

Characterization of top spaces by diffeomorphic Lie groups

M.R. Farhangdoost and M.R. Molaei

Abstract. In this paper we present diffeomorphisms between every two sections of a top space and by using of them we will deduce a characterization for top spaces. We also show that top spaces with finite number of identities are diffeomorphic to smooth Rees matrix semigroups. Moreover a solution for an open problem on top spaces is presented.

M.S.C. 2000: 22E60, 22A15, 58A99.

Key words: Lie group, Lie algebra, Top space, Rees matrix semigroup, MF-semigroup.

1 Introduction

We begin this paper by recalling a class of completely simple semigroups [3, 4]. Let G be a group and let two sets I and Λ be given. If $P : \Lambda \times I \longrightarrow G$ is a mapping, then $I \times G \times \Lambda$ with the product $(i, a, \lambda)(j, b, \mu) = (i, ap(\lambda, j)b, \mu)$ is called Rees matrix semigroup [1, 6] and denoted by $M(G, I, \Lambda, P)$.

Theorem 1.1 [2] *If I and Λ are smooth manifolds, G is Lie group, and $P : I \times \Lambda \longrightarrow G$ is a smooth mapping, then $M(G, I, \Lambda, P)$ is a top space.*

In this paper we present a positive answer to the following problem [2]. Suppose T is a top space with the finite numbers of identities, are there smooth manifolds I and Λ , Lie group G and smooth mapping $P : \Lambda \times I \longrightarrow G$ such that $M(G, I, \Lambda, P)$ and T are diffeomorphic top spaces?

We will also present a solution for another problem sketched in [2]. To describe it let us to recall the next Theorem.

Theorem 1.2 [2] *Let T be a top space and the cardinality of $e(T)$ be a natural number. Then the set of left-invariant vector fields on T is Lie algebra under the Lie bracket operation.*

If T is a top space and $e(T)$ is a finite set then Theorem 1.2 implies that there exists a Lie algebra corresponding to T . According to this Lie algebra there is a Lie group. The second problem is: what is the relation between this Lie group and T ? In the second section we will answer to this problem completely.

Definition 1.3 Let T be a top space (not necessarily connected) and G be a topological group (not necessarily connected), a covering projection $P : T \rightarrow G$ is called top space covering for G , if P satisfies the following conditions:

- (i) $P(t) = e$; for all $t \in e(T)$, where e is identity element;
- (ii) $P(t_1 t_2) = P(t_1) P(t_2)$, for all $t_1, t_2 \in T$.

2 Res matrix semigroups and top spaces

In this section except Theorem 2.4 and Corollary 2.6 we assume T is a top space and $e(T)$ is a finite set with n elements. In fact

$$\{e(t_i) | i = 1, 2, \dots, n\}.$$

Theorem 2.1 If $G_i = e^{-1}(e(t_i))$, $1 \leq i \leq n$, then $G =$

$$\left\{ \left(e(t_1), \dots, e(t_{p-1}), t, e(t_{p+1}), \dots, e(t_n) \right) \mid e(t) = e(t_p), \text{ for some } p \in \{1, \dots, n\} \right\}$$

with the following product

$$g_1 * g_2 := \begin{cases} \left(e(t_1), \dots, e(t_n) \right) & ; s, t \in e(T) \\ g_1 & ; t \in e(T), s \notin e(T) \\ g_2 & ; s \in e(T), t \notin e(T) \\ \left(e(t_1), \dots, \underbrace{ts}_m, \dots, e(t_n) \right) & ; t, s \notin e(T), e(ts) = e(t_m) \end{cases}$$

is a Lie group, where $g_1, g_2 \in G$ and

$$g_1 = \left(e(t_1), \dots, \underbrace{t}_p, \dots, e(t_n) \right), \quad g_2 = \left(e(t_1), \dots, \underbrace{s}_q, \dots, e(t_n) \right).$$

Proof. The straightforward calculations show that G is a Lie group with the identity $e = (e(t_1), \dots, e(t_n))$. \square

Theorem 2.2 For the Lie group G presented in Theorem 2.1, there exist two smooth manifolds Λ, I and a smooth mapping $P : \Lambda \times I \rightarrow G$, which top space T is diffeomorphic with a subset of $M(G, I, \Lambda, P)$.

Proof. Let $\Lambda' = \{\lambda_{e(t)} | e(t) \in T\}$ and $I' = \{i_{e(t)} | e(t) \in T\}$. It is clear that the cardinalities of Λ' and I' are equal to n .

Now we define two relations \sim and \approx on I' and Λ' respectively.

$i_{e(t_p)} \sim i_{e(t_q)}$ if and only if there exists $t_0, s_0 \in T$ such that $e(t_p) = e(t_q)e(t_0)$ or $e(t_q) = e(t_p)e(s_0)$ and $\lambda_{e(t_p)} \approx \lambda_{e(t_q)}$ if and only if there exist $t, s \in T$ such that $e(t_p) = e(t)e(t_q)$ or $e(t_q) = e(s)e(t_p)$. One can check \sim, \approx are two equivalence relations. If $I = I' / \sim$ and $\Lambda = \Lambda' / \approx$ then I and Λ have manifold structures. Let $P : \Lambda \times I \rightarrow G$ be the constant mapping $P(\lambda, i) = e$ and $\phi : T \rightarrow I \times G \times \Lambda$ be defined by

$$\phi(t) = \left(\left[i_{e(t)} \right], \left(e(t_1), \dots, \underbrace{t}_q, \dots, e(t_n) \right), \left[\lambda_{e(t)} \right] \right),$$

when $e(t) = e(t_q)$. We show that ϕ is diffeomorphic to its rang. We only prove the following case.

Let $t, t' \in T$, $e(t) = e(t_q)$, $e(t') = e(t_p)$, $1 \leq p, q \leq n$ and $\phi(t) = \phi(t')$. Then $\left(e(t_1), \dots, \underbrace{t}_q, \dots, e(t_n) \right) = \left(e(t_1), \dots, \underbrace{t'}_p, \dots, e(t_n) \right)$. So $t = e(t_q)$ and $t' = e(t_p)$ (I), moreover since $[i_{e(t)}] = [i_{e(t')}]$ and $[\lambda_{e(t)}] = [\lambda_{e(t')}]$ then there exist $s_i, r_i \in T$, ($i = 1, 2$) such that

$$e(t) = e(t')e(s_1) \quad \text{or} \quad e(t') = e(t)e(r_1)$$

and

$$e(t) = e(s_2)e(t') \quad \text{or} \quad e(t') = e(r_2)e(t).$$

Thus we achieve to the following four cases:

- (i) $e(t) = e(t')e(s_1)$ and $e(t) = e(s_2)e(t')$
- (ii) $e(t) = e(t')e(s_1)$ and $e(t') = e(r_2)e(t)$
- (iii) $e(t') = e(t)e(r_1)$ and $e(t) = e(s_2)e(t')$
- (iv) $e(t') = e(t)e(r_1)$ and $e(t') = e(r_2)e(t)$

Case (i):

$e(t) = e(t')e(s_1)$ implies $e(t)e(t) = e(t')e(s_1)e(t)$. Since $e(t) = e(s_2)e(t')$ then $e(t) = e(t')e(s_1)(e(s_2)e(t')) = e(t')$. Thus $e(t) = e(t')$

Case (ii):

Since $e(t) = e(t')e(s_1)$ then $e(t)e(t') = e(t')e(s_1)e(t') = e(t')$. The condition $e(t') = e(r_2)e(t)$ implies $e(t)(e(r_2)e(t)) = e(t')$. Thus $e(t) = e(t')$.

Proof of other cases is the same as (i) or (ii). Thus $e(t) = e(t')$, by (I) we have $t = t'$. This show that ϕ is an injective mapping.

Now we show that ϕ is a homomorphism.

Let $t, t' \in T$, $t, t' \notin e(T)$, $e(t') = e(t_p)$, $e(t) = e(t_q)$ and $e(tt') = e(t_m)$, where $1 \leq p, q, m \leq n$. If $g_1 = \left(e(t_1), \dots, \underbrace{t}_p, \dots, e(t_n) \right)$ and

$g_2 = \left(e(t_1), \dots, \underbrace{t'}_q, \dots, e(t_n) \right)$ then

$$\begin{aligned} \phi(t)\phi(t') &= \left([i_{e(t)}], g_1, [\lambda_{e(t)}] \right) \left([i_{e(t')}], g_2, [\lambda_{e(t')}] \right) \\ &= \left([i_{e(t)}], g_1 P \left([\lambda_{e(t)}], [i_{e(t)}] \right) g_2, [\lambda_{e(t')}] \right) \\ &= \left([i_{e(t)}], g_1 g_2, [\lambda_{e(t')}] \right) \\ &= \left([i_{e(t)}], \left(e(t_1), \dots, \underbrace{tt'}_m, \dots, e(t_n) \right), [\lambda_{e(t')}] \right) \end{aligned}$$

. Since $[i_{e(t)}] = [i_{e(tt')}]$, $[\lambda_{e(t')}] = [\lambda_{e(tt)}]$ and $e(tt') = e(t_m)$ then $\phi(t)\phi(t') = \phi(tt')$. So ϕ is a homomorphism.

It is clear that ϕ and ϕ^{-1} are differentiable maps. Thus ϕ is a diffeomorphic to its rang. \square

Theorem 2.3 *Let T be a top space. Then for $t_i, t_j \in T$, $e^{-1}(e(t_i))$ is diffeomorphic with $e^{-1}(e(t_j))$.*

Proof. Let $\phi_{ij} : e^{-1}(e(t_i)) \longrightarrow e^{-1}(e(t_j))$ be defined by $\phi_{ij}(s) = e(t_j)se(t_j)$. Then we show that ϕ_{ij} is a diffeomorphism between $e^{-1}(e(t_i))$ and $e^{-1}(e(t_j))$. Let $s_1, s_2 \in e^{-1}(e(t_i))$ be given. Then

$$\begin{aligned}\phi_{ij}(s_1 s_2) &= e(t_j)s_1 s_2(t_j) = e(t_j)s_1 e(t_i)s_2 e(t_j) \\ &= e(t_j)s_1 e(t_i t_j t_i)s_2 e(t_j) \\ &= e(t_j)(s_1 e(t_i))e(t_j)e(t_j)(e(t_i)s_2)e(t_j) \\ &= e(t_j)s_1 e(t_j)e(t_j)s_2 e(t_j) = \phi_{ij}(s_1)\phi_{ij}(s_2).\end{aligned}$$

So, ϕ_{ij} is a homomorphism.

If $\phi_{ij}(s_1) = \phi_{ij}(s_2)$, then $e(t_j)s_1 e(t_j) = e(t_j)s_2 e(t_j)$.

Thus

$$e(t_j)(e(t_i)s_1 e(t_i))e(t_j) = e(t_j)(e(t_i)s_2 e(t_i))e(t_j).$$

Hence

$$e(t_j t_i)s_1 e(t_i t_j) = e(t_j t_i)s_2 e(t_i t_j).$$

So

$$e(t_i)e(t_j t_i)s_1 e(t_i t_j)e(t_i) = e(t_i)e(t_j t_i)s_2 e(t_i t_j)e(t_i).$$

Thus $s_1 = s_2$, and this shows that ϕ_{ij} is an injective mapping. By the same process we show that ϕ_{ji} is also an injective homomorphism. Since $\phi_{ij}^{-1} = \phi_{ji}$ and ϕ_{ij}, ϕ_{ji} are smooth maps, then $e^{-1}(e(t_i))$ is diffeomorphic with $e^{-1}(e(t_j))$. \square

Corollary 2.4 *If T is a top space and D is its MF-semigroup [5], then D is isomorphic to $\pi_1(T_{e(t_0)}, e(t_0))$, for an arbitrary element $t_0 \in T$, where $\pi_1(T_{e(t_0)}, e(t_0))$ is the fundamental group of $T_{e(t_0)}$ with the base point $e(t_0)$ and $T_{e(t_0)} = \{s \in T | e(t_0) = e(s)\}$.*

Theorem 2.5 *Let $e(T) = \{e(t_i) | i = 1, 2, \dots, n\}$, $G_i = e^{-1}(e(t_i))$; ($i = 1, 2, \dots, n$), and I be the manifold constructed in Theorem 2.1 and let $\Lambda = \{1\}$, and $P_i : \Lambda \times I \longrightarrow G_i$ be defined by $P_i(1, j) = e(t_i)$, for all $j \in I$. Then T is diffeomorphic with $M(G_i, I, \Lambda, P_i)$.*

Proof. Let $\psi : T \longrightarrow T \times G_i \times \Lambda$ be defined by $\psi(t) = ([i_{e(t_q)}], \phi_{qi}(t), 1)$ where $t \in T$, $e(t) = e(t_q)$ and ϕ_{qi} is defined in Theorem 2.4. We show that ψ is a diffeomorphism. The previous Theorem implies ψ is an injective homomorphism, it is enough to show that ψ is a surjective map. Let $([i_{e(t_q)}], g_i, 1) \in I \times G_i \times \Lambda$ be given. Moreover let $s = \phi_{iq}(g_i)$. Since

$$\begin{aligned}e(t_i)se(t_i) &= e(t_i)\left(e(t_q)se(t_q)\right)e(t_i) \\ &= e(t_i)e(t_q)e(t_i)g_i e(t_i)e(t_q)e(t_i) \\ &= e(t_i t_q t_i)g_i e(t_i t_q t_i) = g_i\end{aligned}$$

then $\psi(s) = ([i_{e(t_0)}], g_i, 1)$. The definition of ψ implies that ψ and ψ^{-1} are smooth maps. Thus ψ is a diffeomorphism between T and $M(G_i, I, \Lambda, P_i)$. \square

Corollary 2.6 *If T is a top space with the finite number of identities, then there exist a topological group G and a top space covering $P : T \rightarrow G$.*

Proof. Let $t_0 \in T$, $G = G_{t_0} = e^{-1}(e(t_0))$ and let $P : T \rightarrow G$ be defined by $P(t) = e(t_0)te(t_0)$; for all $t \in T$. Then Theorem 2.7 implies that P is a top space covering for T . \square

One can deduce the following corollaries of Theorem 2.5.

Corollary 2.7 *With the assumptions of Corollary 2.8, if $G = e^{-1}(e(t))$ is a connected set then $g, \tilde{\mathfrak{S}}$ and \mathfrak{J} are equal, where $g, \tilde{\mathfrak{S}}$ and \mathfrak{J} are the Lie algebras of G, \tilde{G} and T respectively, where \tilde{G} is the universal covering of G .*

Corollary 2.8 *In the Corollary 2.7, if T is connected and $e(t_0) \in T$, then there exists a unique Lie group structure on T such that $e(t_0)$ is its identity element and the Lie algebra of T as a Lie group, is equal to the Lie algebra of T as a top space.*

3 Conclusion

One of the problems which considered in this paper was: suppose T is a top space, are there any smooth manifold I and Λ , Lie group G and smooth mapping $P : \Lambda \times I \rightarrow G$ such that the top spaces $M(D, I, \Lambda, P)$ and T are diffeomorphic top spaces? We solved this problem when $e(T)$ is a finite set, but the problem is open for the other case.

References

- [1] J.M. Howie, *Fundamental of Semigroup Theory*, Oxford University Press, 1995.
- [2] M.R. Molaei, *Top spaces*, Journal of Interdisciplinary Mathematics, 7, 2 (2004), 173-181.
- [3] M.R. Molaei, *Complete semi-dynamical systems*, Journal of Dynamical Systems and Geometric Theories, 3, 2, (2005).
- [4] M.R. Molaei, *Generalized Structures Based on Completely Simple Semigroups*, Hadronic Press, Florida, 2005.
- [5] M.R. Molaei, M.R. Farhangdoost, *Upper top spaces*, Applied Sciences, 8 (2006), 128-131.
- [6] D. Rees, *On semi-groups*, Proceedings of the Cambridge Philosophical Society, 36 (1940), 387-400.

Authors' addresses:

M.R. Farhangdoost and M.R. Molaei
 Department of Mathematics,
 University of Kerman (Shahid Bahonar),
 76169-14111, Kerman, Iran.
 E-mail:farhang@shirazu.ac.ir
 E-mail:mrmolaei@mail.uk.ac.ir