# Conformal gradient vector fields

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**Abstract.** In this paper we prove that a compact Riemannian manifold with positive constant scalar curvature is isometric to a sphere provided that it admits a nonzero conformal gradient vector field.

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**Key words**: Scalar curvature; conformal vector field; conformal gradient vector field; isometry to a sphere.

## 1 Introduction

Spheres have many interesting geometrical properties among the class of compact connected Riemannian manifolds. That is why, it is an important issue to classify spheres (cf. [1], [2], [4], [5], [6]). An interesting property is the existence of nonconstant functions $f$ on $S^n(c)$ which satisfies $\nabla_X grad\ f = -cfX$, where $grad\ f$ is the gradient of $f$ and $\nabla_X$ is the covarient derivative operator with respect to the smooth vector $X$. Obata showed that a complete connected Riemannian manifold that admits a non constant solution of this differential equation is necessarily isometric to $S^n(c)$ (cf. [5]). Deshmukh and Alsolamy [3] gave an answer for the question: " under what conditions does an $n$-dimensoinal compact and connected Riemannian manifold that admits a nonzero conformal gradient vector field has to be isometric to a sphere $S^n(c)$?", by giving certain bounds for the Ricci curvature which involves the first nonzero eigenvalue of the Laplacian operator on $M$. In this paper we will provide an answer to this question by restricting the scalar curvature to be positive constant as follows:

**Theorem.** *Let $(M, g)$ be an $n$- dimensional compact connected Riemannian manifold of positive constant scalar curvature $n(n-1)c$. If $M$ admits a non zero conformal gradient vector field, then $M$ is isometric to the $n$-sphere $S^n(c)$.*

## 2 Preliminaries

Let $(M, g)$ be a Riemannian manifold with Lie algebra $\mathfrak{X}(M)$ of smooth vector fields on $M$. A vector field $X \in \mathfrak{X}(M)$ is said to be conformal if it satisfies

$$\mathfrak{L}_X g = 2\varphi g \tag{2.1}$$

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for a smooth function $\varphi$:$M \to R$, where $\mathfrak{L}_X$ is the Lie derivative with respect to $X$. If $u = grad\ f$ is the gradient of a smooth function $f$ on $M$ and $u$ is a conformal vector field, then it follows from (2.1) that a conformal vector field $u$ satisfies

$$\nabla_X u = \varphi X, \quad X \in \mathfrak{X}(M) \tag{2.2}$$

For a sphere $S^n(c)$, there exists a non constant function $\varphi \in C^\infty(S^n(c))$ which satisfies

$$\nabla_X \nabla\varphi = -c\varphi X \tag{2.3}$$

where $\nabla\varphi$ is the gradient of $\varphi$ and $\nabla_X$ is the covariant derivative operator with respect to the smooth vector $X$.

The following result is an immediate consequence of the equation (2.2):

**Lemma 2.1** *Let $u$ be a conformal gradient vector field on a compact Riemannian manifold $(M, g)$. Then, for $\varphi = n^{-1}div\ u$,*

$$\int_M \varphi dv = 0.$$

For a smooth function $f$ on $M$, define an operator $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $AX = \nabla_X \nabla f$, where $\nabla f$ is the gradient of $f$. The Ricci operator $Q$ is a symmetric operator defined by

$$Ric(X, Y) = g(Q(X), Y), \quad X, Y \in \mathfrak{X}(M)$$

where $Ric$ is the Ricci tensor of the Riemannian manifold, and hence from the definition of the operator $A$ we have the following relation

$$R(X, Y)\nabla f = (\nabla A)(X, Y) - (\nabla A)(Y, X) \tag{2.4}$$

where $(\nabla A)(X, Y) = \nabla_X AY - A(\nabla_X Y)$, $X, Y \in \mathfrak{X}(M)$. Note that using (2.4) we have the following lemma which gives an important property of the operator $A$.

**Lemma 2.2** *Let $(M, g)$ be Riemannian manifold and $f$ be a smooth function on $M$. Then the operator $A$ corresponding to the fuction $f$ satisfies*

$$\sum_{i=1}^{n} (\nabla A)(e_i, e_i) = \nabla(\Delta f) + Q(\nabla f)$$

*where $\{e_1, e_2, ..., e_n\}$ be a local orthonormal frame on $M$.*

**Lemma 2.3** *Let $u$ be a conformal gradient vector field on an $n$- dimensional Riemannian manifold $(M, g)$. Then the operator $Q$ satisfies*

$$Q(u) = -(n-1)\nabla\varphi,$$

*where $\nabla\varphi$ is the gradient of the smooth function $\varphi = n^{-1}div\ u$.*

# 3 Proof of the Theorem

For an $n$- dimensional compact connected Riemannian manifold $(M, g)$ of positive constant scalar curvature $S = n(n-1)c$, we have that $\Delta\varphi = -nc\varphi$ and from Lemma 2.3 that $\Delta f = n\varphi$. These two relations imply $\Delta\varphi = -c\Delta f$ that is $\Delta(\varphi + cf) = 0$. Thus $\varphi = -cf + \alpha$, where $\alpha$ is a constant. Consequently $\nabla\varphi = -c\nabla f$, which gives $\nabla_X \nabla\varphi = -c\nabla_X \nabla f = -c\varphi X$ that is $\varphi$ satisfies the Obata's differential equation. We claim that $\varphi$ is not constnat. If $\varphi$ is a constant, it will imply that $f$ is a constant which in turn will imply that $u = 0$ and that leads to a contradiction as the statement of the Theorem requires that $u$ is nonzero vector field. Hence by Obata's theorem we get that $M$ is isometric to the *n-sphere* $S^n(c)$.

**Remark.** The compactness condition in the theorem essential, as for the the Riemannian manifold $(R^n, g)$ where $g$ is the Riemannian metric defined by $g = \frac{1}{1+\|x\|^2}\langle \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is the Euclidean metric on $R^n$, choosing $u$ to be the position vector field, the scalar curvature $S$ of $(R^n, g)$ is a positive constant but the manifold is not isometric to a sphere $S^n(c)$.

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