model possesses a flip bifurcation and a Neimark-Sacker bifurcation, after which a stable state is replaced by a (quasi-) periodic motion.

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Key words: consumer sentiment, Hick Samuelson Keynes models, Neimark-Sacker, flip bifurcation, Lyapunov exponent.

1 Introduction

The empirical evidence ([2], [10], [4]) suggests that consumer sentiment influences the household expenditure and thus confirms Keynes'suspicion that consumer "attitudes" and "animalic spirits" may cause fluctuations in the economic activity. Inspired by these observations, we develop a dynamic economic model in which the agents'consumption expenditures depend on their sentiment. As particular cases, the model contains: the Hick-Samuelson ([9]), Puu ([9]), Keynes ([11]) models, as well as the models from [3] and [11]. The Hick and Solow models have been studied in [1] and [5].

The model possesses a flip and Neimark-Sacker bifurcation, if the autonomous consumption is variable.

The paper is organized as follows. In Section 2, we describe the dynamical model with discrete time using the investment, consumption, sentiment and saving functions. For different values of the model parameters we obtain known dynamical models. In Section 3, we analyze the behavior of the dynamical system in the fixed point's neighborhood for the associated map. We establish asymptotical stability conditions for the flip and Neimark-Sacker bifurcations. In the case of flip and Neimark-Sacker bifurcations, the normal forms are described in Section 4. Using the QR method, the algorithm for the determination of the Lyapunov exponents is presented in Section 5. Finally, the numerical simulation is done.

The analysis of the model from the present paper shows its complexity and allows the description of the different scenarios which depend on the autonomous consumption.

2 The mathematical model with discrete time and consumer sentiment.

Let $y_t = y(t)$, $t \in \mathbf{N}$ be the income at time step t and the functions:

1. the investment function $I_t = I(t)$, $t \in \mathbf{N}$:

$$(2.1) \quad I_t = v(y_{t-1} - y_{t-2}) - w(y_{t-1} - y_{t-2})^3, \quad v > 0, w \geq 0;$$

2. the consumption function $C_t = C(t)$, $t \in \mathbf{N}$:

$$(2.2) \quad C_t = a + y_{t-1}(b + cS_{t-1}), \quad a \geq 0, b > 0, c \geq 0,$$

where $S_t = S(t)$ is the sentiment function given by:

$$(2.3) \quad S_t = \frac{1}{1 + \varepsilon \exp(y_{t-1} - y_t)}, \quad \varepsilon \in [0, 1];$$

3. the saving function $E_t = E(t)$, $t \in \mathbf{N}$:

$$(2.4) \quad E_t = d(y_{t-2} - y_{t-1}) + mS_{t-1}, \quad d \geq 0, m \geq 0.$$

The mathematical model is described by the relation:

$$(2.5) \quad y_t = I_t + C_t + E_t.$$

From (2.1), (2.2), (2.3), (2.4) and (2.5) the mathematical model with discrete time and consumer sentiment is given by:

$$(2.6) \quad y_t = a + dy_{t-2} + (b - d)y_{t-1} + v(y_{t-1} - y_{t-2}) - w(y_{t-1} - y_{t-2})^3 + \frac{cy_{t-1} + m}{1 + \varepsilon \exp(y_{t-2} - y_{t-1})}, \quad t \in \mathbf{N}.$$

The parameters from (2.6) have the following economic interpretations. The parameter a represents the autonomous expenditures. The parameter d is the control, $d \in [0, 1]$ and it characterizes a part of the difference between the incomes obtained at two time steps $t-2$ and $t-1$, which is used for consumption or saving in the time step t . The parameter c , $c \in [0, 1]$, is the trend towards consumption. The parameter m , $m \in [0, 1]$, is the trend towards the saving. The parameter b , $b \in (0, 1)$, represents the consumer's reaction against the increase or decrease of his income. When the income

(strongly) decreases, the consumer becomes pessimistic and consumes $0 < b < 1$ of his income. When the income (strongly) increases, the consumer becomes optimistic and consumes $b < b + c < 1$ of his income. Note that Souleles ([10]) finds, in fact, that higher consumer confidence is correlated with less saving and increases in relation to expected future resources. The parameters v and w , $v > 0$, $w \geq 0$ describe the investment function. If $w = 0$ the investment function is linear. The parameter ε , $\varepsilon \in [0, 1]$, describes a family of the sentiment functions.

For different values of the model parameters, we obtain the following classical models:

1. for $a = 0$, $b = 1 - s$, $d = 0$, $m = 0$, $\varepsilon = 0$, $s \in (0, 1)$ from (2.6) we obtain the Hick-Samuels model ([9]):

$$y_t = (1 + v - s)y_{t-1} - vy_{t-2}, \quad v > 0, s \in (0, 1);$$

2. for $v = w = 0$, $\varepsilon = 0$, $m = 0$, (2.6) gives us the Keynes model ([9]):

$$y_t = d(y_{t-2} - y_{t-1}) + a + by_{t-1};$$

3. for $v = w = 0$, $\varepsilon = 1$, (2.6) leads to the model from [3]:

$$y_t = d(y_{t-2} - y_{t-1}) + a + by_{t-1} + \frac{cy_{t-1} + m}{1 + \exp(y_{t-2} - y_{t-1})}, \quad t \in \mathbf{N};$$

4. for $a = 0$, $b = s - 1$, $d = 0$, $m = 0$, $w = 1 + v$, $\varepsilon = 0$, from (2.6) we get the Puu model ([9]):

$$y_t = v(y_{t-1} - y_{t-2}) - (1 + v)(y_{t-1} - y_{t-2})^3 - (1 - s)y_{t-1}.$$

3 The dynamical behavior of the model (2.6).

Using the method from Kusnetsov [6] and [7], we will analyze the system (2.6), considering the parameter a as bifurcation parameter.

The associated map of (2.6) is $F : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ given by:

$$(3.1) \quad F \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} a + (b - d + v)y + (d - v)z - w(y - z)^3 + \frac{cy + m}{1 + \varepsilon \exp(z - y)} \\ y \end{pmatrix}$$

Using the methods from [6], [7], the map (7) has the following properties:

Proposition 1. (i) If $(1 + \varepsilon)(1 - b) - c > 0$, then, for the map (7), the fixed point with the positive components is $E_0(y_0, z_0)$, where:

$$(3.2) \quad y_0 = p_1 a + p_2, z_0 = y_0$$

and

$$p_1 = \frac{1 + \varepsilon}{(1 + \varepsilon)(1 + b) - c}, p_2 = \frac{m}{(1 + \varepsilon)(1 - b) - c};$$

(ii) The Jacobi matrix of the map F in E_0 is given by:

$$(3.3) \quad A = \begin{pmatrix} a_{11} & a_{12} \\ 1 & 0 \end{pmatrix}$$

where

$$a_{11} = p_3 a - p_4 + p_5, a_{12} = -p_3 a + p_4$$

$$p_3 = \frac{\varepsilon c}{(1 + \varepsilon)^2} p_1, p_4 = d - v - \frac{\varepsilon m}{(1 + \varepsilon)^2} - \frac{\varepsilon c}{(1 + \varepsilon)^2} p_2, p_5 = b + \frac{c}{1 + \varepsilon};$$

(iii) The characteristic equation of the matrix A is:

$$(3.4) \quad \lambda^2 - a_{11}\lambda - a_{12} = 0;$$

(iv) If the model parameters $d, v, \varepsilon, b, c, m$ satisfy the following inequality:

$$(3.5) \quad (1 + d - v)(1 + \varepsilon)((1 + \varepsilon)(1 - b) - c) - m\varepsilon(1 - b) > 0,$$

then, for equation (3.4), the roots have the modulus less than 1, if and only if $a \in (a_1, a_2)$, where

$$(3.6) \quad a_1 = \frac{2p_4 - p_5 - 1}{2p_3}, a_2 = \frac{1 + p_4}{p_3};$$

(v) If the model parameters $d, v, \varepsilon, b, c, m$ satisfy the inequality (3.5) and $a = a_1$ then, one of equation (3.4)'s roots is -1 , while the other one has the modulus less than 1;

(vi) If the model parameters $d, v, \varepsilon, b, c, m$ satisfy the inequality (3.5) and $a = a_2$ then, the equation (3.4) has the roots $\mu_1(a) = \mu(a)$, $\mu_2(a) = \bar{\mu}(a)$, where $|\mu(a)| = 1$.

The following statement results from Proposition 1. If the inequality (3.5) holds and $2p_4 - p_5 - 1 < 0$, then for equation (3.4), with $a \in (0, a_2)$, the roots have the modulus less than 1 and for $a = a_2$, it has the roots $\mu_1(a) = \mu(a)$, $\mu_2(a) = \bar{\mu}(a)$, where $|\mu(a)| = 1$. If the inequality (3.5) holds and $2p_4 - p_5 - 1 \geq 0$, then for equation (3.4), with $a \in (a_1, a_2)$, the roots have the modulus less than 1 and for $a = a_1$, one of equation (3.4)'s roots is -1 , while the other one has the modulus less than 1; for $a = a_2$ it has the roots $\mu_1(a) = \mu(a)$, $\mu_2(a) = \bar{\mu}(a)$, where $|\mu(a)| = 1$.

Using [8] and Proposition 1, with respect to parameter a , the asymptotical stability conditions of the fixed point, the conditions for the existence of the flip and Neimark-Sacker bifurcations are presented in:

Proposition 2. (i) If $(1 + \varepsilon)(1 - b) - c > 0$, the inequality (3.5) holds and $2p_4 - p_5 - 1 > 0$, then for $a \in (a_1, a_2)$ the fixed point E_0 is asymptotically stable. If $(1 + \varepsilon)(1 - b) - c > 0$, the inequality (3.5) holds and $2p_4 - p_5 - 1 < 0$, then for $a \in (0, a_2)$ the fixed point E_0 is asymptotically stable;

(ii) If $(1 + \varepsilon)(1 - b) - c > 0$, inequality (3.5) holds and $2p_4 - p_5 - 1 > 0$, then $a = a_1$ is a flip bifurcation and $a = a_2$ is a Neimark-Sacker bifurcation;

(iii) If $(1 + \varepsilon)(1 - b) - c > 0$, inequality (3.5) holds and $2p_4 - p_5 - 1 < 0$, then $a = a_2$ is a Neimark-Sacker bifurcation.

4 The normal form for flip and Neimark-Sacker bifurcations.

In this section, we describe the normal form in the neighborhood of the fixed point E_0 , for the cases $a = a_1$ and $a = a_2$.

We consider the transformation:

$$(4.1) \quad u_1 = y - y_0, u_2 = z - z_0$$

where y_0, z_0 are given by (3.2). With respect to (4.1), the map (3.1) is $G : \mathbf{R}^2 \rightarrow \mathbf{R}^2$, where

$$(4.2) \quad G(u_1, u_2) = \begin{pmatrix} g_1(u_1, u_2) \\ g_2(u_1, u_2) \end{pmatrix}$$

and

$$\begin{aligned} g_1(u_1, u_2) &= -\frac{r}{1+\varepsilon} + (b-d+v)u_1 + (d-v)u_2 - w(u_1 - u_2)^3 + \\ &\quad + \frac{cu_1 + r}{1 + \varepsilon \exp(u_2 - u_1)} \\ g_2(u_1, u_2) &= u_1, r = cy_0 + m. \end{aligned}$$

The map (4.2) has $O(0, 0)$ as fixed point.

We consider:

$$\begin{aligned} a_{11} &= \frac{\partial g_1}{\partial u_1}(0, 0), a_{12} = \frac{\partial g_1}{\partial u_2}(0, 0), l_{20} = \frac{\partial^2 g_1}{\partial u_1^2}(0, 0), l_{11} = \frac{\partial^2 g_1}{\partial u_1 \partial u_2}(0, 0) \\ l_{02} &= \frac{\partial^2 g_1}{\partial u_2^2}(0, 0), l_{30} = \frac{\partial^3 g_1}{\partial u_1^3}(0, 0), l_{21} = \frac{\partial^3 g_1}{\partial u_1^2 \partial u_2}(0, 0), l_{12} = \frac{\partial^3 g_1}{\partial u_1 \partial u_2^2}(0, 0) \\ l_{03} &= \frac{\partial^3 g_1}{\partial u_1^3}(0, 0). \end{aligned}$$

We develop the function $G(u), u = (u_1, u_2)^T$ in the Taylor series until the third order and obtain:

$$G(u) = Au + \frac{1}{2}B(u, u) + \frac{1}{2}C(u, u, u) + O(|u|^4)$$

where A is given by (3.3) and

$$(4.3) \quad B(u, u) = (B^1(u, v), 0)^T, C(u, u, u) = (C^1(u, u, u), 0)^T$$

$$\begin{aligned} B^1(u, u) &= u^T \begin{pmatrix} l_{20} & l_{11} \\ l_{11} & l_{02} \end{pmatrix} u, \\ C^1(u, u, u) &= u^T \left(u_1 \begin{pmatrix} l_{30} & l_{21} \\ l_{21} & l_{12} \end{pmatrix} + u_2 \begin{pmatrix} l_{21} & l_{12} \\ l_{12} & l_{03} \end{pmatrix} \right) u. \end{aligned}$$

For $a = a_1$, given by (3.6) with the condition (v) from Proposition 1, we have:

Proposition 3. (i) The eigenvector $q \in \mathbf{R}^2$, given by $Aq = -q$, has the components:

$$q_1 = 1, q_2 = -1.$$

(ii) The eigenvector $h \in \mathbf{R}^2$, given by $h^T A = -h^T$, has the components:

$$h_1 = \frac{1}{1 + a_{12}}, h_2 = -\frac{a_{12}}{1 + a_{12}}.$$

The relation $\langle q, h \rangle = 1$ holds.

(iii) The normal form of the map (3.1) on the center manifold in $O(0, 0)$ is given by:

$$\eta \rightarrow -\eta + \frac{1}{6}\nu\eta^3 + O(\eta^4)$$

where

$$\nu = \frac{1}{1 + a_{12}}(l_{30} - 3l_{21} + 3l_{12} - l_{03}) + \frac{3}{1 - a_{11} - a_{12}}(l_{20} - 2l_{11} + l_{02})^2.$$

The proof results from straight calculus using the formula [8]:

$$\nu = \langle r, C(q, q, q) \rangle + 3B(q, (I - A)^{-1}B(q, q)), \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $a = a_2$, given by (3.6), with condition (vi) from Proposition 1, we have:

Proposition 4. (i) The eigenvector $q \in C^2$ given by $Aq = \mu_1 q$, where μ_1 is the eigenvalue of the matrix A , has the components:

$$(4.4) \quad q_1 = 1, q_2 = \mu_2 = \bar{\mu}_1;$$

(ii) The eigenvector $h \in C^2$ given by $h^T A = \mu_2 h^T$, where $\mu_2 = \bar{\mu}_1$ is given by:

$$h_1 = \frac{1}{1 + \mu_1^2 a_{12}}, h_2 = \frac{a_{12} \mu_1}{1 + \mu_1^2 a_{12}}.$$

The relation $\langle q, \bar{h} \rangle = 1$ holds.

Using (4.3) and (4.4) we have:

$$\begin{aligned} B^1(q, q) &= l_{20} + 2l_{11}\mu_2 + l_{02}\mu_2^2 \\ B^1(q, \bar{q}) &= l_{20} + l_{11}(\mu_1 + \mu_2) + l_{02}\mu_1\mu_2 \\ B^1(\bar{q}, \bar{q}) &= l_{20} + 2l_{11}\mu_1 + l_{02}\mu_1^2. \end{aligned}$$

We denoted by:

$$(4.5) \quad \begin{aligned} g_{20} &= v_1 B^1(q, q), g_{11} = v_1 B^1(q, \bar{q}), g_{02} = v_1 B^1(\bar{q}, \bar{q}) \\ h_{20}^1 &= (1 - v_1 - \bar{v}_1)g_{20}, h_{11}^1 = (1 - v_1 - \bar{v}_1)g_{11}, h_{02}^1 = (1 - v_1 - \bar{v}_1)g_{02} \\ h_{20}^2 &= -(v_1\mu_2 + \bar{v}_1\mu_1)B^1(q, q), h_{11}^2 = -(v_1\mu_2 + \bar{v}_1\mu_1)B^1(q, \bar{q}), \\ h_{02}^2 &= -(v_1\mu_2 + \bar{v}_1\mu_1)B^1(q, \bar{q}), \end{aligned}$$

$$(4.6) \quad \begin{aligned} w_{20} &= (\mu_1^2 I - A)^{-1} \begin{pmatrix} h_{20}^1 \\ h_{20}^2 \end{pmatrix}, w_{11} = (I - A)^{-1} \begin{pmatrix} h_{11}^1 \\ h_{11}^2 \end{pmatrix}, \\ w_{02} &= (\mu_2^2 - A)^{-1} \begin{pmatrix} h_{02}^1 \\ h_{02}^2 \end{pmatrix}, \end{aligned}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and A is given by (3.3),

and

$$(4.7) \quad \begin{aligned} r_{20} &= B^1(q, w_{20}), r_{11} = B^1(q, w_{11}), \\ C_0 &= C^1(q, q, \bar{q}) = l_{30} + (\mu_1 + 2\mu_2)l_{21} + \mu_2(\mu_2 + 2\mu_1)l_{12} + \mu_1\mu_2^2l_{03}, \\ g_{21} &= v_2(r_{20} + 2r_{11} + C_0). \end{aligned}$$

Using the normal form for the Neimark-Sacker bifurcation of the dynamical systems with discrete time [6] and (4.5), (4.6), (4.7), we obtain:

Proposition 5. *The solution of the system (2.6) in the neighborhood of the fixed point $(y_0, z_0) \in \mathbf{R}^n$ is given by:*

$$\begin{aligned} y_t &= y_0 + \mu_2 x_t + \mu_1 \bar{x}_t + \frac{1}{2} w_{20}^2 x_t^2 + w_{11}^2 x_t \bar{x}_t + \frac{1}{2} w_{02}^2 \bar{x}_t^2 \\ y_{t-1} = z_t = z_0 &+ x_t + \bar{x}_t + \frac{1}{2} w_{20}^1 x_t^2 + w_{11}^1 x_t \bar{x}_t + \frac{1}{2} w_{02}^1 \bar{x}_t^2 \end{aligned}$$

where x_t is the solution of the following equation:

$$(4.8) \quad x_{t+1} = \mu x_t + \frac{1}{2} g_{20} x_t^2 + g_{11} x_t \bar{x}_t + \frac{1}{2} g_{02} \bar{x}_t^2 + \frac{1}{2} g_{21} x_t^2 \bar{x}_t;$$

(ii) A complex variable transformation exists so that the equation (4.8) becomes:

$$\omega_{t+1} = \mu_1 \omega_t + C_1 \omega_t^2 \bar{\omega}_t + O(|\omega_t|^4),$$

where

$$(4.9) \quad C_1 = \frac{g_{20} g_{11} (\mu_2 - 3 - 2\mu_1)}{2(\mu_1^2 - \mu_1)(\mu_2 - 1)} + \frac{|g_{11}|^2}{1 - \mu_1} + \frac{|g_{20}|^2}{2(\mu_1^2 - \mu_2)} + \frac{g_{21}}{2};$$

(iii) If $l_0 = \operatorname{Re}(e^{-i\theta} C_1) < 0$, with $\theta = \arg(\mu_1)$, then in the neighborhood of the fixed point (y_0, z_0) there is a stable limit cycle.

5 The Lyapunov exponents.

If $a_1 > 0$ then for $a \in (0, a_1)$ or $a \in (a_2, \infty)$ the system (2.6) has a complex behavior and it can be established by computing the Lyapunov exponents.

We will use the algorithm QR from [6] and the determination of the Lyapunov exponents can be obtained by solving the following system:

$$y_{t+1} = a + (b - d + v)y_t + (d - v)z_t - w(y_t - z_t)^3 + \frac{cy_t + m}{1 + \exp(z_t - y_t)}$$

$$z_{t+1} = y_t$$

$$x_{t+1} = \arctan \left(\frac{\cos x_t}{f_{11} \cos x_t - f_{12} \sin x_t} \right)$$

$$\lambda_{t+1} = \lambda_t + \ln(|(f_{11} - \tan x_{t+1}) \cos x_t \cos x_{t+1} - f_{12} \sin x_t \cos x_{t+1}|)$$

$$\mu_{t+1} = \mu_t + \ln(|(f_{11} - \tan x_{t+1} + 1) \sin x_t \cos x_{t+1} + f_{12} \cos x_t \cos x_{t+1}|)$$

with

$$f_{11} = \frac{\partial f_1}{\partial y}(y_t, z_t) = b - d + v - 3w(y_t - z_t)^2 + \frac{c + \varepsilon(c + m + cy_t)\exp(z_t - y_t)}{(1 + \varepsilon\exp(z_t - y_t))^2}$$

$$f_{12} = \frac{\partial f_1}{\partial z}(y_t, z_t) = d - v + 3w(y_t - z_t)^2 - \frac{\varepsilon(m + cy_t)\exp(z_t - y_t)}{(1 + \varepsilon\exp(z_t - y_t))^2}.$$

The Lyapunov exponents are:

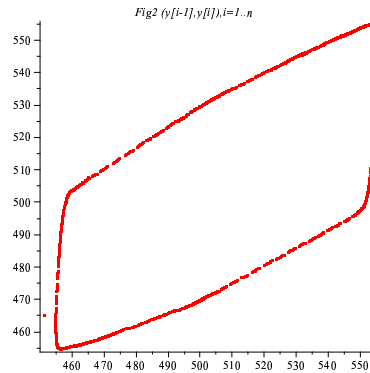
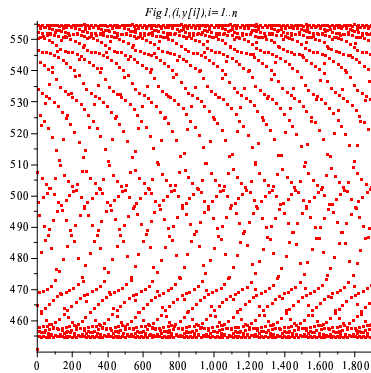
$$L_1 = \lim_{t \rightarrow \infty} \frac{\lambda_t}{t}, L_2 = \lim_{t \rightarrow \infty} \frac{\mu_t}{t}.$$

If one of the two exponents is positive then the system has a chaotic behavior.

6 Numerical simulation.

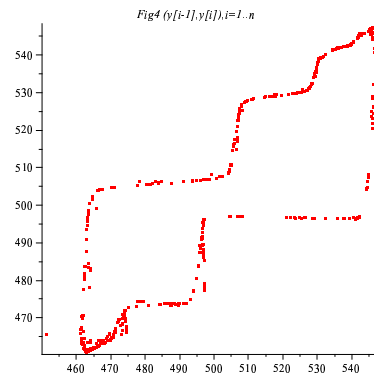
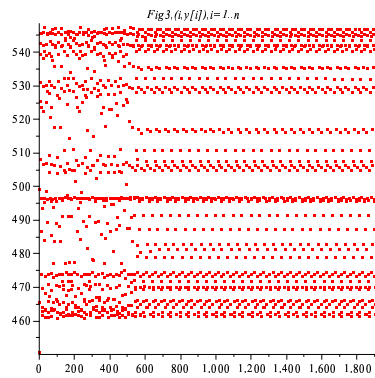
Using a Maple 11 program, the numerical simulation is done. We consider different values for the parameters which are used in the real economic processes. The bifurcation parameter is a .

For $a = 250$, $v = 0.1$, $w = 0$, $c = 0.1$, $b = 0.45$, $\varepsilon = 1$, $m = 0.5$ and $d = 0$ we obtain Fig1 and Fig2, which represent the income y_t and the income variation, with respect to the income y_{t-1} :



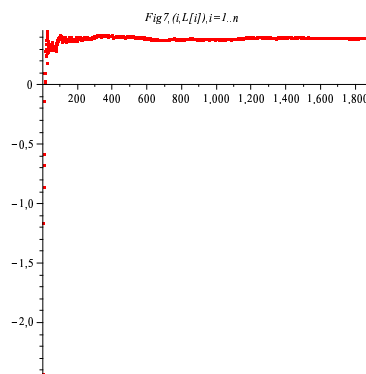
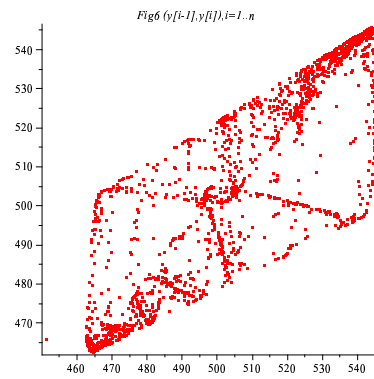
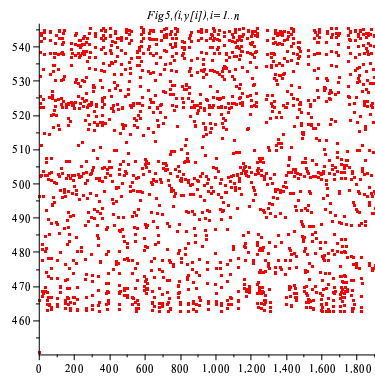
The system is not chaotic.

For $a = 250$, $v = 0.1$, $w = 0$, $c = 0.1$, $b = 0.45$, $\varepsilon = 1$, $m = 0.5$ and $d = 0.6$ we obtain Fig3 and Fig4, which represent the income y_t and the income variation, with respect to the income y_{t-1} :



The system is not chaotic.

For $a = 250$, $v = 0.1$, $w = 0$, $c = 0.1$, $b = 0.45$, $\varepsilon = 1$, $m = 0.5$ and $d = 0.8$ we obtain Fig5, Fig6 and Fig7, which represent the income y_t and the income variation, with respect to the income y_{t-1} and the Lyapunov exponent. In this case the system has a chaotic behavior.



For different values of the parameters, we can obtain a Neimark-Sacker bifurcation point or a flip bifurcation point and a Neimark-Sacker bifurcation point.

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