

Symmetry analysis for a new form of the vortex mode equation

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Abstract. Giving a new form of the vortex mode equation by a proper change of parameter, our aim is to analyze the point and contact symmetries of the new equation. Fundamental invariants and a form of general solutions of point transformations along with some specific examples are also derived.

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1 Introduction

Investigation of nonlinear phenomena appearing in a very wide area of pure and applied sciences has met extremely extensive progresses and developments. These studies which split into numerical and analytical considerations are essentially and in most cases related to some nonlinear differential equations. Among those nonlinear systems, a few interesting open problems concern the hydrodynamic type of equations governing fluid motions. Especially the Euler and Navier-Stokes equations which reveal a mysterious behavior are being intensively studied in two main considerations: The incompressible motion mostly dealing with vortex dynamics and the compressible flow concerning the appearance of discontinuities shocks (see [7] and related references therein). In [7] after a brief derivation of relativistic ideal fluid equations, a multidimensional simple wave ansatz is substituted into these equations and various modes (for instance the vortex mode) and phase velocities relative to the laboratory (fixed) frame are found.

The vortex mode equation [1, 7, 9] is defined as a first order ODE

$$(1.1) \quad \frac{dk}{d\varphi} \cdot \left(n - \frac{k \cdot n}{k^2 + w} k \right) = 0,$$

where w is a constant, φ is still treated as the wave phase, and $k = (k_1, k_2, k_3)$ and $n = (n_1, n_2, n_3)$ are some vectors in \mathbb{R}^3 with the physical meaning of κ in [7] and

unit normal vector to the wave front resp. A symmetry analysis of eq. (1.1) up to both point and contact transformations has performed in [7]. But in this paper, we investigate symmetry properties of a new form of eq. (1.1) as a simple form of the vortex mode equation. As is well known, under change of coordinates the symmetry group of a system of differential equations remains unchanged. But since the jacobian of the following change of parameter is zero, so symmetry analysis of the vortex mode equation and the new form are not necessarily the same. By applying the following change of parameter

$$t := \frac{1}{2} \ln(k^2 + w),$$

we find the new form as follows

$$(1.2) \quad n \cdot \left(\frac{dk}{dt} - k \right) = 0.$$

Since eq. (1.1) is a homogeneous linear equation with respect to n , so we consider it to be of arbitrary length and not necessarily unit.

Eq. (1.2) is in fact an expression of the vortex mode equation that provides an in depth study of eq. (1.1). Roughly speaking, it leads to slightly simpler calculations for finding exact solutions of the vortex mode equation. But for reaching to this goal, we investigate symmetry properties of eq. (1.2) which plays a key role in finding general solutions, fundamental invariants, invariant solutions and etc. Moreover, knowledge of a symmetry group of eq. (1.2) allows us to construct new solutions from old ones [2, 3, 5, 6, 4, 8]. Therefore in this study, we concern with the latter equation to find its point and contact symmetry properties and also give its fundamental invariants and a form of general solutions.

Throughout this paper we assume that indices i, j varies between 1 and 3 and each index of a function implies the derivation of the function with respect to it, unless specially stated otherwise.

2 The point symmetry of the equation

To find the symmetry group of eq. (1.2) by Lie infinitesimal method, we follow the method presented in [5]. We find infinitesimal generators of the equation and also the Lie algebra structure of the symmetry group of (1.2). In this section, we are concerned with the action of the point transformation group.

The equation is a relation among with the variables of 1-jet space $J^1(\mathbb{R}, \mathbb{R}^6)$ with (local) coordinate $(t, k, n, q, p) = (t, k_i, n_j, q_r, p_s)$ (for $1 \leq i, j, r, s \leq 3$), where this coordinate involving an independent variable t and 6 dependent variables k_i, n_j and their derivatives q_r, p_s of first order with respect to t resp.

Let \mathcal{M} be the total space of independent and dependent variables. The solution space of eq. (1.2), (if it exists) is a subvariety $S_\Delta \subset J^1(\mathbb{R}, \mathbb{R}^6)$ of the first order jet bundle of one-dimensional submanifolds of \mathcal{M} .

We define a point transformation on \mathcal{M} with relations

$$\tilde{t} = \phi(t, k_i, n_j), \quad \tilde{k}_r = \chi_r(t, k_i, n_j), \quad \tilde{n}_s = \psi_s(t, k_i, n_j).$$

where ϕ, χ_r and ψ_s are some smooth functions. Let

$$v := T \frac{\partial}{\partial t} + \sum_{i=1}^3 \left(K_i \frac{\partial}{\partial k_i} + N_i \frac{\partial}{\partial n_i} \right)$$

be the general form of infinitesimal generators that signify the Lie algebra \mathfrak{g} of the symmetry group G of eq. (1.2). In this relation, T, K_i and N_j are smooth functions of variables t, k_i and n_j . The first order prolongation [5, 4, 8] of v is as follows

$$v^{(1)} := v + \sum_i K_i^t \frac{\partial}{\partial q_i} + \sum_j N_j^t \frac{\partial}{\partial p_j},$$

where $K_i^t = D_t Q_1^i + T q_{i,t}$ and $N_j^t = D_t Q_2^j + T p_{j,t}$, in which D_t is total derivative and $Q_1^i = K_i - T q_i$ and $Q_2^j = N_j - T p_j$ are characteristics of vector field v [5, 4, 8]. By effecting $v^{(1)}$ on (1.2), we obtain the following expression

$$(2.1) \quad \sum_i \left[n_i (K_{it} - K_i) - N_i k_i + p_i \sum_j n_j K_{jn_i} + q_i \left(N_i - n_i T_t + \sum_j n_j K_{jk_i} \right) - q_i^2 n_i T_{k_i} \right] - \sum_{i \neq j} q_i q_j (n_i T_{k_j} + n_j T_{k_i}) - n_i \sum_{i,j} q_i p_j T_{n_j} = 0,$$

in which, each index (exception for determined indices) signifies the derivation with respect it.

We can prescribe t, k_i, n_j, q_r, p_s ($1 \leq i, j, r, s \leq 3$) arbitrarily, and functions K_i and N_j only depend on t, k_i, n_j ($i, j = 1, 2, 3$). So, eq. (2.1) will be satisfied if and only if we have the following equations

$$(2.2) \quad N_i - n_i T_t + n_1 K_{1k_i} + n_2 K_{2k_i} + n_3 K_{3k_i} = 0,$$

$$(2.3) \quad n_1 K_{1n_i} + n_2 K_{2n_i} + n_3 K_{3n_i} = 0,$$

$$(2.4) \quad n_i T_{k_i} = 0, \quad n_i T_{n_j} = 0, \quad n_i T_{k_j} + n_j T_{k_i} = 0,$$

$$(2.5) \quad \sum_{i=1}^3 \left(n_i (K_{it} - K_i) - N_i k_i \right) = 0,$$

when $1 \leq i, j \leq 3$ and (1.2) is satisfied. These equations are called the determining equations. From (2.3), for each i , we have

$$(2.6) \quad N_i = n_i T_t - (n_1 K_1 + n_2 K_2 + n_3 K_3)_{k_i}.$$

Since $n \neq 0$, without less of generality, one may assume that $n_1 \neq 0$. Then by eqs. (2.4) we conclude that T just depends on t :

$$T = T(t).$$

By solving eqs. (2.3) along with eq. (2.5) in respect to K_1, K_2 and K_3 , then we

deduce the following relations (provided by MAPLE)

$$(2.7) \quad K_1 = k_1 T + e^t F^1$$

$$(2.8) \quad \begin{aligned} K_{2t} &= K_2 - k_1 K_{2k_1} - k_2 K_{2k_2} - k_3 K_{2k_3} + k_2 T_t, \\ n_1 K_{2n_1} &= e^t F_4^1, \quad n_2 K_{2n_2} = -e^t F_4^1 - n_3 K_{2n_3}, \end{aligned}$$

$$(2.9) \quad \begin{aligned} K_{3t} &= K_3 - k_1 K_{3k_1} - k_2 K_{3k_2} - k_3 K_{3k_3} + k_3 T_t, \\ n_1 K_{3n_1} &= e^t F_5^1, \quad K_{3n_2} = K_{2n_3} \quad n_3 K_{3n_3} = -e^t F_5^1 - n_2 K_{2n_3} \end{aligned}$$

for arbitrary function $F^1 = F^1(k_1 e^{-t}, k_2 e^{-t}, k_3 e^{-t}, \frac{n_2}{n_1}, \frac{n_3}{n_1})$. In these relations, F_i^1 implies the derivation of F^1 in respect to the i^{th} coefficient.

Eqs. (2.8) lead to the following relations

$$\begin{aligned} K_{2t} + k_1 K_{2k_1} + k_2 K_{2k_2} + k_3 K_{2k_3} - K_2 - k_2 T_t &= 0, \\ n_1 K_{2n_1} + n_2 K_{2n_2} + n_3 K_{2n_3} &= 0, \end{aligned}$$

so that after solving determines the form of K_2 as

$$(2.10) \quad K_2 = k_2 T + e^t F^2,$$

when $F^2 = F^2(k_1 e^{-t}, k_2 e^{-t}, k_3 e^{-t}, \frac{n_2}{n_1}, \frac{n_3}{n_1})$ is an arbitrary smooth function. Also, from eqs. (2.9), we find that

$$\begin{aligned} K_{3t} + k_1 K_{3k_1} + k_2 K_{3k_2} + k_3 K_{3k_3} - K_3 - k_3 T_t &= 0, \\ n_1 K_{3n_1} + n_2 K_{3n_2} + n_3 K_{3n_3} &= 0, \end{aligned}$$

which these expressions tend to the following solution of K_3 with respect to arbitrary smooth function $F^3 = F^3(k_1 e^{-t}, k_2 e^{-t}, k_3 e^{-t}, \frac{n_2}{n_1}, \frac{n_3}{n_1})$:

$$(2.11) \quad K_3 = k_3 T + e^t F^3.$$

But by satisfying F^2 and F^3 resp. in the two last relations of (2.8) and three last relations of (2.9), we find that

$$\begin{aligned} F^1 &= -\frac{1}{n_1}(n_2 F^2 + n_3 G) - n_2 \int \frac{1}{n_1^2} F^2 dn_1 - n_3 \int \frac{1}{n_1^2} G dn_1 + \frac{n_2 n_3}{n_1} \int \frac{1}{n_1^2} F_5^2 dn_1 \\ &\quad + \frac{1}{n_1} \int \left(F^2 - \frac{n_3}{n_1} F_5^2 \right) dn_2 + \frac{1}{n_1} \int G dn_3 + H, \\ F^3 &= \frac{1}{n_1} \int F_5^2 dn_2 - n_2 \int \frac{1}{n_1^2} F_5^2 dn_1 + G, \end{aligned}$$

when F^2 , $G = G(k_1 e^{-t}, k_2 e^{-t}, k_3 e^{-t}, \frac{n_3}{n_1})$ and $H = H(k_1 e^{-t}, k_2 e^{-t}, k_3 e^{-t})$ are arbitrary functions and F_i^j denotes the derivation of F^j with respect to its i^{th} coefficient (similar statement is valid for G_i and H_i in subsequent relations).

Substituting the new forms of K_1, K_2 and K_3 in eqs. (2.3), we obtain the following relations

$$(2.12) \quad \frac{n_2 n_3}{n_1} \int \frac{1}{n_1^2} F_5^2 dn_1 + \frac{1}{n_1} (n_2 F^2 + n_3 G) = 0,$$

$$(2.13) \quad F^2 + n_1 n_3 \int \frac{1}{n_1^3} F_5^2 dn_1 = 0,$$

$$(2.14) \quad n_2(n_1 + n_3) F_5^2 + n_2^2 F_4^2 + n_1^3 n_2 \int \frac{1}{n_1^3} F_5^2 dn_1 - n_1^2 G = n_1^2 n_2 \int \frac{1}{n_1^2} F_5^2 dn_1.$$

By applying the last relations in F^1 and F^3 , we attain the following relations (for arbitrary F^2)

$$(2.15) \quad F^1 = -\frac{2}{n_1} (n_2 F^2 + n_3 G) - n_2 \int \frac{1}{n_1^2} F^2 dn_1 + \frac{1}{n_1} \int \left(F^2 - \frac{n_3}{n_1} F_5^2 \right) dn_2 - n_3 \int \frac{1}{n_1^2} G dn_1 + \frac{1}{n_1} \int G dn_3 + H,$$

$$(2.16) \quad F^3 = \frac{1}{n_1} \int F_5^2 dn_2 + \frac{n_2}{n_3} F^2 + 2G,$$

in which, we assumed that $n_3 \neq 0$. Otherwise, from eq. (2.13), $F^2 = 0$ and hence by eq. (2.14), $G = 0$ and therefore we have $F^1 = H$ and $F^3 = 0$.

We continue our investigation of symmetry group with the condition $n_3 \neq 0$. From eqs. (2.12)–(2.14), we infer the following relation

$$(2.17) \quad n_2(n_1 + n_3) F_5^2 + n_2^2 F_4^2 = 0.$$

On the other hand, by replacing $F_5^2 = F_4^3$ from relation $K_{3n_2} = K_{2n_3}$ of (2.9), in (2.17), we find that $n_2 F^2 = -2n_3 G$. This relation along with relation (2.17) leads to an equation that its solution determines the form of G as following

$$G = \frac{1}{n_3} (n_1 + n_3) L,$$

where $L = L(k_1 e^{-t}, k_2 e^{-t}, k_3 e^{-t})$ is an arbitrary smooth function. Hence, $F^2 = -\frac{2}{n_2} (n_1 + n_3) L$ and from eqs. (2.15) and (2.16), we have (we suppose that $n_2 \neq 0$, otherwise from eq. (2.12), $G = 0$ and hence $L = 0$)

$$F^1 = \frac{1}{n_1} \left(2(n_1 + n_2 + n_3) + n_1 \ln(n_1 n_3) \right) L + H, \quad F^3 = -2 \ln(n_2) L.$$

Applying the last forms of K_i s in (2.3) for $i = 3$, we find $\frac{n_1}{n_3} L = 0$. We have assumed that $n_1 \neq 0$, therefore $L = 0$ and the forms of K_i and N_j satisfy

$$\begin{aligned} K_1 &= k_1 T + e^t H, & K_2 &= k_2 T, \\ K_3 &= k_1 T, & N_j &= n_i (T_t - T) - n_1 H_i. \end{aligned}$$

Finally, the general form of infinitesimal generators as elements of point symmetry algebra of eq. (1.2), which we call *point infinitesimal generators*, is described by the

$[,]$	v_T	v_H
v_T	0	0
v_H	0	0

Table 1: The commutators table of \mathfrak{g} for eq. (1.2)

following relation, which holds true for arbitrary functions T and H

$$(2.18) \quad v = T \left(\frac{\partial}{\partial t} + k_2 \frac{\partial}{\partial k_2} + k_3 \frac{\partial}{\partial k_3} \right) + (k_1 T + e^t H) \frac{\partial}{\partial k_1} + \sum_{i=1}^3 \left(n_i (T_t - T) - n_1 H_i \right) \frac{\partial}{\partial n_i}.$$

One may divide v into the following infinitesimal generators

$$(2.19) \quad v_T = T \left(\frac{\partial}{\partial t} + \sum_{i=1}^3 k_i \frac{\partial}{\partial k_i} \right) + (T_t - T) \sum_{i=1}^3 n_i \frac{\partial}{\partial n_i},$$

$$v_H = e^t H \frac{\partial}{\partial k_1} - n_1 \sum_{i=1}^3 H_i \frac{\partial}{\partial n_i}.$$

The Lie bracket (commutator) of the vector fields (2.19) results as straightforward linear combination of these fields. The table of commutators is given in Table 1. Hence, the Lie algebra $\mathfrak{g} = \langle v_T, v_H \rangle$ of point symmetry group G is an abelian Lie algebra.

Theorem 1. *The set of all point infinitesimal generators in the forms of (2.19) is the infinite dimensional abelian Lie algebra of the point symmetry group of equation (1.2).*

According to Theorem 2.74 of [5, 4, 8], the invariants $u = I(t, k_1, k_2, k_3, n_1, n_2, n_3)$ of one-parameter group with infinitesimal generators in the forms of (2.19) satisfy the linear, homogeneous partial differential equations of first order:

$$v[I] = 0.$$

The solutions of the latter, are found by the method of characteristics (See [5] and [2] for details). So we can replace the last equation by the following characteristic system of ordinary differential equations ($1 \leq i, j \leq 3$)

$$(2.20) \quad \frac{dt}{T} = \frac{dk_i}{K_i} = \frac{dn_j}{N_j}.$$

By solving eqs. (2.20) of the differential generator (2.19), we (locally) find the follow-

ing general solutions

$$(2.21) \quad \begin{aligned} I_1(t, k, n) &= k_2^{-1}(k_1 T + H) = d_1, \\ I_i(t, k, n) &= \ln(k_i) - t = d_i, \quad (\text{for } i = 2, 3) \\ I_4(t, k, n) &= (T_t - T - H_1) \ln(k_1) - T \ln(n_1) = d_4, \\ I_j(t, k, n) &= (T_t - T) \ln(k_j) - T \ln(n_j(T_t - T) - n_1 H) = d_j, \quad (\text{for } j = 5, 6). \end{aligned}$$

when d_i s are some constants. The functions I_1, I_2, \dots, I_6 form a complete set of functionally independent invariants of one-parameter group generated by (2.19) (see [5]).

Similar to the theorem of section 4.3.3 of [2], the derived invariants (2.16) as independent first integrals of the characteristic system of the infinitesimal generator (2.14), provide the general solution

$$S(t, k, n) := \mu(I_1(t, k, n), I_2(t, k, n), \dots, I_6(t, k, n)),$$

with an arbitrary function μ , which satisfies in the equation $v[\mu] = 0$.

This theorem can be extended for each finite set of independent first integrals (invariants) of characteristic system provided with an infinitesimal generator.

In the following, we give some examples provided with different selections of coefficients of eq. (2.18) for better studying, and we assume that each appeared coefficient of vector fields be non-zero.

Example 1. If we assume that $T = 1$ and $H = 0$, then the infinitesimal operator (2.13) reduces to the following vector field

$$v_1 = \frac{\partial}{\partial t} + \sum_{i=1}^3 k_i \frac{\partial}{\partial k_i} - \sum_{j=1}^3 n_j \frac{\partial}{\partial n_j}.$$

and the group transformations (or flows) for the parameter s are expressible as $(t, k_i, n_j) \rightarrow (t + s, k_i e^s, n_j e^{-s})$, that form the (local) symmetry group of v_1 .

The derived invariants in this case will be as follows

$$I_i = \ln(k_i) - t, \quad I_{j+3} = \ln(n_j) + t, \quad i, j = 1, 2, 3.$$

Therefore, the general solution corresponding to v_1 , when μ is an arbitrary function, will be $S(t, k, n) = \mu(\ln(k_i) - t, \ln(n_j) + t)$.

Example 2. Let $T = t$ and $H = 0$, then the infinitesimal generator is

$$v_2 = t \frac{\partial}{\partial t} + \sum_{i=1}^3 k_i t \frac{\partial}{\partial k_i} + \sum_{j=1}^3 n_j (1 - t) \frac{\partial}{\partial n_j},$$

Then, the flows of v_2 for various values of parameter s are

$$(t, k_i, n_j) \rightarrow (t e^s, k_i e^{t(e^s - 1)}, n_j e^{s - t(e^s - 1)}).$$

Also, we have the below invariants

$$I_i = \ln(k_i) - t, \quad I_{j+3} = \ln\left(\frac{n_j}{t}\right) + t, \quad \text{for } i, j = 1, 2, 3,$$

and the general solution of eq. (1.2) as $S(t, k, n) = \mu\left(\ln(k_i) - t, \ln\left(\frac{n_i}{t}\right) + t\right)$ when μ is an arbitrary function.

Example 3. For the case which $T = 0$ and $H = k_1 e^{-t}$, the infinitesimal generator (2.18) changes to

$$v_3 = k_1 \frac{\partial}{\partial k_1} - \sum_{j=1}^3 n_j \frac{\partial}{\partial n_j}.$$

The derived group transformations of v_3 for parameter s are

$$(t, k_i, n_j) \rightarrow (t, k_1 e^s, k_2, k_3, n_1 e^{-s}, n_2 e^{-s} - n_1 + n_2, n_3 e^{-s} - n_1 + n_3).$$

Thus, the (modified) invariants are (we suppose that $k_2, k_3 \neq 0$, otherwise k_2, k_3 will be two invariants)

$$\begin{aligned} I_1 &= t, & I_2 &= k_2, & I_3 &= k_3, \\ I_4 &= \frac{k_1}{n_1}, & I_5 &= \ln(k_1) - \frac{n_2}{n_1}, & I_6 &= \ln(k_1) - \frac{n_3}{n_1}, \end{aligned}$$

and for arbitrary function μ , the general solution has the form

$$S(t, k, n) = \mu\left(t, k_2, k_3, \frac{k_1}{n_1}, \ln(k_1) - \frac{n_2}{n_1}, \ln(k_1) - \frac{n_3}{n_1}\right).$$

Example 4. If we suppose $T = t$ and $H = (k_1 + k_2 + k_3) e^{-t}$, then we have the following vector field

$$v_4 = t \frac{\partial}{\partial t} + t(2k_1 + k_2 + k_3) \frac{\partial}{\partial k_1} + t \sum_{i=2}^3 k_i \frac{\partial}{\partial k_i} + \sum_{j=1}^3 (n_j(1-t) - n_1 t) \frac{\partial}{\partial n_j},$$

with group transformations of different parameters s , that transform (t, k_i, n_j) to

$$\begin{aligned} P(s) &= \left(t e^s, -(k_2 + k_3) e^{t(e^s-1)} + (k_1 + k_2 + k_3) e^{2t(e^s-1)}, k_p e^{t(e^s-1)}, \right. \\ &\quad \left. n_1 e^{s-2t(e^s-1)}, n_1 e^{s-2t(e^s-1)} + (n_q - n_1) e^{s-t(e^s-1)} \right), \end{aligned}$$

where $p, q = 2, 3$. Its independent invariants are

$$\begin{aligned} I_1 &= (1-2t) \ln(k_3) - t \ln(n_1(1-2t)), & I_2 &= 2t - \ln(2k_1 + k_2 + k_3), \\ I_3 &= (1-t) \ln(k_2) - t \ln(n_2(1-t) - n_1 t), & I_4 &= t - \ln(k_2) \\ I_5 &= (1-t) \ln(k_3) - t \ln(n_3(1-t) - n_1 t), & I_6 &= k_2 k_3^{-1}, \end{aligned}$$

and hence the general solution of (1.2) in respect to infinitesimal operator v_4 is an arbitrary function of these invariants. Indeed, if $u = f(t, k_i, n_j)$ be a solution of eq. (1.2) then also is $u = f(P(s))$ for each s .

3 The contact symmetry of the equation

In continuation, we change the group action and find symmetry group and invariants of eq. (1.2) up to the contact transformation groups. According to Bäcklund theorem [5], if the number of dependent variables be greater than one (like our problem), then each contact transformation is the prolongation of a point transformation. But in this section, we directly earn the structure of infinitesimal generators of contact transformations

We suppose that the general form of a contact transformation be as following

$$\begin{aligned}\tilde{t} &= \phi(t, k_i, n_j, q_r, p_s), & \tilde{k}_l &= \chi_l(t, k_i, n_j, q_r, p_s), & \tilde{n}_m &= \psi_m(t, k_i, n_j, q_r, p_s), \\ \tilde{q}_n &= \eta_n(t, k_i, n_j, q_r, p_s), & \tilde{p}_u &= \zeta_u(t, k_i, n_j, q_r, p_s),\end{aligned}$$

where i, j, l, m, n and u varies between 1 and 6; and $\phi, \chi_l, \psi_m, \eta_n$ and ζ_u are arbitrary smooth functions. In this case of group action, an infinitesimal generator which is a vector field in $J^1(\mathbb{R}, \mathbb{R}^6)$, has the following general form

$$v := T \frac{\partial}{\partial t} + \sum_{i=1}^3 \left[K_i \frac{\partial}{\partial k_i} + N_i \frac{\partial}{\partial n_i} + Q_i \frac{\partial}{\partial q_i} + P_i \frac{\partial}{\partial p_i} \right],$$

for arbitrary smooth functions T, K_l, N_m, Q_m, P_u ($l = 1, 2$ and $1 \leq m, n, u \leq 3$).

Since our computations are done in 1-jet space, so we do not need to lift v to higher jet spaces and hence we act v (itself) on the eq. (1.2), then we find the following relation

$$\sum_i [n_i(Q_i - K_i) + N_i(q_i - k_i)] = 0.$$

Since $n \neq 0$, so without less of generality, we can suppose that $n_1 \neq 0$, then the solution to this equation for would be

$$K_1 = Q_1 + \frac{1}{n_1} \left[\sum_{i=2,3} n_i(Q_i - K_i) + \sum_j N_j(q_j - k_j) \right].$$

Therefore, the infinitesimal generator which we call it as *contact infinitesimal generator* is in the following form

$$\begin{aligned}v &= T \frac{\partial}{\partial t} + \sum_{i=2}^3 K_i \left(\frac{\partial}{\partial k_i} - \frac{n_i}{n_1} \frac{\partial}{\partial k_1} \right) + \sum_{j=1}^3 \left[Q_j \left(\frac{\partial}{\partial q_i} + \frac{n_j}{n_1} \frac{\partial}{\partial k_1} \right) \right. \\ (3.1) \quad & \left. + N_j \left(\frac{\partial}{\partial n_j} + \frac{1}{n_1} (q_j - k_j) \frac{\partial}{\partial k_1} \right) + P_j \frac{\partial}{\partial p_j} \right].\end{aligned}$$

One may divide the latter form to following vector fields, to consist a basis for Lie

$[,]$	v_i	v_j
v_i	0	$v_i + v_j$
v_j	$-v_i - v_j$	0

Table 2: The commutators table provided by contact symmetry.

algebra $\mathfrak{g} = \langle v \rangle$ of contact symmetry group G

$$(3.2) \quad \begin{aligned} v_1 &= T \frac{\partial}{\partial t}, & v_2 &= K_2 \left(\frac{\partial}{\partial k_2} - \frac{n_2}{n_1} \frac{\partial}{\partial k_1} \right), \\ v_3 &= K_3 \left(\frac{\partial}{\partial k_3} - \frac{n_3}{n_1} \frac{\partial}{\partial k_1} \right), & v_4 &= Q_1 \left(\frac{\partial}{\partial q_1} + \frac{\partial}{\partial k_1} \right), \\ v_5 &= Q_2 \left(\frac{\partial}{\partial q_2} + \frac{n_2}{n_1} \frac{\partial}{\partial k_1} \right), & v_6 &= Q_3 \left(\frac{\partial}{\partial q_3} + \frac{n_3}{n_1} \frac{\partial}{\partial k_1} \right), \\ v_7 &= N_1 \left(\frac{\partial}{\partial n_1} + \frac{1}{n_1} (q_1 - k_1) \frac{\partial}{\partial k_1} \right), & v_8 &= N_2 \left(\frac{\partial}{\partial n_2} + \frac{1}{n_1} (q_2 - k_2) \frac{\partial}{\partial k_1} \right), \\ v_9 &= N_3 \left(\frac{\partial}{\partial n_3} + \frac{1}{n_1} (q_3 - k_3) \frac{\partial}{\partial k_1} \right), & v_{10} &= P_1 \frac{\partial}{\partial p_1}, \\ v_{11} &= P_2 \frac{\partial}{\partial p_2}, & v_{12} &= P_3 \frac{\partial}{\partial p_3}. \end{aligned}$$

The commutators $[v_i, v_j]$ for $1 \leq i, j \leq 12$ are linear combinations of v_i themselves, and hence these vector fields construct a basis for Lie algebra \mathfrak{g} of contact symmetry group G . The commutator table is given in Table 2. In this table, when commutator of two vector fields has a part in the form of a v_i , then we used v_i instead of it. As is indicated in this table, for each $1 \leq i, j \leq 12$, the Lie bracket of v_i and v_j has two parts, one part in the form of v_i , and another part in the form of v_j .

Theorem 2. *The contact symmetry group of (1.2) is an infinite dimensional Lie algebra and its Lie algebra is generated by contact infinitesimal operators (3.2) with the commutators table 2.*

4 Conclusions

A symmetry analysis for a new form of the vortex mode equation led to find the structure of point and contact infinitesimal generators as well as fundamental invariants of the new equation. In addition a form of general solutions implied by these invariants was obtained. Also we presented some examples for the point transformation case which tend to a precise determination of related symmetry groups. In the special case of our problem, the contact and point symmetry group of the vortex mode equation were both found to be infinite dimensional Lie groups when the normal vector to wave front is not necessarily unit.

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