Complete hypersurfaces in a hyperbolic space

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Abstract. In this paper, we characterize the n-dimensional ($n \geq 3$) complete hypersurfaces M^n in a hyperbolic space $\mathbb{H}^{n+1}(-1)$ with constant scalar curvature and with two distinct principal curvatures one of which is simple and the other λ has no zero points. We show that M^n is a locus of moving $(n-1)$-dimensional submanifold $M_1^{n-1}(s)$, along $M_1^{n-1}(s)$ the principal curvature λ of multiplicity $n-1$ is constant and $M_1^{n-1}(s)$ is umbilical in $\mathbb{H}^{n+1}(-1)$ and is contained in an $(n-1)$-dimensional sphere $S^{n-1}(c(s)) = E^n(s) \cap \mathbb{H}^{n+1}(-1)$ and is of constant curvature $\{\frac{d\{\log|\lambda^2-(R+1)|^{1/n}\}}{ds}\}^2 + \lambda^2 - 1$, where s is the arc length of an orthogonal trajectory of the family $M_1^{n-1}(s)$, $E^n(s)$ is an n-dimensional linear subspace in $\mathbb{R}^{n+2}$ which is parallel to a fixed $E^n(s_0)$ and $u = |\lambda^2 - (R+1)|^{-\frac{1}{n}}$ satisfies the ordinary differential equation of order 2, $\frac{d^2u}{ds^2} - u(\pm\frac{n-2}{2}\frac{1}{u^n} - R) = 0$.

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1 Introduction

Let $\mathbb{R}^{n+1}(c)$ be an $(n+1)$-dimensional connected Riemannian mamifold with constant sectional curvature c. According to $c > 0, c = 0$ and $c < 0$, it is called sphere space, Euclidean space or hyperbolic space, respectively, and it is denoted by $\mathbb{S}^{n+1}(c)$, $\mathbb{R}^{n+1}$ or $\mathbb{H}^{n+1}(c)$. As it is well known there are many rigidity results for hypersurfaces with constant mean curvature or with constant scalar curvature in $\mathbb{S}^{n+1}(c)$ or $\mathbb{R}^{n+1}$, for example, see[1], [2], [3], [4], [5] and [7] etc., but less are obtained for hypersurfaces immersed into a hyperbolic space. To our best knowledge, there are almost no intrinsic rigidity results for the hypersurfaces in a hyperbolic space until Morvan-Wu [6], Wu [9] proved some rigidity theorems for complete hypersurfaces M^n in a hyperbolic space $\mathbb{H}^{n+1}(c)$ under the assumption that the mean curvature is constant and the Ricci curvature is non-negative. In [8], we investigated the complete hypersurfaces M^n in a hyperbolic space $\mathbb{H}^{n+1}(-1)$ with constant scalar curvature and with two distinct principal curvatures whose multiplicities are greater than 1. We showed that

Theorem 1.1 ([8]). *Let M^n be an n-dimensional complete hypersurface in a hyperbolic space $\mathbb{H}^{n+1}(-1)$ with constant scalar curvature $n(n-1)R$ and with*

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two distinct principal curvatures. If the multiplicities of these two distinct principal curvatures are greater than 1, then M^n is isometric to the Riemannian product $S^k(r) \times H^{n-k}(-1/(r^2+1))$, for some $r > 0$.

As we know that Otsuki [7] characterized the minimal hypersurfaces in a Riemannian manifold $\bar{M}$ of constant curvature $\bar{c}$ with two distinct principal curvatures one of which is simple and Cheng [3] investigated the n-dimensional oriented complete hypersurfaces ($n \geq 3$) in Euclidean space $\mathbb{R}^{n+1}$ with constant scalar curvature and with two distinct principal curvatures one of which is simple. It is natural and important to investigate the complete hypersurfaces M^n in a hyperbolic space $\mathbb{H}^{n+1}(-1)$ with constant scalar curvature and with two distinct principal curvatures one of which is simple. In this paper, we obtain the following:

Theorem 1.2. *Let M^n be an n-dimensional ($n \geq 3$) complete hypersurface in a hyperbolic space $\mathbb{H}^{n+1}(-1)$ with constant scalar curvature $n(n-1)R$ and with two distinct principal curvatures one of which is simple and the other λ has no zero points, then M^n is a locus of moving $(n-1)$-dimensional submanifold $M_1^{n-1}(s)$, along $M_1^{n-1}(s)$ the principal curvature λ of multiplicity $n-1$ is constant and $M_1^{n-1}(s)$ is umbilical in $\mathbb{H}^{n+1}(-1)$ and is contained in an $(n-1)$-dimensional sphere $S^{n-1}(c(s)) = E^n(s) \cap \mathbb{H}^{n+1}(-1)$ and is of constant curvature $\{\frac{d\{\log|\lambda^2-(R+1)|^{1/n}\}}{ds}\}^2 + \lambda^2 - 1$, where s is the arc length of an orthogonal trajectory of the family $M_1^{n-1}(s)$, $E^n(s)$ is an n-dimensional linear subspace in $\mathbb{R}^{n+2}$ which is parallel to a fixed $E^n(s_0)$ and $u = |\lambda^2 - (R+1)|^{-\frac{1}{n}}$ satisfies the ordinary differential equation of order 2*

$$\frac{d^2u}{ds^2} - u(\pm\frac{n-2}{2}\frac{1}{u^n} - R) = 0.$$

2 Preliminaries

Let M^n be an n-dimensional hypersurface in $\mathbb{H}^{n+1}(-1)$. We choose a local orthonormal frame $e_1, \cdots, e_{n+1}$ in $\mathbb{H}^{n+1}(-1)$ such that $e_1, \cdots, e_n$ are tangent to M^n. Let $\omega_1, \cdots, \omega_{n+1}$ be the dual coframe. We use the following convention on the range of indices:

$$1 \leq A, B, C, \cdots \leq n+1; \quad 1 \leq i, j, k, \cdots \leq n.$$

The structure equations of $\mathbb{H}^{n+1}(-1)$ are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}, \tag{2.2}$$

where

$$\Omega_{AB} = -\frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D, \tag{2.3}$$

$$K_{ABCD} = -(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.4}$$

Restricting to M^n,

$$\omega_{n+1} = 0. \tag{2.5}$$

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.6}$$

The structure equations of M^n are

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{2.7}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.8}$$

$$R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.9}$$

$$R_{ij} = -(n-1)\delta_{ij} + nHh_{ij} - \sum_k h_{ik}h_{kj}, \tag{2.10}$$

$$n(n-1)(R+1) = n^2H^2 - |h|^2, \tag{2.11}$$

where $n(n-1)R$ is the scalar curvature, H is the mean curvature and $|h|^2$ is the squared norm of the second fundamental form of M^n.

The Codazzi equation and the Ricci identity are

$$h_{ijk} = h_{ikj}, \tag{2.12}$$

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}, \tag{2.13}$$

where h_{ijk} and h_{ijkl} denote the first and the second covariant derivatives of h_{ij}.

We choose $e_1, \cdots, e_n$ such that $h_{ij} = \lambda_i\delta_{ij}$. From (2.6), we have

$$\omega_{n+1i} = \lambda_i\omega_i, \quad i = 1, 2, \cdots, n. \tag{2.14}$$

Hence, we have from the structure equations of M^n

$$d\omega_{n+1i} = d\lambda_i \wedge \omega_i + \lambda_i d\omega_i = d\lambda_i \wedge \omega_i + \lambda_i \sum_j \omega_{ij} \wedge \omega_j. \tag{2.15}$$

On the other hand, we have on the curvature forms of $\mathbb{H}^{n+1}(-1)$,

$$\begin{aligned} \Omega_{n+1i} =& -\frac{1}{2}\sum_{C,D} K_{n+1iCD}\omega_C \wedge \omega_D \\ =& \frac{1}{2}\sum_{C,D}(\delta_{n+1C}\delta_{iD} - \delta_{n+1D}\delta_{iC})\omega_C \wedge \omega_D \\ =& \omega_{n+1} \wedge \omega_i = 0. \end{aligned} \tag{2.16}$$

Therefore, from the structure equations of $\mathbb{H}^{n+1}(-1)$, we have

$$d\omega_{n+1i} = \sum_j \omega_{n+1j} \wedge \omega_{ji} + \omega_{n+1n+1} \wedge \omega_{n+1i} + \Omega_{n+1i} = \sum_j \lambda_j \omega_{ij} \wedge \omega_j. \tag{2.17}$$

From (2.15) and (2.17), we obtain

$$d\lambda_i \wedge \omega_i + \sum_j (\lambda_i - \lambda_j)\omega_{ij} \wedge \omega_j = 0. \tag{2.18}$$

Putting

$$\psi_{ij} = (\lambda_i - \lambda_j)\omega_{ij}. \tag{2.19}$$

Then $\psi_{ij} = \psi_{ji}$, (2.18) can be written as

$$\sum_j (\psi_{ij} + \delta_{ij} d\lambda_j) \wedge \omega_j = 0. \tag{2.20}$$

By E. Cartan's Lemma, we get

$$\psi_{ij} + \delta_{ij} d\lambda_j = \sum_k Q_{ijk}\omega_k, \tag{2.21}$$

where Q_{ijk} are uniquely determined functions such that

$$Q_{ijk} = Q_{ikj}. \tag{2.22}$$

3 Proof of theorem

We firstly have the following Proposition 3.1 due to Otsuki ([7]).

Proposition 3.1 ([7]). *Let M^n be a hypersurface in a hyperbolic space $\mathbb{H}^{n+1}(-1)$ such that the multiplicities of the principal curvatures are constant. Then the distribution of the space of the principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of the principal vectors.*

Proof of theorem 1.2. Let M^n be an n-dimensional complete hypersurface with constant scalar curvature and with two distinct principal curvatures one of which is simple, that is, without lose of generality, we may assume

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda, \quad \lambda_n = \mu,$$

where λ_i for $i = 1, 2, \cdots, n$ are the principal curvatures of M^n. Since the scalar curvature $n(n-1)R$ is constant, from (2.11), we obtain

$$n(n-1)(R+1) = (n-1)(n-2)\lambda^2 + 2(n-1)\lambda\mu. \tag{3.1}$$

Since we assume $\lambda \neq 0$ on M^n, from (3.1) we have

$$\mu = \frac{n(R+1)}{2\lambda} - \frac{(n-2)\lambda}{2}. \tag{3.2}$$

Therefore, we get

$$\lambda - \mu = n\frac{\lambda^2 - (R+1)}{2\lambda} \neq 0.$$

we know $\lambda^2 - (R+1) \neq 0$.

Let $u = |\lambda^2 - (R+1)|^{-\frac{1}{n}}$. We denote the integral submanifold through $x \in M^n$ corresponding to λ by $M_1^{n-1}(x)$. Putting

$$d\lambda = \sum_{k=1}^{n} \lambda_{,k}\, \omega_k, \qquad d\mu = \sum_{k=1}^{n} \mu_{,k}\, \omega_k. \tag{3.3}$$

From Proposition 3.1, we have

$$\lambda_{,1} = \lambda_{,2} = \cdots = \lambda_{,n-1} = 0 \quad \text{on} \quad M_1^{n-1}(x). \tag{3.4}$$

From (3.2), we have

$$d\mu = [-\frac{n(R+1)}{2\lambda^2} - \frac{n-2}{2}]d\lambda. \tag{3.5}$$

Hence, we also have

$$\mu_{,1} = \mu_{,2} = \cdots = \mu_{,n-1} = 0 \quad \text{on} \quad M_1^{n-1}(x). \tag{3.6}$$

In this case, we may consider locally λ is a function of the arc length s of the integral curve of the principal vector field e_n corresponding to the principal curvature μ. From (2.21) and (3.4), we have for $1 \leq j \leq n-1$,

$$d\lambda = d\lambda_j = \sum_{k=1}^{n} Q_{jjk}\omega_k = \sum_{k=1}^{n-1} Q_{jjk}\omega_k + Q_{jjn}\omega_n = \lambda_{,n}\, \omega_n. \tag{3.7}$$

Therefore, we have

$$Q_{jjk} = 0, \quad 1 \leq k \leq n-1, \quad \text{and} \quad Q_{jjn} = \lambda_{,n}\, . \tag{3.8}$$

By (2.21) and (3.6), we have

$$\begin{aligned} d\mu =& d\lambda_n = \sum_{k=1}^{n} Q_{nnk}\omega_k \\ &= \sum_{k=1}^{n-1} Q_{nnk}\omega_k + Q_{nnn}\omega_n = \sum_{i=1}^{n} \mu_{,i}\, \omega_i = \mu_{,n}\, \omega_n. \end{aligned} \tag{3.9}$$

Hence, we obtain

$$Q_{nnk} = 0, \quad 1 \leq k \leq n-1, \quad \text{and} \quad Q_{nnn} = \mu_{,n}\, . \tag{3.10}$$

From (3.5), we get

$$Q_{nnn} = \mu_{,n} = [-\frac{n(R+1)}{2\lambda^2} - \frac{n-2}{2}]\lambda_{,n}\,. \tag{3.11}$$

From the definition of ψ_{ij}, if $i \neq j$, we have $\psi_{ij} = 0$ for $1 \leq i \leq n-1$ and $1 \leq j \leq n-1$. Therefore, from (2.21), if $i \neq j$ and $1 \leq i \leq n-1$ and $1 \leq j \leq n-1$ we have

$$Q_{ijk} = 0, \quad \text{for any} \quad k. \tag{3.12}$$

By (2.21), (3.8), (3.10), (3.11) and (3.12), we get

$$\psi_{jn} = \sum_{k=1}^{n} Q_{jnk}\omega_k = Q_{jjn}\omega_j + Q_{jnn}\omega_n = \lambda_{,n}\,\omega_j. \tag{3.13}$$

Since λ and μ are two distinct principal curvatures of M^n, we have

$$\lambda - \mu = n\frac{\lambda^2 - (R+1)}{2\lambda} \neq 0.$$

From (2.19), (3.2)and (3.13)we have

$$\begin{aligned} \omega_{jn} &= \frac{\psi_{jn}}{\lambda - \mu} = \frac{\lambda_{,n}}{\lambda - \mu}\omega_j = \frac{\lambda_{,n}}{\lambda - [\frac{n(R+1)}{2\lambda} - \frac{n-2}{2}\lambda]}\omega_j \\ &= \frac{2\lambda\lambda_{,n}}{n[\lambda^2 - (R+1)]}\omega_j. \end{aligned} \tag{3.14}$$

Therefore, from the structure equations of M^n we have

$$d\omega_n = \sum_{k=1}^{n-1} \omega_k \wedge \omega_{kn} + \omega_{nn} \wedge \omega_n = 0.$$

Therefore, we may put $\omega_n = ds$. By (3.7) and (3.9), we get

$$d\lambda = \lambda_{,n}\,ds, \qquad \lambda_{,n} = \frac{d\lambda}{ds},$$

and

$$d\mu = \mu_{,n}\,ds, \qquad \mu_{,n} = \frac{d\mu}{ds}.$$

Then we have

$$\begin{aligned} \omega_{jn} &= \frac{2\lambda\lambda_{,n}}{n[\lambda^2 - (R+1)]}\omega_j = \frac{2\lambda\frac{d\lambda}{ds}}{n[\lambda^2 - (R+1)]}\omega_j \\ &= \frac{d\{\log|\lambda^2 - (R+1)|^{\frac{1}{n}}\}}{ds}\omega_j. \end{aligned} \tag{3.15}$$

(3.15) shows that the integral submanifold $M_1^{n-1}(x)$ corresponding to λ and s is umbilical in M^n and $\mathbb{H}^{n+1}(-1)$. From (3.15) and the structure equations of $\mathbb{H}^{n+1}(-1)$, we have

$$\begin{aligned}d\omega_{jn} &= \sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_{kn}+\omega_{jn}\wedge\omega_{nn}+\omega_{jn+1}\wedge\omega_{n+1n}+\Omega_{jn}\\&=\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_{kn}+\omega_{jn+1}\wedge\omega_{n+1n}+\omega_j\wedge\omega_n\\&=\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_k-(\lambda\mu-1)\omega_j\wedge ds.\end{aligned}$$

From (3.15), we have

$$\begin{aligned}d\omega_{jn} &= \frac{d^2\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds^2}ds\wedge\omega_j+\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}d\omega_j\\&=\frac{d^2\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds^2}ds\wedge\omega_j+\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n}\omega_{jk}\wedge\omega_k\\&=\{-\frac{d^2\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds^2}+[\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}]^2\}\omega_j\wedge ds\\&+\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_k.\end{aligned}$$

From the above two equalities, we have

$$\frac{d^2\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds^2}-\{\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}\}^2-(\lambda\mu-1)=0. \tag{3.16}$$

From (3.2) we get

$$\begin{aligned}&\frac{d^2\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds^2}-\{\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}\}^2\\&\qquad+\frac{n-2}{2}[\lambda^2-(R+1)]-R=0.\end{aligned} \tag{3.17}$$

Since we define $u=|\lambda^2-(R+1)|^{-\frac{1}{n}}$, we obtain from the above equation

$$\frac{d^2u}{ds^2}-u(\pm\frac{n-2}{2}\frac{1}{u^n}-R)=0. \tag{3.18}$$

Since $\mathbb{H}^{n+1}(-1)$ is an $(n+1)$-dimensional connected hypersurface in $\mathbb{R}^{n+2}$. We consider the frame $e_1,e_2,\cdots,e_n,e_{n+1},e_{n+2}$ in $\mathbb{R}^{n+2}$. By (2.14), (3.15) and (3.16), we have

$$\begin{aligned}de_i &= \sum_{j=1}^{n-1}\omega_{ij}e_j+\omega_{in}e_n+\omega_{in+1}e_{n+1}-\omega_ie_{n+2}\\&=\sum_{j=1}^{n-1}\omega_{ij}e_j+\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}\omega_ie_n-\lambda\omega_ie_{n+1}-\omega_ie_{n+2}\\&=\sum_{j=1}^{n-1}\omega_{ij}e_j+[\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}e_n-\lambda e_{n+1}-e_{n+2}]\omega_i.\end{aligned}$$

On the other hand, by means of (3.16), we get

$$\begin{aligned}
d\{&\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}e_n-\lambda e_{n+1}-e_{n+2}\}=d\{\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}\}e_n\\
&+\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}de_n-d\lambda e_{n+1}-\lambda de_{n+1}-de_{n+2}\\
=&\{\frac{d^2\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds^2}e_n-\frac{d\lambda}{ds}e_{n+1}\}ds\\
&+\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}(\sum_{j=1}^{n-1}\omega_{nj}e_j+\omega_{nn+1}e_{n+1}+\omega_{nn+2}e_{n+2})\\
&-\lambda(\sum_{j=1}^{n-1}\omega_{n+1j}e_j+\omega_{n+1n}e_n+\omega_{n+1n+2}e_{n+2})+\sum_{j=1}^{n-1}\omega_ie_j+\omega_ne_n\\
=&\{\frac{d^2\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds^2}e_n-\frac{d\lambda}{ds}e_{n+1}\}ds+\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}(\sum_{j=1}^{n-1}\omega_{nj}e_j\\
&-\mu\omega_ne_{n+1}-\omega_ne_{n+2})-\lambda(\lambda\sum_{j=1}^{n-1}\omega_je_j+\mu\omega_ne_n)+\sum_{j=1}^{n-1}\omega_ie_j+\omega_ne_n\\
\equiv&[\frac{d^2\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds^2}-\lambda\mu+1]e_n\omega_n\\
&-\{\frac{d\lambda}{ds}+\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}\mu\}e_{n+1}\omega_n\\
&-\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}e_{n+2}\omega_n\qquad(mod\{e_1,\cdots,e_{n-1}\})\\
=&\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}\{\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}e_n-\lambda e_{n+1}-e_{n+2}\}ds.
\end{aligned}$$

Putting

$$W=e_1\wedge\cdots\wedge e_{n-1}\wedge\{\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}e_n-\lambda e_{n+1}-e_{n+2}\},$$

we have

$$dW=\frac{d\{\log|\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds}Wds. \tag{3.19}$$

(3.19) shows that n-vector W in $\mathbb{R}^{n+2}$ is constant along $M_1^{n-1}(x)$. Hence, there exists an n-dimensional linear subspace $E^n(s)$ in $\mathbb{R}^{n+2}$ containing $M_1^{n-1}(x)$. By (3.19), the n-vector field W depends only on s and by integrating it, we get

$$W=\{\frac{\lambda^2(s)-(R+1)}{\lambda^2(s_0)-(R+1)}\}^{\frac{1}{n}}W(s_0).$$

Hence, we have that $E^n(s)$ is parallel to $E^n(s_0)$ in $\mathbb{R}^{n+2}$.

Since $\Omega_{ij} = -\omega_i \wedge \omega_j$, from (2.2), the curvature of $M_1^{n-1}(x)$ is given by

$$\begin{aligned} & d\omega_{ij} - \sum_{k=1}^{n-1} \omega_{ik} \wedge \omega_{kj} \\ &= \omega_{in} \wedge \omega_{nj} + \omega_{in+1} \wedge \omega_{n+1j} + \omega_i \wedge \omega_j \\ &= -\{(\frac{d\{\log |\lambda^2 - (R+1)|^{\frac{1}{n}}\}}{ds})^2 + \lambda^2 - 1\}\omega_i \wedge \omega_j. \end{aligned}$$

Therefore we know that the curvature of $M_1^{n-1}(x)$ is $(\frac{d\{\log |\lambda^2-(R+1)|^{\frac{1}{n}}\}}{ds})^2 + \lambda^2 - 1$ and $M_1^{n-1}(x)$ is contained in an $(n-1)$-dimensional sphere $S^{n-1}(c(s)) = E^n(s) \cap \mathbb{H}^{n+1}(-1)$ of the intersection of $\mathbb{H}^{n+1}(-1)$ and an n-dimensional linear subspace $E^n(s)$ in $\mathbb{R}^{n+2}$ which is parallel to a fixed $E^n(s_0)$. This completes the proof of Theorem 1.2.

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