On the stability and control of the Schimizu-Morioka system of dynamical equations

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Abstract. In this paper, the stability analysis of the Schimizu-Morioka system of dynamical equations is performed by applying the Routh-Hurwitz criterion to the solutions of the system near to equilibrium points. The control analysis of the linearized version of the system of equations is then made by adding a control term. As well, phase portraits are analyzed in terms of values of a control parameter.

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1 Introduction

Schimizu-Morioka [9] constructed a model system by dynamical equations which received much attention due to its ability to describe bifurcation of the associated Lorenz like attractors. The model is described by the dynamical system

$$\begin{cases} \dot{X} = Y \\ \dot{Y} = X - \lambda Y - XZ \\ \dot{Z} = -\alpha Z + X^2, \end{cases} \tag{1.1}$$

where $\alpha,\ \lambda > 0$.
They studied the complex behavior of the trajectories of the system by means of computer simulation. The system of equations (1.1) was used as a model to investigate the well known Lorenz system [6, 4] for the cases when the Rayleigh number is large. Shilnikov [7, 8] pointed out that the boundary of the region of existence of a Lorenz like attractor includes two 2-dimensional points, say $Q_\sigma(\alpha = 0.608,\ \lambda = 1.0499)$ and $Q_A(\alpha = 0.549,\ \lambda = 0.605)$. Recently, Kozlov [5] presented an algorithm for the construction of solutions of the ordinary differential equations with power asymptotics. He also applied the technique involved in it to study the stability of the system. Chernousko *et al.* [3] presented the method of dynamical system's phase state evaluation and building of control laws on the basis of the estimates. Controlled dynamical systems subject to perturbations are considered under uncertainties and

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constraints in this study. In this paper, we therefore take interest first to examine the stability of the motion of the system by applying the Routh-Hurwitz criterion to the solutions about the three equilibrium points of the system, and then to investigate the effects of some control parameters on the dynamical system under the influence of a perturbation scheme through Lyapunov's first method.

2 Stability Analysis

In case that from the dynamical system of equations (1.1) follows that

$$\frac{\partial \dot{X}}{\partial X} + \frac{\partial \dot{Y}}{\partial Y} + \frac{\partial \dot{Z}}{\partial Z} = -(\lambda + \alpha) < \ 0,$$

then the system is known to be a dissipative one. For equilibrium (critical or fixed) points, $\dot{X} = \dot{Y} = \dot{Z} = 0$. These will yield $(0,\ 0,\ 0)$, $(\sqrt{\alpha},\ 0,\ 1)$ and $(-\sqrt{\alpha},\ 0,\ 1)$ as three equilibrium points. Let us denote these equilibrium points by $(X_e,\ Y_e,\ Z_e)$. In order to perform the stability analysis of the system about the equilibrium points, we shall introduce small perturbation as follows

$$\left\{ \begin{array}{l} X = X_e + x \\ Y = Y_e + y \\ Z = Z_e + z, \end{array} \right. \tag{2.1}$$

where $x,\ y,\ z$ are small quantities. Using (2.1) in (1.1) and linearizing, we easily get the system of equations

$$\left\{ \begin{array}{l} \dot{x} = Y_e + y \\ \dot{y} = (X_e - \lambda Y_e - X_e Z_e) + (1 - Z_e)x - \lambda y - X_e z \\ \dot{z} = (-\alpha Z_e + X_e^2) + 2X_e x - \alpha z. \end{array} \right. \tag{2.2}$$

We shall further apply the Routh-Hurwitz criterion to the solutions of the system of equations about the three equilibrium points, viz.–$(0,\ 0,\ 0)$, $(\sqrt{\alpha},\ 0,\ 1)$ and $(-\sqrt{\alpha},\ 0,\ 1)$, and discuss its consequences.

Case 1. For the equilibrium points $(X_e,\ Y_e,\ Z_e) = (0,\ 0,\ 0)$, the system of equations (2.2) reduces to

$$\left\{ \begin{array}{l} \dot{x} = y \\ \dot{y} = x - \lambda y \\ \dot{z} = -\alpha z. \end{array} \right. \tag{2.3}$$

We assume a solution for the system of equations (2.3) of the form

$$\left\{ \begin{array}{l} x = A_0\ e^{\mu_0 t} \\ y = B_0\ e^{\mu_0 t} \\ z = C_0\ e^{\mu_0 t} \end{array} \right. \tag{2.4}$$

where $A_0,\ B_0,\ C_0$ are nonzero constants. With the help of (2.4), the system of equations (2.3) generates

$$\begin{array}{l} A_0\mu - B_0 = 0 \\ A_0 - (\mu + \lambda)B_0 = 0 \\ (\mu + \alpha)C_0 = 0. \end{array}$$

For non-zero values of $A_0,\ B_0,\ C_0$ we have

$$\begin{vmatrix} \mu_0 & -1 & 0 \\ 1 & -(\mu_0+\lambda) & 0 \\ 0 & 0 & \mu_0+\alpha \end{vmatrix} = 0,$$

which on simplification yields

$$\mu_0^3 + (\lambda+\alpha)\mu_0^2 + (\alpha\lambda - 1)\mu_0 - \alpha = 0. \tag{2.5}$$

If the values of μ_0 obtained from the equation (2.5) have negative real parts, then by the Lyapunov's stability theorem the dynamical system (1.1) assure the stability near the equilibrium point $(0,\ 0,\ 0)$. The Routh-Hurwiz criterion states that the necessary and sufficient conditions for the equation $a_0^*\mu^3 + a_1^*\mu^2 + a_2^*\mu + a_3^* = 0$, to have negative real parts are $a_0^* > 0,\ a_1^* > 0,\ a_2^* > 0,\ a_3^* > 0$, and, $a_1^*a_2^* - a_0^*a_3^* > 0$. Comparing these with the equation (2.5) we obtain $\lambda + \alpha > 0$, $\lambda\alpha - 1 > 0$, $-\alpha > 0$ and $(\lambda+\alpha)(\lambda\alpha - 1) + \alpha > 0$. Since α is positive, $-\alpha > 0$ does not hold. Thus the system of equations (1.1) is not stable near the equilibrium point $(0,\ 0,\ 0)$.

Case 2. For the equilibrium point $(X_e,\ Y_e,\ Z_e) = (\sqrt{\alpha},\ 0,\ 1)$, the system of equations (2.2) reduces to

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\lambda y - \sqrt{\alpha} z \\ \dot{z} = 2\sqrt{\alpha} x - \alpha z. \end{cases} \tag{2.6}$$

The system of equations (2.6) will have solutions of the form

$$\begin{cases} x = A_1\ e^{\mu_1 t} \\ y = B_1\ e^{\mu_1 t} \\ z = C_1\ e^{\mu_1 t}, \end{cases}$$

where $A_1,\ B_1,\ C_1$ are constants and $A_1B_1C_1 \neq 0$, if

$$\begin{vmatrix} -\mu_1 & 1 & 0 \\ 0 & -(\mu_1+\lambda) & -\sqrt{\alpha} \\ 2\sqrt{\alpha} & 0 & -(\mu_1+\alpha) \end{vmatrix} = 0.$$

After simplification, this reduces to

$$\mu_1^3 + (\lambda+\alpha)\mu_1^2 + (\alpha\lambda)\mu_1 - 2\alpha = 0. \tag{2.7}$$

Applying the Lyapunov's stability theorem and the Routh-Hurwiz criterion, the dynamical system (1.1) will be stable near the equilibrium point $(\sqrt{\alpha},\ 0,\ 1)$ if

$$\alpha + \lambda > 0,\ \alpha\lambda > 0,\ -2\alpha > 0 \text{ and } (\alpha+\lambda)\alpha\lambda + 2\alpha > 0.$$

Since $\alpha > 0,\ -2\alpha < 0$, it follows that the dynamical system (1.1) is not stable near the equilibrium point $(\sqrt{\alpha},\ 0,\ 1)$.

Case 3. For the equilibrium point $(X_e,\ Y_e,\ Z_e) = (-\sqrt{\alpha},\ 0,\ 1)$, the system of equations (2.2) becomes

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\lambda y + \sqrt{\alpha} z \\ \dot{z} = -2\sqrt{\alpha} x - \alpha z. \end{cases} \tag{2.8}$$

If the system (2.8) has a solution of the form

$$\begin{cases} x = A_2\ e^{\mu_2 t} \\ y = B_2\ e^{\mu_2 t} \\ z = C_2\ e^{\mu_2 t}, \end{cases}$$

where $A_2,\ B_2,\ C_2$ are constants and $A_2 B_2 C_2 \neq 0$, then we have

$$\mu_2^3 + (\lambda + \alpha)\mu_2^2 + (\alpha\lambda)\mu_2 + 2\alpha = 0. \tag{2.9}$$

Similar with the cases 1 and case 2, the dynamical system (1.1) will be stable near the equilibrium point $(-\sqrt{\alpha},\ 0,\ 1)$ if

$$\alpha + \lambda > 0,\ \alpha\lambda > 0,\ 2\alpha > 0 \text{ and } (\alpha + \lambda)\alpha\lambda - 2\alpha > 0$$

are satisfied simultaneously for $\alpha > 0,\ \lambda > 0$.

The last inequality can be re-written as $\lambda^2\alpha + \lambda\alpha^2 - 2\alpha > 0$. This holds true if $\alpha^4 + 8\alpha^2 < 0$, which is impossible for real α. Hence the equilibrium point $(-\sqrt{\alpha},\ 0,\ 1)$ is also unstable for the dynamical system (1.1).

3 Control of the linearized Schimizu-Morioka system

It is seen from the previous section that the equilibrium points $(X_e,\ Y_e,\ Z_e)$ e.g., $(0,\ 0,\ 0)$, $(\sqrt{\alpha},\ 0,\ 1)$, $(-\sqrt{\alpha},\ 0,\ 1)$ are all unstable.

To study these cases, we apply the following transformations

$$z_1 = X - X_e,\ z_2 = Y - Y_e,\ z_3 = Z - Z_e \tag{3.1}$$

to the original dynamical system (1.1), where $z_1,\ z_2,\ z_3$ are small quantities. Linearising the transformed equations and adding a control term u to the right-hand side of the second equation, we obtain

$$\begin{cases} \dot{z_1} = Y_e + z_2 \\ \dot{z_2} = (X_e - \lambda Y_e - X_e Z_e) + (1 - Z_e)z_1 - \lambda z_2 - X_e z_3 + u \\ \dot{z_3} = (X_e^2 - \alpha Z_e) + 2X_e z_1 - \alpha z_3. \end{cases} \tag{3.2}$$

The system of equations (3.2) can be written as

(3.3)

$$\begin{pmatrix} \dot{z_1} \\ \dot{z_2} \\ \dot{z_3} \end{pmatrix} = \begin{pmatrix} Y_e \\ X_e - \lambda Y_e - X_e Z_e \\ X_e^2 - \alpha Z_e \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 - Z_e & -\lambda & -X_e \\ 2X_e & 0 & -\alpha \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u.$$

It is seen that at each of the equilibrium points $(X_e,\ Y_e,\ Z_e)$,

$$\begin{pmatrix} Y_e \\ X_e - \lambda Y_e - X_e Z_e \\ X_e^2 - \alpha Z_e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, (3.3) is reduced to the form

$$\dot{W} = A'W + B'u, \tag{3.4}$$

where

$$W = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbf{R}^3;\ A' = \begin{pmatrix} 0 & 1 & 0 \\ 1 - Z_e & -\lambda & -X_e \\ 2X_e & 0 & -\alpha \end{pmatrix}$$

$$B' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix};\quad u\ \in \mathbf{R}.$$

Now, the system is controllable if and only if (Balachandran & Dauer, [2])

$$\mathrm{Rank}(B',\ A'B', (A')^2 B') = 3.$$

Simple calculations yield,

$$(B',\ A'B', (A')^2 B') = \begin{pmatrix} 0 & 1 & -\lambda \\ 1 & -\lambda & 1 - Z_e + \lambda^2 \\ 0 & 0 & 2X_e \end{pmatrix}.$$

Hence, the unstable equilibrium point $(X_e,\ Y_e,\ Z_e) = (0,\ 0,\ 0)$ is not controllable whereas the equilibrium points $(X_e,\ Y_e,\ Z_e) = (\pm\sqrt{\alpha},\ 0,\ 1)$ are controllable.

Following Vincent & Yu [11], the control term may be put as $u = -kz_1$, where k is an appropriate gain.

Case 1. At the unstable equilibrium point $(X_e,\ Y_e,\ Z_e) = (\sqrt{\alpha},\ 0,\ 1)$, the system of equations (3.4) yields

$$\begin{cases} \dot{z_1} = z_2 \\ \dot{z_2} = -kz_1 - \lambda z_2 - \sqrt{\alpha} z_3 \\ \dot{z_3} = 2\sqrt{\alpha} z_1 - \alpha z_3. \end{cases} \tag{3.5}$$

Let us assume the solution of the system of equations (3.5) in the form

$$\begin{cases} z_1 = E_1\ e^{\mu_3 t} \\ z_2 = E_2\ e^{\mu_3 t} \\ z_3 = E_3\ e^{\mu_3 t};\quad (E_1 E_2 E_3 \neq 0). \end{cases}$$

The characteristic equation for the system of equations (3.5) is obtained as

$$\begin{vmatrix} -\mu_3 & 1 & 0 \\ -k & -(\mu_3 + \lambda) & -\sqrt{\alpha} \\ 2\sqrt{\alpha} & 0 & -(\mu_3 + \alpha) \end{vmatrix} = 0$$

which yields,

$$\mu_3^3 + (\lambda + \alpha)\mu_3^2 + (\alpha\lambda + k)\mu_3 + \alpha(2 + k) = 0. \tag{3.6}$$

Since the system is unstable near $(\sqrt{\alpha},\ 0,\ 1)$, we choose the control parameter k in such a way that all the roots of (3.6) have negative real parts. Applying the Routh-Hurwitz criterion, equation (3.6) will have all roots with negative real parts if

$$\lambda + \alpha > 0, \quad \lambda\alpha + k > 0, \quad \alpha(2 + k) > 0 \text{ and } (\lambda + \alpha)(\lambda\alpha + k) - \alpha(2 + k) > 0.$$

Since λ and α are both positive, the first inequality is true. The second, third and forth inequalities hold if $k > -\lambda\alpha$, $k > -2$ and $k > \frac{\alpha}{\lambda}\{2 - \lambda(\lambda + \alpha)\}$, respectively. Thus, if we choose

$$k > \max\{-\lambda\alpha, -2, \frac{\alpha}{\lambda}[2 - \lambda(\lambda + \alpha)]\}, \tag{3.7}$$

then the original unstable system near the critical point $(\sqrt{\alpha},\ 0,\ 1)$ becomes asymptotically stable.

Case 2. At the unstable equilibrium point $(X_e,\ Y_e,\ Z_e) = (-\sqrt{\alpha},\ 0,\ 1)$, the system of equations (3.4) yields

$$\begin{cases} \dot{z_1} = z_2 \\ \dot{z_2} = -kz_1 - \lambda z_2 + \sqrt{\alpha} z_3 \\ \dot{z_3} = -2\sqrt{\alpha} z_1 - \alpha z_3. \end{cases} \tag{3.8}$$

Considering a solution of the system (3.8) of the form

$$\begin{cases} z_1 = F_1\ e^{\mu_4 t} \\ z_2 = F_2\ e^{\mu_4 t} \\ z_3 = F_3\ e^{\mu_4 t}; \quad (F_1 F_2 F_3 \neq 0). \end{cases}$$

We have the characteristic equation as

$$\mu_4^3 + (\lambda + \alpha)\mu_4^2 + (\alpha\lambda + k)\mu_4 + \alpha(2 + k) = 0. \tag{3.9}$$

Like in case 1 of section 3, the unstable equilibrium point $(-\sqrt{\alpha},\ 0,\ 1)$ becomes asymptotically stable if we choose the control parameter k as

$$k > \max\{-2, -\lambda\alpha, \frac{\alpha}{\lambda}[2 - \lambda(\lambda + \alpha)]\}. \tag{3.10}$$

4 Results and discussions

It is seen from Section 2 that the equilibrium points $(0,\ 0,\ 0)$ and $(\pm\sqrt{\alpha},\ 0,\ 1)$ are all unstable for all positive values of the parameters α and λ. But in the beginning of Section 3 it has been shown that the unstable equilibrium points $(\pm\sqrt{\alpha},\ 0,\ 1)$ can be made controlable by adding a suitable control term in the given system. A control term $-kz_1$ is added to the second equation of the system of equations (3.5) in case 1 and also in the second equation of the system of equations (3.8) in case 2. In each case it is seen that if $k > \max\{-\lambda\alpha, -2, \frac{\alpha}{\lambda}[2 - \lambda(\lambda + \alpha)]\}$, then both the unstable equilibrium points become stable provided that $\lambda(\lambda + \alpha) < 2$.

We take the values of the parameters $\lambda = 0.605$ and $\alpha = 0.549$. The time evolution of the components $z_1, \ z_2, \ z_3$ of the uncontrolled ($k = 0$) system are displayed in Figs. 1(a), 2(a), 3(a), and of those with control are displayed in Figs. 1(b,c), 2(b,c), 3(b,c). Equation (3.7) or (3.10) gives $k > 1.181$, when $\lambda = 0.605$ and $\alpha = 0.549$, for stability. Figs. 1(b), 2(b), 3(b) show that the dynamical system (1.1) remains unstable for $k < 1.181$ whereas Figs. 1(c), 2(c), 3(c) show that the dynamical system (1.1) can be made stable by adding control term and satisfying the inequality (3.7) or (3.10).

The phase portraits (trajectories) in two dimensional planes, e.g., $(z_1, \ z_2)$, $(z_1, \ z_3)$ and $(z_2, \ z_3)$ of the uncontrolled (for $k = 0$) as well as of the controlled motions (for $k = 1$ and $k = 1.3$) are drawn in Figs. 4, 5 and 6. The behavior of the trajectories as depicted by the $z_1, \ z_2$ and z_3 components are different from each other, but all are heading to limit cycles. If the condition (3.7) or (3.10) is satisfied by the control parameter k, then the limit cycles become stable - otherwise they are unstable. Within the literature, related results can be found in [10, 1].

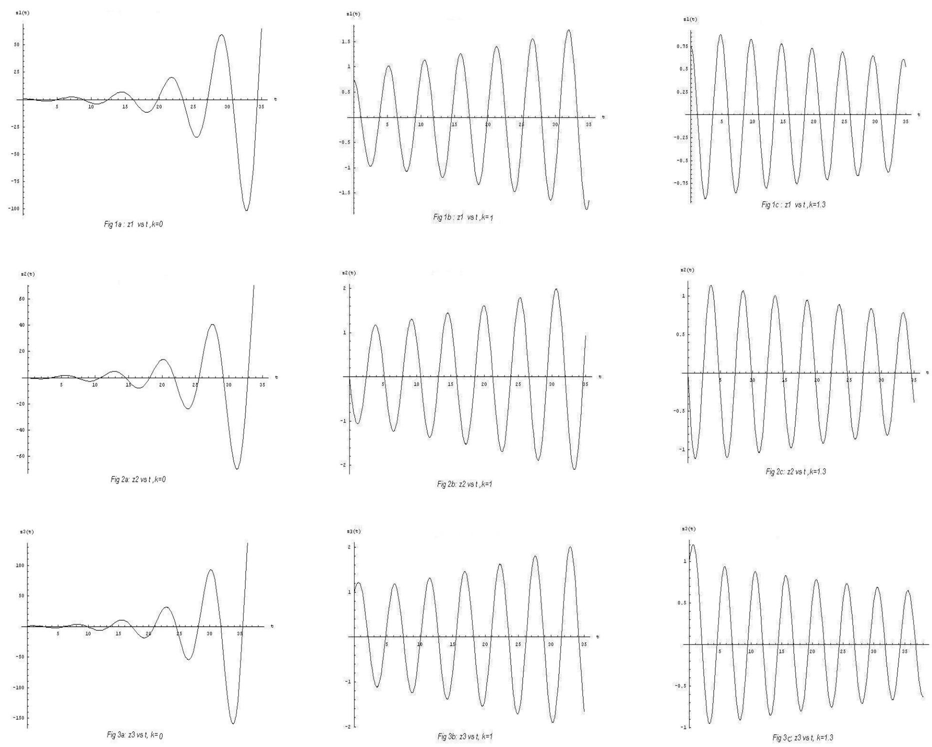

Fig 1a : z1 vs t ,k=0 — Fig 1b : z1 vs t ,k=1 — Fig 1c : z1 vs t ,k=1.3

Fig 2a: z2 vs t ,k=0 — Fig 2b: z2 vs t ,k=1 — Fig 2c: z2 vs t ,k=1.3

Fig 3a: z3 vs t, k=0 — Fig 3b: z3 vs t, k=1 — Fig 3c: z3 vs t, k=1.3

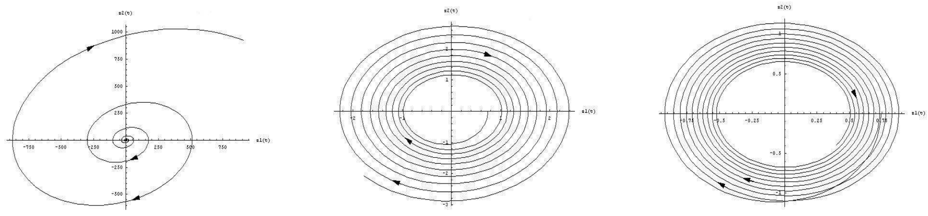

Fig 4a: z2 vs z1 ,k=0 Fig 4b: z2 vs z1 ,k=1 Fig 4c: z2 vs z1 ,k=1.3

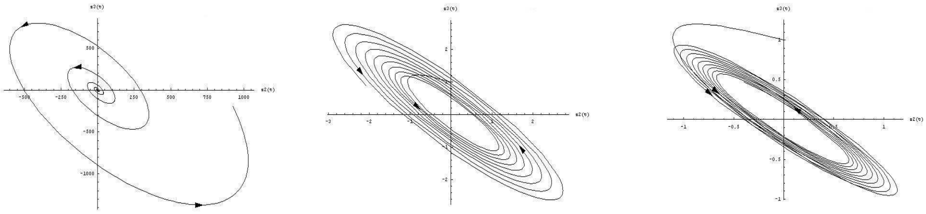

Fig 5a: z3 vs z2 , k=0 Fig 5b: z3 vs z2 ,k=1 Fig 5c: z3 vs z2 ,k=1.3

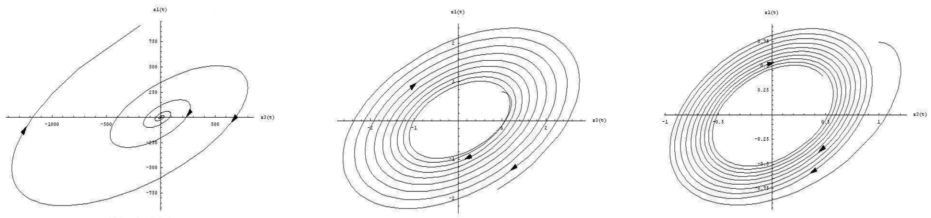

Fig 6a: z1 vs z3 ,k=0 Fig 6b: z1 vs z3, k=1 Fig 6c: z1 vs z3 ,k= 1.3

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