Minimal volume of the connected sum of Euclidean spaces

Hongyu Wang and Haifeng Xu

Abstract. In this paper, we give a detailed proof of the result that the minimal volume of the connected sum of $\mathbb{R}^n$ is zero for $n \geq 2$. The approach is to construct a sequence of explicit complete metrics on them such that the curvatures are bounded in absolute value by 1 and the volumes tend to zero uniformly.

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Key words: minimal volume, connected sum, smooth gluing of metrics.

1 Introduction

The definition of minimal volume of a C^∞ manifold M (without boundary) was first introduced by Mikhael Gromov [6]. It is a geometric invariant. As stated in [9], the minimal volume does depend on the smooth structure of the manifold (also see [3]).

J. Cheeger and M. Gromov introduced in ([4, 6]) the concepts of F-structure and T-structure and obtained some results about F-structure and minimal volume. F-structure is a natural generalization of the effective torus action on manifold M. For precise definitions about T-structure, F-structure and more general the $\tilde{g}$-structure, we refer to [4]. The definitions are stated in the language of sheaves. Roughly speaking, a $\tilde{g}$-structure on M is a connected topological group sheaf on M and a complete local action on M such that M is covered by open saturated subsets $\{V(x)\}$ corresponding to such action and the mappings from $\tilde{V}(x)$ to the open saturated sets $V(x)$ are normal coverings and the sheaf when restricted on the orbits is a local constant sheaf. An F-structure on a manifold M is a $\tilde{g}$-structure such that the covering maps are finite and every stalk $G_x,\ x \in M$ is isomorphic to a torus group. If the covering maps are trivial, then the F-structure is called a T-structure. An F-structure is called polarized if the torus actions defined on the finite coverings are locally-free. For example, S^4 and $\mathbb{C}P^2$ admit T-structures although they cannot admit any polarized F-structure.

J. Cheeger and M. Gromov [4] proved that if M admits a polarized F-structure then the minimal volume of M vanishes. The graph manifold is a 3-manifold which admit a polarized T- structure. So graph manifold is a special T-manifold. Thus the minimal volume of graph manifold is zero.

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Furthermore, T. Soma proved in [10] that the connected sum of two graph manifold is still a graph manifold. In [6] Gromov pointed out that this result holds for odd dimensional manifolds with T-structures (under the original definition given by Gromov [6], i.e. which is polarized.). Paternain and Petean proved in [9] that the result also holds for the family of manifolds which admit general T-structures and for any dimension greater than 2. There is a little difference between the original definition of T-structure given in [6] and that given in ([4, 9]). The latter asks the torus action be effective but not necessarily locally free.

Hence we could not apply the theorem of Paternain and Petean to answer the question that whether the minimal volume of the connected sum of Euclidean spaces $\mathbb{R}^n\#\mathbb{R}^n$ $(n > 2)$ is zero. It is not a special example of the theorem of Paternain and Petean.

So we take the direct method to compute the minimal volume. The work is by constructing metrics on $\mathbb{R}^n\#\mathbb{R}^n$ we show that the minimal volume is equal to zero without using the concepts mentioned above. In this paper, we always equip disk D^2 and "Y-piece" Y with smooth metric j_ε and l_ε respectively, as shown in the following pictures (Figure 1 and 2). Y, called "Y-pieces", is a compact topological surface with

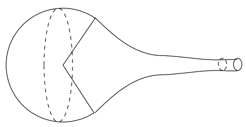

Figure 1: (D^2, j_ε)

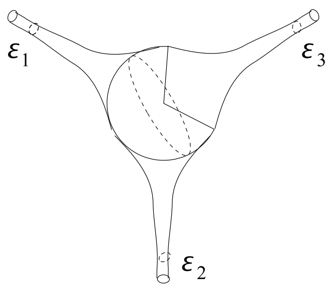

Figure 2: (Y, l_ε)

boundary obtained from a 2 dimensional sphere by cutting away the interior of 3 disjoint closed topological disks. For the method to construct such metrics on the two objects, we refer to [8]. Generally, the two surfaces should satisfy the following conditions:

1. *The curvature is bounded by a constant which is independent of* ε.
2. *When restricted to a small neighborhood of a boundary circle, the metric is flat. For this purpose we take a product metric on this neighborhood.*
3. *The length of boundary circle is* $2\pi\varepsilon$.

In particular, the three boundary circles in (Y, l_ε) should have different perimeters in the metric construction of $\mathbb{R}^3$. Details can be found in [8]. Hence, the metric of Y should be denoted by $l_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$.

Similarly, $(D^2, j_{\varepsilon f(t)})$ means that the surface $(D^2, j_{\varepsilon f(t)})$ satisfies the conditions (1), (2) and the following (3'):

(3') *The length of the boundary circle is* $2\pi\varepsilon f(t)$.

For convenience, in this paper, we will say that manifold M is collapsing, if it can be equipped with a series of smooth metrics g_ε such that the curvatures are uniformly bounded in absolute value by constant 1, and at the same time the volumes tend to zero. Here constant 1 may be replaced by a constant $C > 1$ which is independent of ε, since by scaling if necessary, the metrics can be made satisfying the inequality $|K_{g_\varepsilon}| \leq 1$ while the volumes $\mathrm{Vol}(M, g_\varepsilon)$ still tend to zero as $\varepsilon \to 0$. Of course, we still say that the metrics g_ε are collapsing.

Surfaces of revolution in Euclidean 3-space have been extensively studied. For surfaces of revolution in Minkowski 3-space, we refer to [7].

2 Minimal Volumes of S^{2n+1} for $n \geq 1$

Proposition 2.1. *MinVol*$(S^{2n+1}) = 0$ *for* $n \geq 1$.

Proof. By [11], the m-dimensional sphere S^m can be obtained as follows

$$S^m = (S^1 \times D^{m-1}) \cup_{Id} (D^2 \times S^{m-2}),$$

where the boundaries are attached by the identity map Id. Hence,

$$S^3 = (S^1 \times D^2) \cup_{Id} (D^2 \times S^1).$$

Let

$$g_\varepsilon^3 := \varepsilon^2 d\theta^2 + j_\varepsilon$$

be a series of metrics on $S^1 \times D^2$. Since j_ε is a product metric on a small neighborhood of the boundary circle, the metrics g_ε^3 can be glued (or attached) smoothly by the identity map Id along the boundaries $\partial(S^1 \times D^2)$ and $\partial(D^2 \times S^1)$. Therefore we get a series of global metrics h_ε^3 on S^3. It is easy to verify that $\left|K_{h_\varepsilon^3}\right|$ are uniformly bounded by a constant C and $\mathrm{Vol}(S^3, h_\varepsilon^3) \to 0$ as $\varepsilon \to 0$. Hence $\mathrm{MinVol}(S^3) = 0$.

For S^5, we have

$$\begin{aligned} S^5 &= (S^1 \times D^4) \cup_{Id} (D^2 \times S^3) \\ &= (S^1 \times D^2 \times D^2) \cup_{Id} (D^2 \times S^3). \end{aligned}$$

Let $\varepsilon^2 d\theta^2 + j_\varepsilon + j_\varepsilon$ be the product metric on $S^1 \times D^2 \times D^2$. On the other hand, let $j_\varepsilon + h_\varepsilon^3$ be the product metric on $D^2 \times S^3$.

It is easy to see that, under metric $j_\varepsilon + j_\varepsilon$, the small neighborhood of $\partial(D^2 \times D^2) = S^3 \times I$ is endowed with the product metric $h_\varepsilon^3 + dt^2$.

Hence, along the boundaries, the metric on $S^1 \times D^2 \times D^2$ can be attached smoothly with the metric on $D^2 \times S^3$ by the identity map Id. Also, we get a series of global smooth metrics h_ε^5 on S^5 which are collapsing. For simplicity, here we omit the verification about curvatures.

Thus, the metrics h_ε^3 may yield metrics h_ε^5 on S^5, and h_ε^5 may yield metrics h_ε^7 on S^7, and so on . . . . Therefore, we have proved that $\mathrm{MinVol}(S^{2n+1}) = 0$. □

Remark 2.2. $S^{2n+1} (n \geq 1)$ is a principal S^1-bundle over $\mathbb{C}P^n$. There exists a free S^1-action on S^{2n+1}, which is the simplest example of polarized T-structure. It is easy to construct metrics on such manifolds (with a free action of circle S^1) such that the metrics will infer $\mathrm{MinVol}(S^{2n+1}) = 0$(See Berger's book [2, P.549]). As an example, we can give metrics on S^3 as follows. Let

$$g_\varepsilon = \frac{4(dx_1^2 + dx_2^2)}{(1 + x_1^2 + x_2^2)^2} + \varepsilon^2\theta^2, \tag{2.1}$$

where

$$\theta = \frac{x_1 dx_2 - x_2 dx_1}{1 + x_1^2 + x_2^2} + dx_3. \tag{2.2}$$

It is easy to verify that such g_ε are global metrics on S^3, and the corresponding curvatures are uniformly bounded in absolute value by a constant $C > 0$, and the volumes tend to zero as $\varepsilon \to 0$.

But the similar form

$$\frac{4\mathrm{Re}(dz_0 d\bar{z}_0 + \cdots + dz_n d\bar{z}_n)}{(|z_0|^2 + \cdots + |z_n|^2)}$$

is not a global metric on $\mathbb{C}P^n$. We should take the Fubini-Study metric on $\mathbb{C}P^n$ [5].

For any principal S^1-bundle P over a closed Riemannian manifold (M, g), we construct a sequence of metrics $\tilde{g}_\varepsilon$ on P as follows: Let

$$\tilde{g}_\varepsilon = \pi^* g + \varepsilon^2\theta^2 = (\widetilde{\omega}^1)^2 + \cdots + (\widetilde{\omega}^n)^2 + (\widetilde{\omega}^{n+1})^2, \tag{2.3}$$

where

$$\widetilde{\omega}^i = \pi^*\omega^i, \quad \widetilde{\omega}^{n+1} = \varepsilon\theta. \tag{2.4}$$

and $i\theta$ is a S^1-connection on P. Suppose

$$d\theta = \frac{1}{2} F_{ij}\widetilde{\omega}^i \wedge \widetilde{\omega}^j, \quad i, j = 1, 2, \ldots, n, \tag{2.5}$$

where $F_{ij} = -F_{ji}$(see [5]). By a direct calculation, we have

$$\widetilde{R}^i_{jji} = R^i_{jji} + \frac{3}{4}\varepsilon^2 (F_{ji})^2. \tag{2.6}$$

Hence the curvatures under the sequence of metrics are uniformly bounded in absolute value by a constant $C > 0$ and

$$\mathrm{Vol}(P, \tilde{g}_\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0. \tag{2.7}$$

Complementary results about the metric above can be found in [1].

3 The minimal volume of $S^n \times \mathbb{R}$ for $n \geq 1$

In this section, we will show that $\mathrm{MinVol}(S^n \times \mathbb{R}) = 0$ for $n \geq 1$, that is $\mathrm{MinVol}(\mathbb{R}^n \# \mathbb{R}^n) = 0$ for $n \geq 2$. Let us consider some examples first.

Example 3.1. *$\mathrm{MinVol}(S^1 \times \mathbb{R}) = 0$.*

Proof. Let $g_\varepsilon = \varepsilon^2 f^2(t)d\theta^2 + dt^2$ be the metrics on $S^1 \times \mathbb{R}$, where $\theta \in [0, 2\pi],\ t \in \mathbb{R}$. Suppose that function $f(t) \in C^\infty(\mathbb{R})$ satisfies the following conditions:

$$\begin{cases} \dfrac{|f''(t)|}{f(t)} \le 1, \\ \displaystyle\int_{-\infty}^{+\infty} f(t)dt < +\infty, \\ f(t) > 0, \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

For instance, set

$$f(t) := \begin{cases} 1 - e^{-\frac{24}{t^2}}, & t \ne 0; \\ 1, & t = 0. \end{cases}$$

Then $|K| = \frac{|f''(t)|}{f(t)} \le 1$, and $\mathrm{Vol}(S^1 \times \mathbb{R}, g_\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence $\mathrm{MinVol}(S^1 \times \mathbb{R}) = 0$.
□

Example 3.2. *MinVol*$(S^2 \times \mathbb{R}) = 0$.

Proof. Since $S^2 \times \mathbb{R}$ is homomorphic to $\mathbb{R}^3 \# \mathbb{R}^3$, we only need to construct metrics on the connected sum $\mathbb{R}^3 \# \mathbb{R}^3$ by using of the metrics on $\mathbb{R}^3$ which has been constructed in [8, Section 3]. In other words, we will make use of the torus decomposition of $\mathbb{R}^3$. However, $\mathbb{R}^3$ can be viewed as the infinite connected sum of S^3.

$$\mathbb{R}^3 = S^3 \# S^3 \# \cdots \# S^3 \# \cdots . \tag{3.1}$$

Hence

$$\mathbb{R}^3 \# \mathbb{R}^3 = (\cdots \# S^3 \# \cdots \# S^3 \# S^3) \# (S^3 \# S^3 \# \cdots \# S^3 \# \cdots). \tag{3.2}$$

Since connected sum is a local operation, by the Heegaard splitting of S^3, we have

$$S^3 \# S^3 = \big((S^1 \times D^2) \cup_{Id} (D^2 \times S^1)\big) \# \big((S^1 \times D^2) \cup_{Id} (D^2 \times S^1)\big).$$

So special attention is paid to the connected sum

$$(D^2 \times S^1) \# (S^1 \times D^2).$$

This is the connected sum of two solid torus, which is well described by Figure 3 [10]. It is homeomorphic to $Y \times S^1 \cup_{Id} S^1 \times D^2$. Hence, the torus decomposition of $\mathbb{R}^3$ and the description formula (3.1) are just the same thing.

Take metrics on the building blocks $Y \times S^1$ and $S^1 \times D^2$ as follows:

$$l_{\varepsilon_{i1}, \varepsilon_{i2}, \varepsilon_{i3}} + \left(\frac{\varepsilon}{2^i}\right)^2 d\theta^2, \qquad \left(\frac{\varepsilon}{2^i}\right)^2 d\theta^2 + j_{\frac{\varepsilon}{2^i}}.$$

Here we take $\varepsilon_{i1} = \frac{\varepsilon}{2^{i-1}}$, $\varepsilon_{i2} = \frac{\varepsilon}{2^i}$, and $\varepsilon_{i3} = \frac{\varepsilon}{2^{i+1}}$. In Figure 4, we also use the letter Y to indicate the three boundary circles in the "Y-piece". So by gluing the metrics along boundaries as shown in Figure 4, we have constructed a series of global smooth metrics on $\mathbb{R}^3 \# \mathbb{R}^3$, and we have $\mathrm{MinVol}(S^2 \times \mathbb{R}) = 0$. For detailed picture of metric construction on $\mathbb{R}^3$, we refer to [8]. □

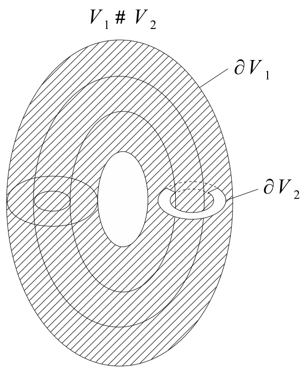

Figure 3: $V_1\#V_2$

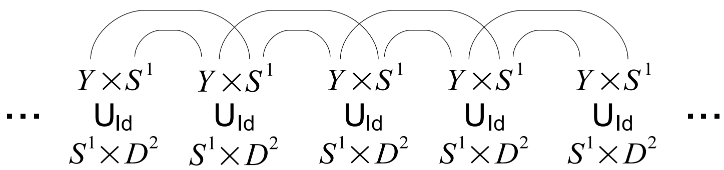

Figure 4: metric construction on $\mathbb{R}^3\#\mathbb{R}^3$

Example 3.3. *MinVol*$(S^3 \times \mathbb{R}) = 0$.

Proof.

$$S^3 \times \mathbb{R} = \big((S^1 \times D^2) \cup_{Id} (D^2 \times S^1)\big) \times \mathbb{R}.$$

Suppose $S^1 \times D^2 \times \mathbb{R}$ and $D^2 \times S^1 \times \mathbb{R}$ are endowed with metrics

$$\varepsilon^2 f^2(t)d\theta^2 + j_{\varepsilon f(t)} + dt^2,$$

and

$$j_{\varepsilon f(t)} + \varepsilon^2 f^2(t)d\theta^2 + dt^2$$

respectively, where $\theta \in [0, 2\pi],\ t \in \mathbb{R}$.

Note that, along the boundaries, the two metrics can be attached smoothly by the identity map Id. And by the calculation and argument in [8, Section 4] (which is rather complex), $S^3 \times \mathbb{R}$ is collapsing under such metrics. □

Proposition 3.4. *MinVol*$(S^{2n+1} \times \mathbb{R}) = 0$.

Proof. We have proved this result in case $n = 0$ and $n = 1$. When $n = 2$, we have

$$S^5 \times \mathbb{R} = \big((S^1 \times D^4) \cup_{Id} (D^2 \times S^3)\big) \times \mathbb{R}. \tag{3.3}$$

Again, S^1, D^2, S^3, $\mathbb{R}$ are endowed with metrics $\varepsilon^2 f^2(t)d\theta^2$, $j_{\varepsilon f(t)}$, $h^3_{\varepsilon f(t)}$ and dt^2 respectively. For $D^4 \cong D^2 \times D^2$, we assign it the metric $j_{\varepsilon f(t)} + j_{\varepsilon f(t)}$. Then, along the boundaries $\partial(S^1 \times D^4)$ and $\partial(D^2 \times S^3)$, the metrics can be attached smoothly by the identity map Id.

In detail, on $S^1 \times D^2 \times D^2 \times \mathbb{R}$, the metric is given by

$$\varepsilon^2 f^2(t) d\theta^2 + j_{\varepsilon f(t)} + j_{\varepsilon f(t)} + dt^2.$$

In terms of function G (For details see [8, Section 4]), the metric above can be written as

$$\begin{aligned}\varepsilon^2 f^2(t) d\theta^2 + &\big(1 + (G'_x(x,t))^2\big) dx^2 + G^2(x,t) d\alpha^2 \\ &+ \big(1 + (G'_y(y,t))^2\big) dy^2 + G^2(y,t) d\beta^2 + dt^2.\end{aligned}$$

Let

$$\begin{aligned}\omega^1 = \varepsilon f(t) d\theta, \quad \omega^2 &= \sqrt{1 + {G'_x}^2}\, dx, \quad \omega^3 = G(x,t) d\alpha, \\ \omega^4 &= \sqrt{1 + {G'_y}^2}\, dy, \quad \omega^5 = G(y,t) d\beta, \quad \omega^6 = dt.\end{aligned}$$

Then, by the structure equations, we get the connection form matrix $(\omega_i^j)_{6\times 6}$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\varepsilon f'(t) d\theta \\ & 0 & \frac{G'_x}{\sqrt{1+{G'_x}^2}} d\alpha & 0 & 0 & -\frac{G'_x G''_{xt}}{\sqrt{1+{G'_x}^2}} dx \\ & & 0 & 0 & 0 & -G'_t(x,t) d\alpha \\ & & & 0 & \frac{G'_y}{\sqrt{1+{G'_y}^2}} d\beta & -\frac{G'_y G''_{yt}}{\sqrt{1+{G'_y}^2}} dy \\ & & & & 0 & -G'_t(y,t) d\beta \\ & & & & & 0 \end{bmatrix}$$

It is benefit to make a comparison with the connection form matrix which is shown below (see [8, Section 4]):

$$(\omega_i^j)_{4\times 4} = \begin{bmatrix} 0 & 0 & 0 & -\varepsilon f'(t) d\theta \\ & 0 & \frac{G'_x}{\sqrt{1+{G'_x}^2}} d\alpha & -\frac{G'_x G''_{xt}}{\sqrt{1+{G'_x}^2}} dx \\ & & 0 & -G'_t(x,t) d\alpha \\ & & & 0 \end{bmatrix}$$

By a direct calculation, we get the curvatures $\{K_{ij}\}$:

$$\begin{aligned}
K_{12} &= -\frac{f'(t)}{f(t)} \cdot \frac{G'_x G''_{xt}}{1 + {G'_x}^2}, \quad K_{13} = -\frac{f'(t)}{f(t)} \cdot \frac{G'_t(x,t)}{G(x,t)}, \\
K_{14} &= -\frac{f'(t)}{f(t)} \cdot \frac{G'_y G''_{yt}}{1 + {G'_y}^2}, \quad K_{15} = -\frac{f'(t)}{f(t)} \cdot \frac{G'_t(y,t)}{G(y,t)}, \\
K_{16} &= -\frac{f''(t)}{f(t)}, \\
K_{23} &= -\left(\frac{G''_{xx}}{G(x,t)(1 + {G'_x}^2)^2} + \frac{G'_x G'_t(x,t) G''_{xt}}{G(x,t)(1 + {G'_x}^2)} \right), \\
K_{24} &= -\frac{G'_x G''_{xt}}{1 + {G'_x}^2} \cdot \frac{G'_y G''_{yt}}{1 + {G'_y}^2}, \quad K_{25} = -\frac{G'_x G''_{xt}}{1 + {G'_x}^2} \cdot \frac{G'_t(y,t)}{G(y,t)}, \\
K_{26} &= -\frac{(G''_{xt})^2 + G'_x G'''_{xtt}(1 + {G'_x}^2)}{(1 + {G'_x}^2)^2},
\end{aligned}$$

$$K_{34} = -\frac{G'_t(x,t)}{G(x,t)} \cdot \frac{G'_y G''_{yt}}{1+{G'_y}^2}, \quad K_{35} = -\frac{G'_t(x,t)}{G(x,t)} \cdot \frac{G'_t(y,t)}{G(y,t)},$$
$$K_{36} = -\frac{G''_{tt}(x,t)}{G(x,t)},$$
$$K_{45} = -\left(\frac{G''_{yy}}{G(y,t)(1+{G'_y}^2)^2} + \frac{G'_y G'_t(y,t) G''_{yt}}{G(y,t)(1+{G'_y}^2)}\right),$$
$$K_{46} = -\frac{(G''_{yt})^2 + G'_y G'''_{ytt}(1+{G'_y}^2)}{(1+{G'_y}^2)^2}, \quad K_{56} = -\frac{G''_{tt}(y,t)}{G(y,t)}.$$

If we write these into a matrix, we can make a good comparison with the curvatures in [8, Section 4]:

$$\begin{bmatrix} K_{12} & K_{13} & K_{14} \\ & K_{23} & K_{24} \\ & & K_{34} \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{f'(t)}{f(t)} \cdot \frac{G'_x G''_{xt}}{1+{G'_x}^2} & -\frac{f'(t)}{f(t)} \cdot \frac{G'_t(x,t)}{G(x,t)} & -\frac{f''(t)}{f(t)} \\ & -\left(\frac{G''_{xx}}{G(1+{G'_x}^2)^2} + \frac{G'_x G'_t G''_{xt}}{G(1+{G'_x}^2)}\right) & -\frac{(G''_{xt})^2 + G'_x G'''_{xtt}(1+{G'_x}^2)}{(1+{G'_x}^2)^2} \\ & & -\frac{G''_{tt}}{G} \end{bmatrix}$$

By the calculation and argument in [8, Section 4], we see that not only the curvatures K_{12}, K_{13}, K_{14}, K_{15}, K_{16}, K_{23}, K_{26}, K_{36}, K_{45}, K_{46}, K_{56} are uniformly bounded, but also the curvatures K_{24}, K_{25}, K_{34}, K_{35}, since the following inequalities hold (see [8, Section 4]):

$$\begin{array}{rcl} |G'_x| & \leq & \varepsilon + \varepsilon C, \\ |G'''_{xt}| & \leq & \varepsilon + 4C\varepsilon, \\ \frac{|G'_t|}{G} & < & C, \end{array}$$

where the constant C is independent of ε. Hence, by scaling if necessary, the curvatures will be bounded in absolute value by 1, and the volumes still tend to zero. Therefore, $\mathrm{MinVol}(S^5 \times \mathbb{R}) = 0$.

For $n = 3$,

$$\begin{aligned} S^7 \times \mathbb{R} &= \big((S^1 \times D^6) \cup_{Id} (D^2 \times S^5)\big) \times \mathbb{R} \\ &= \big((S^1 \times D^2 \times D^2 \times D^2) \cup_{Id} (D^2 \times S^5)\big) \times \mathbb{R}. \end{aligned}$$

The argument is analogous to the case $n = 2$. The most important thing is that all the curvatures can be inferred according to the curvatures listed above. Therefore, for all $n \geq 0$, we have

$$\mathrm{MinVol}(S^{2n+1} \times \mathbb{R}) = 0.$$

□

Remark 3.5. An alternate proof is here. Take Riemannian metric $\tilde{g}_\varepsilon$ on S^{2n+1} as in Remark 2.2. It can be written as a decomposition into the vertical and horizontal parts in local:

$$\tilde{g}_\varepsilon = \pi^* g + \varepsilon^2 \theta^2,$$

where g is the Fubini-Study metric on $\mathbb{C}P^n$. Then $g(\varepsilon) := \varepsilon^2 f^2(t)\theta^2 + \pi^* g + dt^2$ is the required metrics on $S^{2n+1} \times \mathbb{R}$.

Proposition 3.6. $MinVol(S^{2n} \times \mathbb{R}) = 0$.

Proof. We have proved the proposition in case $n = 1$ in Example 3.2. Similarly, for $\mathbb{R}^5 = S^5 \# S^5 \# \cdots \# S^5 \# \cdots$, we have

$$\mathbb{R}^5 \# \mathbb{R}^5 = (\cdots \# S^5 \# \cdots \# S^5 \# S^5) \# (S^5 \# S^5 \# \cdots \# S^5 \# \cdots).$$

Hence, we only need to consider

$$S^5 \# S^5 = \big((S^1 \times D^4) \cup_{Id} (D^2 \times S^3)\big) \# \big((S^1 \times D^4) \cup_{Id} (D^2 \times S^3)\big)$$

For $(D^2 \times S^3) \# (S^1 \times D^4)$, suppose $D_x^2 \times S^1 \subset S^3$, where $D_x^2 \subset D^2$, x is a fixed point in D and D_x is a neighborhood of x. The following picture (Figure 5) will give a nice description when understanding the metric structure on the connected sum $S^1 \times D^4 \# S^1 \times D^4$. In this picture the solid line with arrows "$\longleftrightarrow$" means gluing,

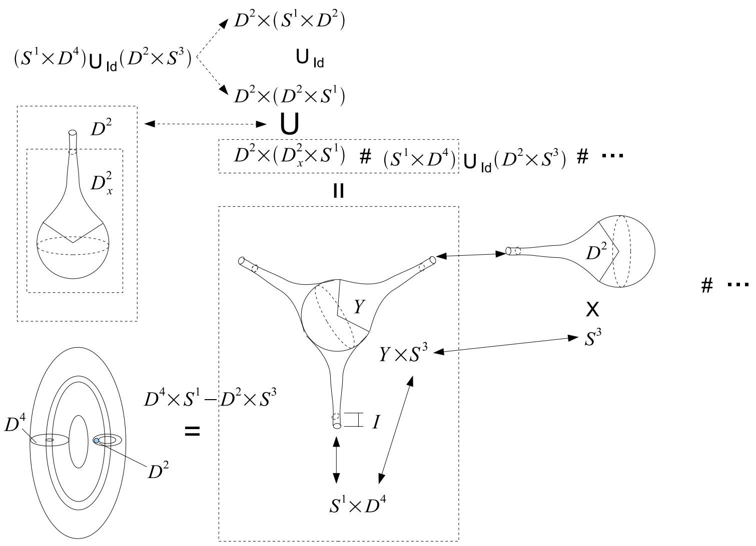

Figure 5: Metric construction on $\mathbb{R}^5$

and Y still denotes the "Y-piece" as in [8].

Then, we can construct the metrics on $\mathbb{R}^5$ and $\mathbb{R}^5 \# \mathbb{R}^5$. Hence $\mathrm{MinVol}(\mathbb{R}^5 \# \mathbb{R}^5) = 0$.

In general,

$$(S^1 \times D^n) \# (S^1 \times D^n) = S^1 \times D^n - S^{n-1} \times D^2.$$

So we can construct metrics on $\mathbb{R}^{2n+1} \# \mathbb{R}^{2n+1}$ in the same way. In this case, the building blocks (Y, l_ε) and (D^2, j_ε) are still used. Another block is S^{2n-1} whose metric has been constructed in the proof of Proposition 2.1. So we can get a similar picture as Figure 5. Therefore, we complete the proof. □

Remark 3.7. For S^{2n-1}, we also can take metrics as in Remark 3.5. We also can explain the connected sum $S^{2n+1}\#S^{2n+1}$ from the point of view of principle S^1-bundle S^{2n+1} on $\mathbb{C}P^n$. Let $U_0 = \{[z_0, z_1, \dots, z_n] \in \mathbb{C}P^n \mid z_0 \neq 0\}$, then $U_0 = \mathbb{C}P^n \setminus \mathbb{C}P^{n-1}$. So locally, $S^{2n+1}|_{U_0} \simeq U_0 \times S^1$. Take a small disk $D^{2n} \subset U_0$, then we only need to consider the connected sum $D^{2n} \times S^1 \# D^{2n} \times S^1$. The left is same as above.

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Authors' addresses:

Hongyu Wang and Haifeng Xu
School of Mathematical Science, Yangzhou University,
180 Siwangting Road, 225002, Yangzhou, Jiangsu, P. R. China
E-mail: hywang@yzu.edu.cn, beard_xu@yahoo.com.cn