Statistical properties of hyperbolic Julia sets

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Abstract. In this article, we consider the family of hyperbolic rational maps $T(z)$ defined on the Riemann sphere and establish almost sure invariance principles on the natural extensions of Julia sets for the function $\log |T'|$. This implies a number of well-known corollaries, including the weak invariance principles and the law of iterated logarithms.

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1 Introduction

Let $\widehat{\mathbf{C}}$ denote the complex sphere, meaning the complex plane along with the point at infinity. Let $T : \widehat{\mathbf{C}} \longrightarrow \widehat{\mathbf{C}}$ be a rational map of degree $d \geq 2$, and denote by $\mathbb{J}$ its Julia set. If $f : \mathbb{J} \longrightarrow \mathbf{R}$ is a Hölder continuous function, we would like to study some statistical properties like the weak invariance principles and the law of iterated logarithm.

It is a classical problem in ergodic theory to understand the statistical properties of typical orbits. For example, the Birkhoff Ergodic Theorem describes the average behaviour of such orbits and the Central Limit Theorem describes the deviation from this average. These results are subsumed by more general invariance principles. In this paper, we prove statistical properties like the weak invariance principles (as in Theorem (6.1)) and the law of iterated logarithm (as in Theorem (6.2)) for the function $\log |T'|$ defined on the natural extension $\overline{\mathbb{J}}$ of the Julia set $\mathbb{J}$ of the considered rational map T. In fact, we prove in this paper, an almost sure invariance principle (as stated in Theorem (3.3)) and we derive the above mentioned properties as corollaries to this technical theorem as explained by Philipp and Stout in [16].

A functional form of the law of iterated logarithm can also be deduced with the help of work done by Denker [4]. Furthermore, problems of this kind have remained a focus of interest amidst Liverani, Saussol and Vaienti [13] and Denker and Philipp [5]. Papers by Young [18] and Isola [11] on central limit theorems for indifferent maps, Field, Melbourne and Török [8] on almost sure invariance principles for flows and skew products and Campanino and Isola [2] on invariance principles for maps admitting an infinite invariant measure provide a good motivation into the subject.

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2 Preliminary Results

In this section, we briefly review some basic definitions and established results that are essential in our analysis.

Let $T : \widehat{\mathbf{C}} \longrightarrow \widehat{\mathbf{C}}$ be a rational map of degree $d > 1$. For any $n \in \mathbf{N}$, we denote by T^n the n-th iterate of T.

Definition 2.1. The set of normality, $\mathbb{F}(T)$ is defined to be the set of points $z \in \widehat{\mathbf{C}}$ such that the sequence $\{T^n\}$ forms a normal family (in the sense of Montel) in some neighbourhood of z. The complement, $\mathbb{J}(T)$ of $\mathbb{F}(T)$ is known as the *Julia set.*

It is easy to observe that $\mathbb{F}(T)$ is open and has the property of complete invariance under T, that is $z \in \mathbb{F}(T) \iff T(z) \in \mathbb{F}(T)$. $\mathbb{J}(T)$ is then obviously closed and completely invariant under T. More details of these and other basic properties of the sets $\mathbb{F}(T)$ and $\mathbb{J}(T)$ can be found in [1].

Definition 2.2. We say the rational map T is *hyperbolic* if there exists $C > 0$ and $\lambda > 1$ such that for all $z \in \mathbb{J}$ and $n \geq 1$, we have $|(T^n)'(z)| \geq C\lambda^n$.

From now on, we shall always assume that the considered rational map T is hyperbolic. We shall be interested in the restriction of the rational map in its Julia set; $T : \mathbb{J} \longrightarrow \mathbb{J}$. We shall write $\mathcal{C}(\mathbb{J})$ for the space of all real-valued continuous functions defined on $\mathbb{J}$ and denote the supremum norm on $\mathbb{J}$ by $\|f\| = \sup_{z\in\mathbb{J}} |f(z)|$. For $\alpha > 0$, we shall write $\mathcal{C}^\alpha(\mathbb{J})$ for the space of real-valued Hölder continuous functions defined on $\mathbb{J}$ with Hölder exponent α. Note that there exists a smallest constant $C_f > 0$ such that $|f(z_1) - f(z_2)| \leq C_f|z_1 - z_2|^\alpha$ for all $z_1, z_2 \in \mathbb{J}$. Then the natural norm on $\mathcal{C}^\alpha(\mathbb{J})$ is denoted by $\|f\|_\alpha := C_f + \|f\|$.

Definition 2.3. Two continuous functions f and g are said to be *cohomologous* to each other with respect to the rational map T if there exists a continuous function $u : \mathbb{J} \longrightarrow \mathbf{R}$ such that $f - g = u \circ T - u$. A function that is cohomologous to the zero function 0 is called a *coboundary.*

Moreover, a Hölder continuous function is a coboundary if and only if $f^n(z) := f(z) + f(T(z)) + f(T^2(z)) + \cdots + f(T^{n-1}(z)) = 0$, for every periodic point z of order n, i.e., $T^n(z) = z$ [12]. For a function $g \in \mathcal{C}(\mathbb{J})$, one defines the transfer operator $\mathcal{L}_g$ by

$$\mathcal{L}_g \ \phi(z) \ = \sum_{w\in T^{-1}z} e^{g(w)} \ \phi(w); \tag{2.1}$$

where ϕ are real-valued continuous functions on $\mathbb{J}$ and $z \in \mathbb{J}$. Then, we know by a classical result [15] on the transfer operators that $\mathcal{L}_g$ has a simple, maximal, positive eigenvalue given by $e^{\mathcal{P}(g)}$, with a corresponding strictly positive eigenfunction h where $\mathcal{P}(g)$ is the pressure of the function g. If g is Hölder continuous, then the corresponding eigenfunction h is also Hölder continuous. Furthermore, if g is Hölder continuous we know from [4] that there exists a unique equilibrium state denoted by μ_g realising the maximum in the variational principle,

$$\mathcal{P}(g) = \sup\left\{h_T(\mu) + \int g d\mu : \mu \in \mathcal{M}_T\right\}. \tag{2.2}$$

Here, $h_T(\mu)$ denotes the entropy of T with respect to the measure μ and $\mathcal{M}_T$ denotes the space of all T-invariant probability measures defined on $\mathbb{J}$.

Note that the eigenfunction h corresponding to the largest eigenvalue $e^{\mathcal{P}(g)}$ is bounded away from 0 and ∞. Denker, Przyticki and Urbanski [6] assert that there exists a constant k so that $\| (\mathcal{L}_g)^n \phi\|_\alpha \leq k\|\phi\|_\alpha$ for all $n \in \mathbf{N}$ and $\phi \in \mathcal{C}^\alpha(\mathbb{J})$. We make use of this estimate in order to normalise the transfer operator $\mathcal{L}_g$. We now introduce a normalised transfer operator, along the lines of Haydn [10], $\widehat{\mathcal{L}} : \mathcal{C}^\alpha(\mathbb{J}) \longrightarrow \mathcal{C}^\alpha(\mathbb{J})$ defined by

$$\widehat{\mathcal{L}}\phi(z) = e^{-\mathcal{P}(g)}\mathcal{L}_{g+\log h-\log h\circ T}\phi(z). \tag{2.3}$$

Note that the principal eigenvalue of the normalised transfer operator $\widehat{\mathcal{L}}$ has been rescaled to 1 and the corresponding eigenfunction being the constant, $\widehat{\mathcal{L}}1 = 1$. Again, we know from the already stated result on transfer operators that the maximal eigenvalue 1 of the normalised transfer operator $\widehat{\mathcal{L}}$ is isolated and simple.

The following result due to Ruelle [17] was observed by Coelho and Parry in [3]. For any two Hölder continuous functions $f, g \in \mathcal{C}^\alpha(\mathbb{J})$ and $|t|$ sufficiently small, if μ_g denotes the equilibrium state of g, then

$$\left.\frac{d}{dt}\mathcal{P}(g+tf)\right|_{t=0} = \int f d\mu_g \;; \tag{2.4}$$

$$\left.\frac{d^2}{dt^2}\mathcal{P}(g+tf)\right|_{t=0} = \lim_{n\to\infty}\frac{1}{n}\int\left(f^n(z) - n\int f d\mu_g\right)^2 d\mu_g. \tag{2.5}$$

The quantity in statement (2.5) is strictly positive unless f is cohomologous to a constant and will henceforth be known as variance.

Recalling (from [14]) that the hyperbolic rational map $T : \mathbb{J} \longrightarrow \mathbb{J}$ is a quotient of an appropriately defined shift map on the shift space, [15] gives us the following bound.

For every Hölder continuous function $\phi \in \mathcal{C}^\alpha(\mathbb{J})$ there exists $0 < \theta < 1$ such that for all $n \geq 0$,

$$\left\|\widehat{\mathcal{L}}^n\phi - g\int \phi d\mu_f\right\| \leq O\left(\theta^n\right). \tag{2.6}$$

When the condition given in (2.6) is satisfied, Coelho and Parry assert in [3] that the function ϕ satisfies the following central limit theorem.

Theorem 2.1 (Central Limit Theorem). *Every Hölder continuous function $\phi : \mathbb{J} \longrightarrow \mathbf{R}$ satisfies the central limit theorem:*
There exists $\sigma \geq 0$ such that

$$\lim_{n\to\infty}\frac{1}{n}\int\left[\phi^n - \int\phi d\mu_f\right]^2 d\mu_f = \sigma^2, \tag{2.7}$$

and if $\sigma^2 > 0$, then for any $t \in \mathbf{R}$,

$$\mu_f\left(\left\{z \in \mathbb{J} : \frac{1}{\sqrt{n\sigma^2}}\left[\phi^n(z) - n\int\phi d\mu_f\right] \leq t\right\}\right) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}}du. \tag{2.8}$$

However, the authors in [15] subsequently improve the upper bound for the transfer operator in comparison to the one above (as given in the equation (2.6)).

Theorem 2.2 ([15]). *Let $f \in \mathcal{C}^{\alpha}(\mathbb{J})$ and let μ_f be its equilibrium state. Then, there exists $0 < \theta < 1$ such that for all $k > 0$, when $\int f d\mu_f = 0$, we have*

$$\|\widehat{\mathcal{L}}^k f\|_{\alpha} = O(\theta^k). \tag{2.9}$$

If we write $U_T f(z) = f(Tz)$ then the condition $\widehat{\mathcal{L}}1 = 1$ implies that $\widehat{\mathcal{L}}$ is a left inverse to U_T, i.e., $\widehat{\mathcal{L}}U_T = I$ where I is the identity.

The result stated in Theorem (2.2) is stronger and is required to prove stronger results like the invariance principles. First, we state and prove a lemma that is useful in the sequel.

Lemma 2.3 (c.f.[18]). *There exists a Hölder continuous function $w \in \mathcal{C}^{\alpha}(\mathbb{J})$ such that if we set $\phi := f + (w - U_T w)$ then*

$$\widehat{\mathcal{L}}\phi = 0 \text{ and } f^k(z) = \phi^k(z) + O(1).$$

Proof. We can define the function $w \in \mathcal{C}^{\alpha}(\mathbb{J})$ by the series

$$w := \sum_{j=1}^{\infty} \widehat{\mathcal{L}}^j f$$

which converges since by Theorem (2.2), we have

$$\|\widehat{\mathcal{L}}^k f\|_{\alpha} = O(\theta^{\alpha k}) \text{ for some } 0 < \theta < 1.$$

Hence,

$$\widehat{\mathcal{L}}w - w = \sum_{j=1}^{\infty} \widehat{\mathcal{L}}^{j+1} f - \sum_{j=1}^{\infty} \widehat{\mathcal{L}}^j f = -\widehat{\mathcal{L}}f.$$

Since $\widehat{\mathcal{L}}$ is a left inverse to U_T, i.e., $\widehat{\mathcal{L}}U_T = I$, we have

$$\widehat{\mathcal{L}}\phi = \widehat{\mathcal{L}}f + \widehat{\mathcal{L}}(w - U_T w) = \widehat{\mathcal{L}}f + (\widehat{\mathcal{L}}w - w) = 0.$$

We also notice that

$$f^k(z) - \phi^k(z) = U_T^k w(z) - w(z),$$

so that

$$|f^k(z) - \phi^k(z)| \leq 2\|w\|. \tag{2.10}$$

□

3 Martingales and the Main Theorem

We begin by stating the following lemma and observe that it is naturally satisfied.

Lemma 3.1. *Let T be a hyperbolic rational map and $\mathbb{J}$ its Julia set. Then, there exists a constant $k > 0$ such that*

$$\log \left| \frac{T'(z_1)}{T'(z_2)} \right| \leq k\ |z_1 - z_2|.$$

We now need to replace the sequence of functions $\phi^k(z)$ with another sequence (on a related space) which forms a martingale. Let us consider the natural extension $\bar{T} : \bar{\mathbb{J}} \longrightarrow \bar{\mathbb{J}}$ of $T : \mathbb{J} \longrightarrow \mathbb{J}$ which is the space consisting of all sequences $\underline{z} = \{z_k\}_{-\infty}^{0}$ in $\mathbb{J}$ satisfying $T(z_{k-1}) = z_k$, for $k \leq 0$. Let $\bar{f} : \bar{\mathbb{J}} \longrightarrow \mathbf{R}$ denote the natural extension of the Hölder continuous function $f : \mathbb{J} \longrightarrow \mathbf{R}$. Further, we denote by $\bar{\mu}_f$ the associated $\bar{T}$-invariant probability measure on $\bar{\mathbb{J}}$. There is a canonical projection from $\bar{\mathbb{J}}$ to $\mathbb{J}$ defined by $\pi(\underline{z}) = z_0$. The σ-algebra $\mathcal{B}$ for $\mathbb{J}$ allows us to associate a natural σ-algebra $\mathcal{B}_0 = \pi^{-1}\mathcal{B}$ on the natural extension. Let $\mathcal{B}_k := \bar{T}^k \mathcal{B}_0$, for $k \geq 0$.

Definition 3.1. Let $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots$, be a nested sequence of σ-algebras. Then, a sequence of functions $F_k : \bar{\mathbb{J}} \longrightarrow \mathbf{R}$ is called an *increasing martingale*, if F_k is $\mathcal{B}_k$-measurable and $E(F_k|\mathcal{B}_{k-1}) = F_{k-1}$. Or, equivalently $E(F_k - F_{k-1}|\mathcal{B}_{k-1}) = 0$.

The function $\phi : \mathbb{J} \longrightarrow \mathbf{R}$ naturally extends to a function $\bar{\phi} : \bar{\mathbb{J}} \longrightarrow \mathbf{R}$ by $\bar{\phi}((z_k)_{-\infty}^{0}) = \phi(z_0)$. Let us now denote

$$\bar{\phi}^k(\underline{z}) := \bar{\phi}(\bar{T}^{-k}(\underline{z})) + \cdots + \bar{\phi}(\bar{T}^{-2}(\underline{z})) + \bar{\phi}(\bar{T}^{-1}(\underline{z})).$$

Since $\bar{\phi}$ is $\mathcal{B}_0$-measurable, it immediately follows that $\bar{\phi}^k$ is $\mathcal{B}_k$ measurable.

Lemma 3.2 (c.f.[18]). *The sequence $\bar{\phi}^k$ is a martingale with respect to the increasing sequence of σ-algebras $\mathcal{B}_k$, $k \geq 1$.*

Proof. For $k \geq 1$, we can write

$$\begin{aligned} E(\bar{\phi}^k - \bar{\phi}^{k-1}|\mathcal{B}_{k-1}) &= E(\bar{\phi} \circ \bar{T}^{-k}|\bar{T}^{k-1}\mathcal{B}_0) \\ &= E(\bar{\phi}|\bar{T}^{-1}\mathcal{B}_0) \circ \bar{T}^k \\ &= E(\phi|T^{-1}\mathcal{B}) \circ T^k \\ &= U_T^{k+1}\hat{\mathcal{L}}\phi \\ &= 0, \end{aligned}$$

since we know from Lemma (2.3) $\widehat{\mathcal{L}}\phi = 0$. In particular, we have the sequence $\bar{\phi}^k$ is a martingale. □

We now state the main result of this paper in the following theorem.

Theorem 3.3. *Let $\bar{f} : \bar{\mathbb{J}} \longrightarrow \mathbf{R}$ be a Hölder continuous function with $\int \bar{f} d\bar{\mu}_f = 0$. Then, there exists a Hölder continuous function $\bar{\phi} : \bar{\mathbb{J}} \longrightarrow \mathbf{R}$, a one-dimensional Brownian motion $W : \Omega \longrightarrow \mathcal{C}(\mathbf{R}^+)$ on some probability space (Ω, ν) such that $W(\cdot)(t)$ has variance $t\sigma^2$ and a sequence of random variables $F_n : \Omega \longrightarrow \mathbf{R}$ such that*

(1) the families $\{F_k\}_{k\geq 1}$ and $\{\bar{\phi}^k\}_{k\geq 1}$ have the same distribution, i.e., for every Borel set $A \subset \mathbf{R}$ we have for all $k \geq 1$,

$$\bar{\mu}_f\left(\left\{z \in \bar{\mathbb{J}} : \bar{\phi}^k(z) \in A\right\}\right) = \nu\left(\{\omega \in \Omega : F_k(\omega) \in A\}\right) \text{ and}$$

(2) $F_k(\cdot) = W(\cdot)(k) + o(\sqrt{k})$, for $k \geq 0$, ν a.e.,

provided $\bar{f}$ is not a coboundary.

4 Proof of Part (1)

Let the variance of the function $\bar{f}$ be denoted by

$$\tilde{\sigma}^2 = \int \bar{f}(z)^2 d\bar{\mu}_f + 2\sum_{j=1}^{\infty} \int \bar{f}(\bar{T}^j z)\bar{f}(z) d\bar{\mu}_f > 0. \tag{4.1}$$

If we replace $\bar{f}$ by the cohomologous function $\bar{\phi}$ then the variance remains unaltered. So,

$$\tilde{\sigma}^2 = \int \bar{\phi}(z)^2 d\bar{\mu}_f + 2\sum_{j=1}^{\infty} \int \bar{\phi}(\bar{T}^j z)\bar{f}(z) d\bar{\mu}_f > 0. \tag{4.2}$$

We note this from alternative characterisations of the variance, as in [15]. We shall now define Brownian motion.

Definition 4.1. Let (Ω, ν) be a probability space. Then, a stochastic process $W : \Omega \longrightarrow \mathcal{C}(\mathbf{R}^+)$ is called a *Brownian motion* only if

(1) $W(\omega)(0) = 0$, a.e. ν;

(2) there exists $\sigma^2 > 0$ such that for each $t_0 > 0$ the values $\omega \longmapsto W(\omega)(t_0) \in \mathbf{R}$ have a normal distribution with variance $t_0\sigma^2$;

(3) for times $t_0 < t_1 < \cdots < t_k$ the differences $\omega \longmapsto W(\omega)(t_{j+1}) - W(\omega)(t_j) \in \mathbf{R}$ are independent random variables.

The following result is a standard one.

Lemma 4.1. *Brownian motion satisfies the law of iterated logarithm, i.e.,*

$$\limsup_{t\to\infty} \frac{|W(\omega)(t)|}{\sqrt{2\sigma^2 t \log\log t}} = 1 \; ; \; \nu \text{ a.e.}$$

Let us now use the results due to Pollicott and Sharp [18] following the analysis of Field, Melbourne and Török, [8] based on a treatment of Philipp and Stout, [16] for the martingale $\bar{\phi}^k$. A key ingredient is the martingale version of the Skorokhod embedding theorem, stated below.

Proposition 4.2 ([9]). *There exists a Brownian motion $W^*(\cdot)$ on a probability space (Ω, ν) such that $W^*(\cdot)(t)$ has variance t, an increasing sequence of σ-algebras $\mathcal{F}_k$ and sequences of random variables $\tau_k : \Omega \longrightarrow \mathbf{R}^+$ such that*

(1) $F_k := W^*(S_k)$, *where* $S_k := \sum_{j=0}^{k-1} \tau_j$ *has the same distribution as* $\bar{\phi}^k$;

(2) F_k *and* S_k *are* $\mathcal{F}_k$*-measurable; and*

(3) $E(\tau_j|\mathcal{F}_{j-1}) = E([F_j - F_{j-1}]^2|\mathcal{F}_{j-1})$, ν *a.e. for every* $j \geq 1$.

The above result immediately implies Part (1) of Theorem (3.3).

5 Proof of Part (2)

Let us now replace S_k by $\tilde{\sigma}^2 k$ where $\tilde{\sigma}^2 = \int_{\mathbb{J}} \bar{\phi}^2 d\bar{\mu}_f > 0$ in order to obtain the estimate in Part (2) of Theorem (3.3). Using Part (3) of Proposition (4.2), we have

$$
\begin{aligned}
S_k - \tilde{\sigma}^2 k = \quad & \sum_{j=0}^{k-1} [\tau_j - E(\tau_j|\mathcal{F}_{j-1})] \\
+ \quad & \sum_{j=0}^{k-1} \left[E\left([F_j - F_{j-1}]^2|\mathcal{F}_{j-1}\right) - [F_j - F_{j-1}]^2\right] \\
+ \quad & \sum_{j=1}^{k-1} [F_j - F_{j-1}]^2 - \tilde{\sigma}^2 k, \ \nu \ a.e.,
\end{aligned} \tag{5.1}
$$

where we set $F_{-1} = 0$. Both the first and the second terms on the right-hand side of the equation (5.1) are martingales because

$$
\begin{aligned}
E(\tau_j - E(\tau_j|\mathcal{F}_{j-1})|\mathcal{F}_j) &= 0 \quad \text{and} \\
E\left(E\left([F_j - F_{j-1}]^2 |\mathcal{F}_{j-1}\right) - [F_j - F_{j-1}]^2 |\mathcal{F}_j\right) &= 0.
\end{aligned}
$$

Let us now invoke the strong law of large numbers for martingales, as done by Feller in [7] for the terms in the equation (5.1) to see that, for any $\delta > 0$,

$$
S_k - \tilde{\sigma}^2 k = \sum_{j=0}^{k-1} [F_j - F_{j-1}]^2 - \tilde{\sigma}^2 k + O(k^{\frac{1}{2}+\delta}), \ \nu \ a.e. \tag{5.2}
$$

We shall now consider the following integral in order to estimate the summation in the equation (5.2). For $\delta > 0$,

$$
I_\delta := \int_\Omega \left(\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{2}+\delta}} \left[(F_j(\omega) - F_{j-1}(\omega))^2 - \tilde{\sigma}^2 \right] \right)^2 d\nu(\omega),
$$

The next lemma relates I_δ to the function $\bar{\phi}$.

Lemma 5.1 (c.f.[18]). *We can write*

$$I_\delta = \int_{\mathbb{J}} \left(\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{2}+\delta}} \left[\left(\bar{\phi}(\bar{T}^j z)\right)^2 - \int_{\mathbb{J}} \bar{\phi}^2 d\bar{\mu}_f \right] \right)^2 d\bar{\mu}_f.$$

For convenience, we now introduce a function Φ defined by

$$\Phi := \bar{\phi}^2(z) - \int_{\mathbb{J}} \bar{\phi}^2 d\bar{\mu}_f.$$

Lemma 5.2 (c.f.[18]). *There exists a constant $c > 0$ and $0 < \theta < 1$ such that*

$$\left| \int_{\mathbb{J}} \Phi \circ \bar{T}^k \Phi d\bar{\mu}_f \right| \leq c\ \theta^k \text{ for } k \geq 0.$$

The above lemma is again a consequence of Theorem (2.2). Consider the expansion

$$\begin{aligned}
I_\delta &:= \sum_{j=1}^{\infty} \sum_{p=1}^{\infty} \frac{1}{j^{\frac{1}{2}+\delta}} \frac{1}{p^{\frac{1}{2}+\delta}} \int_{\mathbb{J}} \Phi \circ \bar{T}^j \Phi \circ \bar{T}^p d\bar{\mu}_f \\
&= \sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{2}+\delta}} \int_{\mathbb{J}} \Phi^2 d\bar{\mu}_f + 2 \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{j^{\frac{1}{2}+\delta}} \frac{1}{(j+m)^{\frac{1}{2}+\delta}} \int_{\mathbb{J}} \Phi \circ \bar{T}^m \Phi d\bar{\mu}_f \\
&\leq \|\Phi^2\|_\infty \left(\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{2}+\delta}} \right) + 2c \left(\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{2}+\delta}} \right) \left(\sum_{m=1}^{\infty} \theta^m \right) \\
&< \infty,
\end{aligned}$$

using Lemma (5.2). In particular, we deduce that for any $\delta > 0$,

$$\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{2}+\delta}} \left([F_j(\omega) - F_{j-1}(\omega)]^2 - \int_{\mathbb{J}} \bar{\phi}^2 d\bar{\mu}_f \right) < \infty \ \nu \ a.e.$$

Applying the Kronecker lemma [9], we can deduce that

$$\sum_{j=1}^{k-1} \left[[F_j(\omega) - F_{j-1}(\omega)]^2 - \int_{\mathbb{J}} \bar{\phi}^2 d\bar{\mu}_f \right] = O(k^{\frac{1}{2}+\delta}). \tag{5.3}$$

Comparing the equations (5.2) and (5.3), we have $S_k = \tilde{\sigma}^2 k + O(k^{\frac{1}{2}+\delta})$ and so $F_k(\cdot) = W^*(S_k) = W^*(\tilde{\sigma}^2 k) + O(k^{\frac{1}{4}+\delta})$. Now, let

$$W(\cdot)(t) = W^*(\cdot)(\sigma^2 t).$$

Then, for $k \geq 0$,

$$\begin{aligned}
W^*(S_k) &= W^*(\tilde{\sigma}^2 k) + O(k^{\frac{1}{4}+\delta}) \\
&= W^* \left(\frac{\tilde{\sigma}^2 k}{\int d\bar{\mu}_f} + o(k) \right) + O(k^{\frac{1}{4}+\delta}) \\
&= W^*(\sigma^2 k) + o(\sqrt{k}) \\
&= W(k) + o(\sqrt{k}).
\end{aligned}$$

With the help of the above defined rescaled Brownian motion, the proof of Part (2) of Theorem (3.3) is also complete.

6 Important Corollaries

Now, we write two main theorems that follows from Theorem 3.3.

Theorem 6.1. *If $\bar{f} : \bar{\mathbb{J}} \longrightarrow \mathbf{R}$ is a Hölder continuous function with $\int \bar{f} d\bar{\mu}_f = 0$, then the weak invariance principle holds provided $\bar{f}$ is not a coboundary.*

The next one describes the growth of the sums $\bar{f}^n$ and asserts that

$$\limsup_{n\to\infty} \frac{1}{\sigma\sqrt{2n\log\log n}} \bar{f}^n(z) = 1, \ \bar{\mu}_f \ a.e.$$

Theorem 6.2. *If $\bar{f} : \bar{\mathbb{J}} \longrightarrow \mathbf{R}$ is a Hölder continuous function with $\int \bar{f} d\bar{\mu}_f = 0$, then the law of iterated logarithm holds provided $\bar{f}$ is not a coboundary.*

The derivation of the two Theorems (6.1) and (6.2) from almost sure invariance principles (like Theorem (3.3)) is well explained by Philipp and Stout in [16].

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