

# Superfields associated to second order dynamical systems of some deformable continuous media

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**Abstract.** The study of first or second order dynamical differential systems, highlights an interesting property, namely we can associate to them fields and propagation waves of these fields ([1], [4], [5]). We generalize these considerations to the evolution of some deformable continuous media, because of their applications, for example in biology.

**M.S.C. 2000:** 35G20, 83C22.

**Key words:** partial differential equations, Maxwell equations.

## Introduction

In what follows we show that to any system of partial differential equations written in main form:  $F_a = A_{ba}^{ij} u_{ij}^b + B_a = 0$ , ( $A_{ab}^{ij} = A_{ab}^{ji}$ ,  $A_{ab}^{ij} = A_{ab}^{ij}(x^h, u^c, u_h^c)$ ,  $B_a = B_a(x^h, u^c, u_h^c)$ ), which models the evolution of a deformable continuous media, we can (locally and) canonically associate differentiable  $(n + 1)$ -forms on the jet space  $J^1(N, M)$  such that the solutions of the system are among the characteristic manifolds. Some of these forms satisfy the additional property to be closed. We call them *superfield forms* and their coefficients *superfield coefficients*. The necessary and sufficient conditions for a form to be closed are called the *Maxwell equations*.

We denote by  $J^2(N, M)$  the space of second order jets from  $N$  to  $M$ , where  $N$  and  $M$  are two smooth manifolds, of dimensions  $n$  and  $m$  respectively.

On  $N$  we consider a volume form:

$$(0.1) \quad \omega_0 = \omega dx^1 \wedge \cdots \wedge dx^n.$$

By the inner product with  $\frac{\partial}{\partial x^i}$ , we obtain the  $(n - 1)$ -forms:

$$(0.2) \quad i_{\frac{\partial}{\partial x^i}} \omega_0 = \omega_i = (-1)^{i-1} \omega dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n, \quad (i = \overline{1, n}),$$

satisfying the properties:  $dx^i \wedge \omega_j = \delta_j^i \omega_0$ ,  $\sum_{i=1}^n dx^i \wedge \omega_i = n \omega_0$ .

## 1 The Lagrange form

### 1.1 The definition of the general Lagrange form.

Let us consider the smooth manifolds  $R^n$  and  $M$ , of dimensions  $n$  and  $m$  respectively and of local coordinates  $(x^i)$  and  $(u^a)$ . On the manifold  $R^n$  we have the canonical volume form  $\omega_0$  and the forms  $\omega_i$ . On the jet space  $J^1(R^n, M)$  we consider a particular  $(n+1)$ -form  $\Omega$ , locally written:

$$(1.1) \quad \begin{aligned} \Omega = & A_{ab}^{ih} du_i^a \wedge du^b \wedge \omega_h + \frac{1}{2} B_{ab}^h du^a \wedge du^b \wedge \omega_h + \\ & + \frac{1}{2} Q_{ab}^{ijh} du_i^a \wedge du_j^b \wedge \omega_h + (E_a du^a - P_a^i du_i^a) \wedge \omega_0, \end{aligned}$$

where:  $B_{ab}^h + B_{ba}^h = 0$ ,  $Q_{ab}^{ijh} + Q_{ba}^{jih} = 0$ ,  $\forall i, j, h = \overline{1, n}$ .

We assume that the conditions:

$$(1.2) \quad \det \Delta_{ijh} = \det \begin{pmatrix} B_{ab}^i & -A_{ba}^{ih} \\ A_{ab}^{ij} & Q_{ab}^{ijh} \end{pmatrix} \neq 0, \quad \forall i, j, h = \overline{1, n},$$

are satisfied. It follows that:

$$(1.3) \quad \begin{aligned} \det \tilde{\Delta}_{ijh} &= \det \begin{pmatrix} B_{ab}^i & -A_{ba}^{ih} & E_a \\ A_{ab}^{ij} & Q_{ab}^{ijh} & -P_a^i \\ -E_b & P_b^j & 1 \end{pmatrix} \neq 0, \quad \forall i, j, h = \overline{1, n}, \\ \det \bar{\Delta}_{ijh} &= \det \begin{pmatrix} B_{ab}^i & -A_{ba}^{ih} & E_a \\ A_{ab}^{ij} & Q_{ab}^{ijh} & -P_a^i \\ -E_b & P_b^j & 0 \end{pmatrix} = 0, \quad \forall i, j, h = \overline{1, n}. \end{aligned}$$

**Definition 1.** We call a Lagrange form on  $J^1(R^n, M)$  an  $(n+1)$ -form written, in a local chart, as (1.1) and verifying the conditions (1.2).

### 1.2 A Cartan type theorem

Let us consider, on  $J^1(R^n, M)$ , the vector fields:  $X = X^i \frac{\partial}{\partial x^i} + X^a \frac{\partial}{\partial u^a} + X_i^a \frac{\partial}{\partial u_i^a}$  and  $X_h = X_h^i \frac{\partial}{\partial x^i} + X_h^a \frac{\partial}{\partial u^a} + X_h^a \frac{\partial}{\partial u_i^a}$ , where  $h = \overline{1, n}$ . To the form  $\Omega$  we associate the function:  $\Omega(X, \underset{1}{X}, \dots, \underset{n}{X})$ .

**Definition 2.** A field  $X$  which satisfies the condition  $\Omega(X, \underset{1}{X}, \dots, \underset{n}{X}) = 0$ ,  $\forall X_h$ ,  $h = \overline{1, n}$ , is called characteristic field associated to the form  $\Omega$ .

By the definition, it follows that the  $n$ -form  $i_X \Omega$ , called *characteristic form*, lead us to the equation  $i_X \Omega = 0$ . We say that  $X$  is a solution of the equation  $\Omega(X, \underset{1}{X}, \dots, \underset{n}{X}) = 0$ , if  $\Omega(X, \underset{1}{X}, \dots, \underset{n}{X}) = 0$ ,  $\forall X_h$ . A necessary condition such that this property to hold is that the following relations to be satisfied:

$$(1.4) \quad \Psi_a = i_{\frac{\partial}{\partial u^a}} \Omega = 0, \quad \Psi_a^i = i_{\frac{\partial}{\partial u_i^a}} \Omega = 0, \quad \forall i = \overline{1, n}, \quad \forall a = \overline{1, m}.$$

Moreover, the relations:

$$(1.4') \quad \Psi^i = i \frac{\partial}{\partial x^i} \Omega = 0, \quad \forall i = \overline{1, n},$$

hold.

The system (1.4) is build up by  $m(n+1)$  conditions and to it belong the  $n$  not essential conditions (1.4'). These relations represent a system of  $m+n+mn$  equations with the same number of unknown differentials:

$$(1.4'') \quad \begin{aligned} \Psi_a &= i \frac{\partial}{\partial u^a} \Omega = B_{ab}^h du^b \wedge \omega_h - A_{ba}^{jh} du_j^b \wedge \omega_h + E_a \omega_0 = 0, \\ \Psi_a^i &= i \frac{\partial}{\partial u^a} \Omega = A_{ab}^{ih} du^b \wedge \omega_h + Q_{ab}^{ijh} du_j^b \wedge \omega_h - P_a^i \omega_0 = 0, \\ \Psi^i &= i \frac{\partial}{\partial x^i} \Omega = -(E_a du^a - P_a^j du_j^a) \wedge \omega_i = 0. \end{aligned}$$

**Definition 3.** We call characteristic manifold a submanifold given by the functions  $u^a = u^a(x^i)$ , if they, together with their partial derivatives, transform the characteristic equations (1.4'') in identities.

The characteristic manifolds lead us to the equations:

$$(1.5) \quad \begin{aligned} B_{ab}^h u_h^b - A_{ba}^{jh} u_{jh}^b + E_a &= 0, \\ A_{ab}^{ih} u_h^b + Q_{ab}^{ijh} u_{jh}^b - P_a^i &= 0, \\ -E_b u_h^b + P_b^j u_{jh}^b &= 0, \end{aligned}$$

also called characteristic equations.

The last set of the equations (1.5<sub>3</sub>) are linear combinations of the equations (1.5<sub>1</sub>) and (1.5<sub>2</sub>). It follows that, on the jet space, the characteristic manifolds are of dimension  $n$ .

**Proposition 4.** To any Lagrange form  $\Omega$  we can (canonically) associate a system of second order partial differential equations so that its solutions are the characteristic manifolds of the form  $\Omega$ .

### 1.3 The Maxwell's principle

The Maxwell's principle is the property of the form  $\Omega$  to be closed ([6]). By  $d\Omega = 0$ , we obtain the *Maxwell equations*:

$$(1.6) \quad \begin{aligned} \sum_{(a,b,c)} \frac{\partial B_{ab}^h}{\partial u^c} &= 0, \quad \sum_{(a,b,c),(i,j,k)} \frac{\partial Q_{ab}^{ijh}}{\partial u_k^c} = 0, \quad \frac{\partial B_{ab}^h}{\partial x^h} + \frac{\partial E_b}{\partial u^a} - \frac{\partial E_a}{\partial u^b} = 0, \\ \frac{\partial Q_{ab}^{ijh}}{\partial x^h} + \frac{\partial P_a^i}{\partial u_j^b} - \frac{\partial P_b^j}{\partial u_i^a} &= 0, \quad \frac{\partial A_{ab}^{ih}}{\partial x^h} + \frac{\partial E_b}{\partial u_i^a} + \frac{\partial P_a^i}{\partial u^b} = 0, \\ \frac{\partial B_{ab}^i}{\partial u_j^c} + \frac{\partial A_{ca}^{ij}}{\partial u^b} - \frac{\partial A_{cb}^{ij}}{\partial u^a} &= 0, \quad \frac{\partial Q_{ab}^{ijh}}{\partial u^c} + \frac{\partial A_{bc}^{jh}}{\partial u_i^a} - \frac{\partial A_{ac}^{ih}}{\partial u_j^b} = 0. \end{aligned}$$

### 1.4 The potential form

Given a Lagrange form (1.1) satisfying the Maxwell's principle, locally there exists an  $n$ -form  $\xi$ , called *potential form*, such that  $d\xi = \Omega$ . Let us denote:

$$\xi = A_a^h du^a \wedge \omega_h + C_a^{ih} du_i^a \wedge \omega_h - V \omega_0.$$

Its exterior differential is:

$$\begin{aligned} d\xi = & \left( \frac{\partial A_b^h}{\partial u_i^a} - \frac{\partial C_a^{ih}}{\partial u^b} \right) du_i^a \wedge du^b \wedge \omega_h + \frac{1}{2} \left( \frac{\partial A_b^h}{\partial u^a} - \frac{\partial A_a^h}{\partial u^b} \right) du^a \wedge du^b \wedge \omega_h + \\ & + \frac{1}{2} \left( \frac{\partial C_b^{jh}}{\partial u_i^a} - \frac{\partial C_a^{ih}}{\partial u_j^b} \right) du_i^a \wedge du_j^b \wedge \omega_h - \\ & - \left[ \left( \frac{\partial A_a^i}{\partial x^i} + \frac{\partial V}{\partial u^a} \right) du^a + \left( \frac{\partial C_a^{ij}}{\partial x^j} + \frac{\partial V}{\partial u_i^a} \right) du_i^a \right] \wedge \omega_0. \end{aligned}$$

By identifying the coefficients of  $d\xi$  with those of  $\Omega$ , it follows that:

$$(1.7) \quad \begin{aligned} A_{ab}^{ih} &= \frac{\partial A_b^h}{\partial u_i^a} - \frac{\partial C_a^{ih}}{\partial u^b}, \quad B_{ab}^i = \frac{\partial A_b^i}{\partial u^a} - \frac{\partial A_a^i}{\partial u^b}, \\ Q_{ab}^{ijh} &= \frac{\partial C_b^{jh}}{\partial u_i^a} - \frac{\partial C_a^{ih}}{\partial u_j^b}, \quad E_a = -\frac{\partial A_a^i}{\partial x^i} - \frac{\partial V}{\partial u^a}, \quad P_a^i = \frac{\partial C_a^{ij}}{\partial x^j} + \frac{\partial V}{\partial u_i^a}. \end{aligned}$$

The potential form is not unique, it is defined via the addition of a form  $d\Phi^i \wedge \omega_i$ , where  $\Phi^i$  are arbitrary functions. We have:

$$d\Phi^i \wedge \omega_i = \left( \frac{\partial \Phi^i}{\partial u^a} du^a + \frac{\partial \Phi^i}{\partial u_j^a} du_j^a \right) \wedge \omega_i + \frac{\partial \Phi^i}{\partial x^i} \omega_0.$$

The functions  $\Phi^i$  can be chosen so that  $\frac{\partial \Phi^i}{\partial x^i} = V$ . In this case the coefficients of the superfield are expressed by the normed potential:

$$(1.7') \quad \begin{aligned} A_{ab}^{ih} &= \frac{\partial A_b^h}{\partial u_i^a} - \frac{\partial C_a^{ih}}{\partial u^b}, \quad B_{ab}^i = \frac{\partial A_b^i}{\partial u^a} - \frac{\partial A_a^i}{\partial u^b}, \\ Q_{ab}^{ijh} &= \frac{\partial C_b^{jh}}{\partial u_i^a} - \frac{\partial C_a^{ih}}{\partial u_j^b}, \quad E_a = -\frac{\partial A_a^i}{\partial x^i}, \quad P_a^i = \frac{\partial C_a^{ij}}{\partial x^j}. \end{aligned}$$

### 1.5 The reduced Lagrange form

**Definition 5.** A Lagrange form  $\Omega$  is called reduced Lagrange form if (and only if) all its coefficients  $Q_{ab}^{ijh}$  are equal to zero,  $\forall(x^i)$ .

Such a form is written as follows:

$$(1.8) \quad \begin{aligned} \Omega = & A_{ab}^{ih} du_i^a \wedge du^b \wedge \omega_h + \frac{1}{2} B_{ab}^h du^a \wedge du^b \wedge \omega_h + \\ & + (E_a du^a - P_a^i du_i^a) \wedge \omega_0. \end{aligned}$$

Its characteristic equations:

$$(1.9) \quad \begin{aligned} \Psi_a &= B_{ab}^h du^b \wedge \omega_h - A_{ba}^{jh} du_j^b \wedge \omega_h + E_a \omega_0 = 0, \\ \Psi_a^i &= A_{ab}^{ih} du^b \wedge \omega_h - P_a^i \omega_0 = 0, \\ \Psi^i &= -(E_a du^a - P_a^j du_j^a) \wedge \omega_i = 0, \end{aligned}$$

lead us to the partial differential equations:

$$(1.10) \quad \begin{aligned} -A_{ba}^{ij} u_{ij}^b + B_{ab}^i u_i^b + E_a &= 0, \\ A_{ab}^{ij} u_j^b - P_a^i &= 0, \\ -E_a u_i^a + P_a^j u_j^a &= 0. \end{aligned}$$

The Maxwell's principle is expressed by:

$$(1.11) \quad \begin{aligned} \sum_{(a,b,c)} \frac{\partial B_{ab}^h}{\partial u^c} &= 0, \quad \frac{\partial B_{ab}^h}{\partial x^h} + \frac{\partial E_b}{\partial u^a} - \frac{\partial E_a}{\partial u^b} = 0, \\ \frac{\partial P_a^i}{\partial u_j^b} - \frac{\partial P_b^j}{\partial u_i^a} &= 0, \quad \frac{\partial A_{ab}^{ih}}{\partial x^h} + \frac{\partial E_b}{\partial u_i^a} + \frac{\partial P_a^i}{\partial u^b} = 0, \\ \frac{\partial B_{ab}^i}{\partial u_j^c} + \frac{\partial A_{ca}^{ij}}{\partial u^b} - \frac{\partial A_{cb}^{ij}}{\partial u^a} &= 0, \quad \frac{\partial A_{bc}^{jh}}{\partial u_i^a} - \frac{\partial A_{ac}^{jh}}{\partial u_j^b} = 0. \end{aligned}$$

## 2 Lagrange forms associated to systems of partial differential equations written in main form. The construction of them

We will define and build general Lagrange  $(n+1)$ -forms associated to the evolution of dynamical systems of deformable continuous media, described by equations written in main form:

$$(2.1) \quad F_a = A_{ba}^{ij} u_{ij}^b + B_a = 0, \quad a = \overline{1, m}, \quad (A_{ab}^{ij} = A_{ab}^{ji}), \quad \det(A_{ab}^{ij}) \neq 0, \quad \forall i, j = \overline{1, n},$$

where  $F : J^2(R^n, M) \rightarrow T^*M$ .

On  $R^n$  we consider the canonical chart and the  $n$ -form  $\omega_0 = \omega dx^1 \wedge \dots \wedge dx^n$  as volume form. The relations (0.2) hold.

A *solution* of the equations (2.1) is a set of functions  $u^a = u^a(x^h)$ , which, together with their partial derivatives  $u_i^a = \frac{\partial u^a}{\partial x^i}$ ,  $u_{ij}^a = \frac{\partial^2 u^a}{\partial x^i \partial x^j}$ , transform the equations (2.1) in identities.

We multiply the left-hand sides of (2.1) with  $\omega_0$  and we obtain, along the solutions of (2.1), the forms:  $F_a \omega_0 = (A_{ba}^{ij} u_{ij}^b + B_a) \omega_0$ ,  $(A_{ab}^{ij} = A_{ab}^{ji})$ . By the definitions of  $\omega_0$  and  $\omega_i$ , we have the  $n$ -forms:

$$(2.2) \quad \psi_a = A_{ba}^{jh} du_j^b \wedge \omega_h + B_a \omega_0.$$

We denote the contact forms by:  $\theta^a = du^a - u_i^a dx^i$ ,  $\theta_i^a = du_i^a - u_{ij}^a dx^j$ .

## 2.1 The simple Lagrange form

To the system (2.1) we associate a *simple Lagrange*  $(n+1)$ -form:  $\Omega = -\theta^a \wedge \psi_a$ , which can be also written as:

$$(2.3) \quad \Omega = A_{ab}^{ih} du_i^a \wedge du^b \wedge \omega_h + (E_a du^a - P_a^i du_i^a) \wedge \omega_0,$$

where  $E_a = -B_a$  (the Lorentz force),  $P_a^i = A_{ab}^{ij} u_j^b$  (the impulse of the system). These relations generalize the Lorentz conditions.

In general, the form  $\Omega$  is not closed. Let us assume that it is closed (the system satisfies the Maxwell's principle). This happens if and only if the *Maxwell conditions*:

$$(2.4) \quad \begin{aligned} \frac{\partial A_{ab}^{ih}}{\partial x^h} + \frac{\partial E_b}{\partial u_i^a} + \frac{\partial P_a^i}{\partial u^b} = 0, \quad \frac{\partial E_b}{\partial u^a} - \frac{\partial E_a}{\partial u^b} = 0, \quad \frac{\partial P_a^i}{\partial u_i^b} - \frac{\partial P_b^j}{\partial u_j^a} = 0, \\ \frac{\partial A_{ca}^{ij}}{\partial u^b} - \frac{\partial A_{cb}^{ij}}{\partial u^a} = 0, \quad \frac{\partial A_{bc}^{jh}}{\partial u_i^a} - \frac{\partial A_{ac}^{ih}}{\partial u_j^b} = 0 \end{aligned}$$

hold. By replacing the functions  $E_a$  and  $P_a^i$  with their expressions, it follows:

**Proposition 6.** *The dynamical systems for which the simple Lagrange form is closed, are selfadjoint.*

## 2.2 The reduced Lagrange form

We build the reduced Lagrange form, associated to the system (2.1), by:

$$(2.5) \quad \Omega = -\theta^a \wedge \psi_a + \frac{1}{2} B_{ab}^h \theta^a \wedge \theta^b \wedge \omega_h,$$

where the coefficients  $B_{ab}^h$  are, for the moment, not determined.

This form is written in coordinates as follows:

$$(2.5') \quad \begin{aligned} \Omega = A_{ab}^{ih} du_i^a \wedge du^b \wedge \omega_h + \frac{1}{2} B_{ab}^h du^a \wedge du^b \wedge \omega_h - \\ - [(B_a + B_{ab}^h u_h^b) du^a + A_{ab}^{ih} u_h^b du_i^a] \wedge \omega_0. \end{aligned}$$

The relations:

$$(2.6) \quad E_a = -B_a - B_{ab}^j u_j^b, \quad P_a^i = A_{ab}^{ij} u_j^b, \quad Q_{ab}^{ijh} = 0$$

follow. They are called Lorentz conditions, with the meaning that  $-B_a = E_a + B_{ab}^j u_j^b$  is the Lorentz superforce and  $P_a^i = A_{ab}^{ij} u_j^b$  represents the superimpulse.

The Maxwell's principle, expressed by the condition  $d\Omega = 0$ , lead us to the system of equations (1.11), called *Maxwell-Helmholtz equations*.

**Theorem 7.** *The necessary condition for a dynamical system to be selfadjoint is that the reduced Lagrange form, associated to it, to be closed.*

In this case, the Maxwell equations become the Helmholtz equations.

*Proof.* By the Lorentz conditions (2.6) and the Maxwell equations (1.11), it follows the relations:

$$B_{ab}^h = \frac{1}{2} \left( \frac{\partial B_b}{\partial u_h^a} - \frac{\partial B_a}{\partial u_h^b} \right).$$

They define the functions  $B_{ab}^h$ .

If we remove the functions  $E_a$ ,  $P_a^i$  and  $B_{ab}^i$  from the equations (1.11), we obtain:

$$(2.7) \quad \begin{aligned} A_{ab}^{ij} &= A_{ba}^{ij}, \quad \frac{\partial A_{ac}^{ih}}{\partial u_j^b} - \frac{\partial A_{bc}^{jh}}{\partial u_i^a} = 0, \\ \frac{\partial B_a}{\partial u_i^b} + \frac{\partial B_b}{\partial u_i^a} &= 2 \left( \frac{\partial}{\partial x^h} + u_h^c \frac{\partial}{\partial u^c} \right) A_{ab}^{ih}, \\ \frac{\partial B_a}{\partial u^b} - \frac{\partial B_b}{\partial u^a} &= \frac{1}{2} \left( \frac{\partial}{\partial x^h} + u_h^c \frac{\partial}{\partial u^c} \right) \left( \frac{\partial B_a}{\partial u_h^b} - \frac{\partial B_b}{\partial u_h^a} \right), \end{aligned}$$

which are the (generalized) Helmholtz equations and it follows that the system (2.1) is selfadjoint.

Given the coefficients  $B_{ab}^i$ ,  $A_{ab}^{ij}$  and  $B_a$ , the coefficients  $E_a$  and  $P_a^i$  follow.

A necessary condition so that the Lagrange form to be closed is that the relation:

$$\frac{\partial B_a}{\partial u_h^b} + \frac{\partial B_b}{\partial u_h^c} + \frac{\partial B_c}{\partial u_h^a} = \frac{\partial B_a}{\partial u_h^c} + \frac{\partial B_b}{\partial u_h^a} + \frac{\partial B_c}{\partial u_h^b} = 0$$

to hold ( $rot_v B = 0$ ).  $\square$

### 2.3 The special Lagrange form

Let assume that we have a system of partial differential equations given in the special form:

$$(2.8) \quad F_a = A_{ba}^{ij} u_{ij}^b + \tilde{B}_{ba}^i u_i^b + C_a = 0, \quad (A_{ab}^{ij} = A_{ab}^{ji}).$$

Such a system is written in the main form (2.1), where:  $B_a = \tilde{B}_{ba}^i u_i^b + C_a$ .

Multiplying the left-hand sides from (2.8) with  $\omega_0$ , we obtain the forms:

$$\psi_a = A_{ba}^{jh} du_j^b \wedge \omega_h + \tilde{B}_{ba}^h du^b \wedge \omega_h + C_a \omega_0.$$

We build the Lagrange form by  $\Omega = -\theta^a \wedge \psi_a$  and it is written as follows:

$$(2.9) \quad \begin{aligned} \Omega &= A_{ab}^{ih} du_i^a \wedge du^b \wedge \omega_h + \frac{1}{2} B_{ab}^h du^a \wedge du^b \wedge \omega_h - \\ &\quad - [(C_a + \tilde{B}_{ab}^h u_h^b) du^a + A_{ab}^{ih} u_h^b du_i^a] \wedge \omega_0, \end{aligned}$$

where:  $B_{ab}^h = \tilde{B}_{ab}^h - \tilde{B}_{ba}^h$ ,  $Q_{ab}^{ijh} = 0$ , and the Lorentz conditions are:

$$-C_a = E_a + \tilde{B}_{ab}^j u_j^b, \quad P_a^i = A_{ab}^{ij} u_j^b.$$

The Maxwell equations are given by (1.11) with the above Lorentz conditions.

## 2.4 The Lagrange form associated to Lagrangian dynamical systems

The evolution of a Lagrangian dynamical system, of deformable continuous media, is described by the Euler-Lagrange equations:

$$(2.10) \quad \frac{\partial^2 L}{\partial u_i^a \partial u_j^b} u_{ji}^b + \frac{\partial^2 L}{\partial u_i^a \partial u^b} u_i^b + \frac{\partial^2 L}{\partial u_i^a \partial x^i} - \frac{\partial L}{\partial u^a} = 0, \quad a = \overline{1, m}.$$

Such a system is written in the special form (2.8) and, thus, we associate to it the Lagrange form (2.9).

The Euler-Lagrange equations are written in the main form (2.1), where:

$$A_{ab}^{ij} = \frac{1}{2} \left( \frac{\partial^2 L}{\partial u_i^a \partial u_j^b} + \frac{\partial^2 L}{\partial u_j^a \partial u_i^b} \right), \quad B_a = \frac{\partial^2 L}{\partial u_i^a \partial u^b} u_i^b + \frac{\partial^2 L}{\partial u_i^a \partial x^i} - \frac{\partial L}{\partial u^a}.$$

We multiply with  $\omega_0$  the left-hand sides of the equations (2.10) and we obtain the forms:

$$\psi_a = \frac{\partial^2 L}{\partial u_j^a \partial u_h^b} du_j^b \wedge \omega_h + \frac{\partial^2 L}{\partial u_h^a \partial u^b} du^b \wedge \omega_h + \left( \frac{\partial^2 L}{\partial u_i^a \partial x^i} - \frac{\partial L}{\partial u^a} \right) \omega_0.$$

The Lagrange form is defined by  $\Omega = -\theta^a \wedge \psi_a$  and locally it is written as follows:

$$\begin{aligned} \Omega = & A_{ab}^{ih} du_i^a \wedge du^b \wedge \omega_h + \frac{1}{2} \left( \frac{\partial^2 L}{\partial u_h^b \partial u^a} - \frac{\partial^2 L}{\partial u_h^a \partial u^b} \right) du^a \wedge du^b \wedge \omega_h + \\ & + (E_a du^a - P_a^i du_i^a) \wedge \omega_0, \end{aligned}$$

which is the form (2.9), where

$$(2.11) \quad \begin{aligned} A_{ab}^{ij} &= \frac{1}{2} \left( \frac{\partial^2 L}{\partial u_i^a \partial u_j^b} + \frac{\partial^2 L}{\partial u_j^a \partial u_i^b} \right), \quad \tilde{B}_{ab}^i = \frac{\partial^2 L}{\partial u_i^a \partial u^b}, \\ B_{ab}^i &= \frac{\partial^2 L}{\partial u_i^b \partial u^a} - \frac{\partial^2 L}{\partial u_i^a \partial u^b}, \quad C_a = \frac{\partial^2 L}{\partial u_i^a \partial x^i} - \frac{\partial L}{\partial u^a}, \\ E_a &= \frac{\partial L}{\partial u^a} - \frac{\partial^2 L}{\partial u_i^a \partial x^i} - \frac{\partial^2 L}{\partial u_i^b \partial u^a} u_i^b, \quad P_a^i = A_{ab}^{ij} u_j^b. \end{aligned}$$

The following properties hold:

1°  $A_{ab}^{ij} = A_{ab}^{ji} = A_{ba}^{ij}$ .

2° The set of coefficients of the Lagrange form can be arranged in three subsets:  $A_{ab}^{ij}$ , which give the geometrical part,  $P_a^i$  representing the kinematical part, and  $E_a$  and  $B_{ab}^i$ , which form the dynamical part.

3° By the main form we deduce that  $\mathcal{F}_a^N = -B_a$  represents the Newtonian force associated to the system.

4° The expression  $\mathcal{F}_a^L = E_a + B_{ab}^i u_i^b$  represents the Lorentz force associated to the superfield.

5° The relation:

$$-B_a = E_a + B_{ab}^i u_i^b = \frac{\partial L}{\partial u^a} - \frac{\partial^2 L}{\partial u_i^a \partial x^i} - \frac{\partial^2 L}{\partial u_i^a \partial u^b} u_i^b,$$

certifies the identity of the two forces.

In general, the Lagrange form associated to a Lagrangian system is not closed. We have:

**Proposition 8.** *The Lagrange form associated to a Lagrangian system is closed if and only if the system is selfadjoint.*

Indeed, the Maxwell equations written for the coefficients given by the relations (2.11), are satisfied if and only if

$$\frac{\partial^2 L}{\partial u_i^a \partial u_j^b} = \frac{\partial^2 L}{\partial u_j^a \partial u_i^b}.$$

In this case the system is selfadjoint and conversely.

## 2.5 The construction of the general Lagrange form

Let consider a system of equations in main form:  $A_{ba}^{ij} u_i^b + B_a = 0$ , ( $A_{ab}^{ij} = A_{ab}^{ji}$ ), describing the evolution of a dynamical system of a deformable continuous media. In order to define the Lagrange  $(n + 1)$ -form, we consider the following auxiliary forms:  $\theta^a = du^a - u_i^a dx^i$ ,  $\theta_i^a = du_i^a - u_{ij}^a dx^j$ , called contact forms, and also the canonical volume form  $\omega_0$  on  $R^n$  and the forms  $\omega_i$ .

We multiply the left-hand sides of (2.1) with  $\omega_0$  and, by the definitions of  $\omega_0$ ,  $\omega_i$  and by the relations  $dx^i \wedge \omega_j = \delta_j^i \omega_0$ , we obtain along the solutions of (2.1), the forms (2.2):  $\psi_a = A_{ba}^{ij} du_j^b \wedge \omega_i + B_a \omega_0$ .

The general Lagrange form associated to the system (2.1) is defined by

$$(2.12) \quad \Omega = -\theta^a \wedge \psi_a + \frac{1}{2} B_{ab}^i \theta^a \wedge \theta^b \wedge \omega_i + \frac{1}{2} Q_{ab}^{ijh} \theta_i^a \wedge \theta_j^b \wedge \omega_h,$$

where the functions  $B_{ab}^h$ ,  $Q_{ab}^{ijh}$  are, for the moment, not determined. The characteristics of  $\Omega$  are given by the equations  $\theta^a = 0$ ,  $\theta_i^a = 0$  and  $\psi_a = 0$ .

Developed, this form is written as follows:

$$(2.12') \quad \Omega = A_{ab}^{ih} du_i^a \wedge du^b \wedge \omega_h + \frac{1}{2} B_{ab}^h du^a \wedge du^b \wedge \omega_h + \frac{1}{2} Q_{ab}^{ijh} du_i^a \wedge du_j^b \wedge \omega_h \\ - [(B_a + B_{ab}^j u_j^b) du^a + (A_{ab}^{ij} u_j^b + Q_{ab}^{ijh} u_{jh}^b) du_i^a] \wedge \omega_0.$$

It is of the general form (1.1), where the relations:

$$(2.13) \quad E_a = -B_a - B_{ab}^j u_j^b, \\ P_a^i = A_{ab}^{ij} u_j^b + Q_{ab}^{ijh} u_{jh}^b$$

hold. We call them, by generalization, *Lorentz conditions*.

The functions  $-B_a = E_a + B_{ab}^i u_i^b$  generalize the Lorentz forces and  $P_a^i = A_{ab}^{ij} u_j^b + Q_{ab}^{ijh} u_{jh}^b$  represent generalized impulses.

The characteristic manifolds are described by the equations (1.5), where the last set of relations are linear combinations of the others.

The Maxwell's principle ask that the form  $\Omega$  to be closed. This condition is expressed by the system of equations (1.6).

In order to integrate the system (1.6), we remove from its equations, by the Lorentz conditions, the functions  $E_a$  and  $P_a^i$ . By (1.6<sub>3</sub>) and (2.13<sub>1</sub>) it follows:

$$(2.14) \quad \left( \frac{\partial}{\partial x^h} + u_h^c \frac{\partial}{\partial u^c} \right) B_{ab}^h + \frac{\partial B_a}{\partial u^b} - \frac{\partial B_b}{\partial u^a} = 0.$$

By (1.6<sub>4</sub>) and (2.13<sub>2</sub>) it follows:

$$\left( \frac{\partial}{\partial x^h} + u_h^c \frac{\partial}{\partial u^c} + u_{hk}^c \frac{\partial}{\partial u_k^c} \right) Q_{ab}^{ijh} + A_{ab}^{ij} - A_{ba}^{ij} = 0.$$

These quasi-linear partial differential equations determine the functions  $B_{ab}^i$  and  $Q_{ab}^{ijh}$  by the coefficients of the equations of the given dynamical system. We reintroduce these functions into the Lorentz conditions (2.13), and we obtain all the coefficients of the superfield and, thus, the general Lagrange form.

Moreover, by (1.6<sub>5</sub>), (2.13) and (1.6<sub>6</sub>) it follows the condition:

$$\left( \frac{\partial}{\partial x^h} + u_h^c \frac{\partial}{\partial u^c} \right) A_{ab}^{ih} - \frac{\partial B_b}{\partial u_i^a} - B_{ba}^i + \frac{\partial Q_{ac}^{ihk}}{\partial u^b} u_{hk}^c = 0,$$

which, by the last condition of (1.6), becomes:

$$(2.15) \quad \left( \frac{\partial}{\partial x^h} + u_h^c \frac{\partial}{\partial u^c} + u_{hk}^c \frac{\partial}{\partial u_k^c} \right) A_{ab}^{ih} - u_{jh}^c \frac{\partial A_{cb}^{jh}}{\partial u_i^a} - \frac{\partial B_b}{\partial u_i^a} - B_{ba}^i = 0.$$

Let assume that the system is selfadjoint. In this case, the coefficients  $Q_{ab}^{ijh} \equiv 0$  and  $A_{ab}^{ij}$  are symmetric. We rewrite the relation (2.15), interchanging the indices  $a$  and  $b$ , and by adding and subtracting respectively these relations, we obtain:

$$(2.16) \quad B_{ab}^i = \frac{1}{2} \left( \frac{\partial B_b}{\partial u_i^a} - \frac{\partial B_a}{\partial u_i^b} \right),$$

$$(2.17) \quad \frac{\partial B_a}{\partial u_i^b} + \frac{\partial B_b}{\partial u_i^a} = 2 \left( \frac{\partial}{\partial x^h} + u_h^c \frac{\partial}{\partial u^c} \right) A_{ab}^{ih}.$$

By (2.14) and (2.16) it follows:

$$(2.18) \quad \frac{\partial B_a}{\partial u^b} - \frac{\partial B_b}{\partial u^a} = \frac{1}{2} \left( \frac{\partial}{\partial x^h} + u_h^c \frac{\partial}{\partial u^c} \right) \left( \frac{\partial B_a}{\partial u_h^b} - \frac{\partial B_b}{\partial u_h^a} \right).$$

Thus, we proved again the proposition that the necessary and sufficient condition so that a system to be selfadjoint is that its Lagrange form to be reduced and closed. In this case the Maxwell equations are reduced to the Helmholtz equations. Related results can be found in [2, 3].

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