On the Legendre transform and Hamiltonian formalism in Berwald-Moor geometry

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Abstract. The Legendre transform is explicitly described in the Berwald-Moor pseudo-Finsler case; differences towards the proper Finslerian framework are emphasized. The fundamental associated geometric objects and structures are constructed and discussed.

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1 The Legendre transform of a Finsler space

The Legendre transform was extended from the Riemannian framework to the Finslerian ansatz in Miron's works ([8], [9]); further, this was subject of careful detailed discussion in the monograph [3].

Definition 1.1 ([3]). Let (M, F) be a proper (positive-definite) Finsler space ([10], [3]) and let U be an open subset in M, domain of the tangent bundle atlas. We call *Legendre transform* of (M, F) the local mapping $\mathcal{L} : TM|_U \to T^*M|_U$ given in coordinates by $L(x, y) = (x, p)$ with

$$p_i = \frac{1}{2}\frac{\partial F^2}{\partial y^i}. \tag{1.1}$$

Remark 1.2. a) We note that $p_i = FF_i = g_{i0}$, where we denoted $F_i = \dot{\partial}_i F$, $g_{ij} = \frac{1}{2}\dot{\partial}^2_{ij} F^2$, $\dot{\partial}_i = \partial/\partial y^i$, and the null index denotes transvection with y.

b) The Jacobian matrix of the fiber action of $\mathcal{L}$ is exactly $\frac{\partial p_i}{\partial y^j} = g_{ij}$. Since g_{ij} is non-degenerate, it follows that $\mathcal{L}$ is a local diffeomorphism.

c) $\mathcal{L}^{-1} : T^*M \to TM$ is given by $\mathcal{L}^{-1}(x, p) = (x, y)$, with y^i satisfying the nonlinear system:

$$g_{ij}(x, y)y^j = p_i, \ i = \overline{1, n}.$$

d) In the proper Finslerian framework, we note that the dual Finsler structure $F_* : T^*M \to \mathbb{R}$ is usually introduced like in functional analysis: the norm F_* is

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defined as ([3, Sec. 14.8, p. 407])

$$F_*(x,p) = \sup_{y \in S_x M} p(y), \text{ where } S_x M = \{y \in T_x M |\ F(x,y) = 1\}$$

and it is proved that since F_* is everywhere defined, the Hamiltonian $\mathcal{H} = \frac{1}{2}F_*^2$ provided by F_* is everywhere defined. Moreover, since $F_* \circ \mathcal{L}(y) = F(y)$, $\mathcal{H}$ coincides with the one constructed in classical mechanics, i.e.,

$$H(p) = p_j y^j - \frac{1}{2}F^2(y) = \frac{1}{2}F^2(y), \text{ for } p = \mathcal{L}(y).$$

2 The Legendre transform of $\mathcal{H}_n$

Regarding the metric Finsler tensor of the Berwald-Moór space $\mathcal{H}_n$ $(n \geq 2)$, one has the following result:

Theorem 2.1 ([7]). *In the Berwald-Moór Finsler space $\mathcal{H}_n = (M = \mathbb{R}^n, F)$, with*

$$F(y) = \rho^{1/n}, \quad \rho = y^1 \cdot \ldots \cdot y^n, \tag{2.1}$$

we have:

a) the metric tensor field has the components

$$g_{ij}(y) = \frac{1}{2}\frac{\partial F^2}{\partial y^i \partial y^j} = \frac{\rho^{2/n}}{n y^i y^j}\left(\frac{2}{n} - \delta_{ij}\right). \tag{2.2}$$

b) the determinant of the Finsler metric tensor field is given by

$$\det(g_{ij}) = (-1)^{n+1}/n^n;$$

c) the dual tensor field associated to g_{ij} has the components

$$g^{ij} = \frac{n y^i y^j}{\rho^{2/n}}\left(\frac{2}{n} - \delta^{ij}\right). \tag{2.3}$$

As pointed out in [2], the signature of g_{ij} given by (2.2) is $(+,-,\ldots,-)$, hence the space $\mathcal{H}_n$ is not proper Finslerian, but *pseudo-Finslerian.* In order to obtain the inverse Legendre transform $\mathcal{L}^{-1}$, we first check the bijectivity of $\mathcal{L}$, in the Berwald-Moór case. To this aim, we have the following

Lemma 2.2. *Let F be the pseudo-Finsler metric given by (2.1). Then the mapping $\varphi : T_x M \to T_x^* M$,*

$$\varphi(y) = (k F^\alpha F_i)_{i=\overline{1,n}},$$

where lower indices of F denote partial derivatives w.r.t y^i, is bijective if and only if $k\alpha \neq 0$.

Proof. The Jacobian matrix of $[\varphi_*]$ has the components

$$k F^{\alpha-1}(\alpha F_i F_j + F F_{ij}) = k F^{\alpha-1}(g_{ij} + (\alpha - 1)F_i F_j).$$

We apply the following

Lemma 2.3. *If the matrix* $[a] = (a_{ij})_{i,j=\overline{1,n}}$ *is invertible and* $b_i \in \mathbb{R}, i \in \overline{1,n}$*, then* $[\tilde{a}] = (\tilde{a}_{ij} = a_{ij} + \varepsilon b_i b_j)_{i,j=\overline{1,n}}$ *is invertible and* $\det[\tilde{a}] = a_*(1+B_*)$*, where* $a_* = \det[a]$*,* $b_* = a^{ij} b_i b_j$*, with* $[a]^{-1} = (a^{ij})_{i,j=\overline{1,n}}$*.*

Hence, for $\varepsilon = 1$, $[a] = [g]$, we obtain

$$\det[\varphi_*] = (kF^{\alpha-1})^n \cdot \det[g] \cdot (1 + (\alpha - 1) g^{ij} F_i F_j).$$

But, using the Lemma 2.2 we get

$$g^{ij} F_i F_j = \frac{1}{F^2} \left[2y^i y^j - n(y^i)^2 \delta^{ij} \right] \cdot \frac{1}{n^2} \frac{F^2}{y^i y^j} = 1.$$

Hence $\det[\varphi_*] = \alpha (kF^{\alpha-1})^n \cdot \dfrac{(-1)^{n+1}}{n^n}$ which is nonzero iff $k\alpha \neq 0$. □

We shall further provide a series of examples, relevant in literature ([3], [5]), which emerge from Theorem 2.1:

a) In [5, p. 89] and in [11] is considered the transformation $\varphi(y) = p$, with $p_i = F^{-1} g_{i0}$, where the null index denotes transvection with y. The Lemma applies, since $p_i = F_i = kF^\alpha F_i$ with $k = 1$ and $\alpha = 0$. Then obviously $\det[\varphi_*] = 0$ and φ is *not* invertible. The matrix $[\varphi_*]$ is exactly the matrix associated to the (degenerate) Lansberg metric. In this case, due to the $0-$homogeneity of F_i, the mapping φ depends only on direction and takes whole rays of vectors to the same image p_i. This homogeneity is obvious, since F is 1-homogeneous, and in the Berwald-Moór case,

$$F_i = \frac{1}{n} \frac{\rho^{1/n}}{y^i} = \frac{1}{n} \sqrt[n]{\frac{y^1 y^2 \dots y^n}{(y^i)^n}}.$$

Hence φ maps a ray $Span(y)$ into the differential dF - which is assimilated to the Euclidean gradient at $y/F(y)$ of the $F-$indicatrix $\Sigma_{\mathcal{H}_n} : F(y) = 1$.

b) In [5, p. 82] is considered the transformation $\varphi(y) = p$, with $p_i = F^{-1} F_i$. In this case we have $\varphi(y) = p$, with $k = 1$, $\alpha = -1$ and the Jacobian of φ is $F^{-2} \det[g] \neq 0$.

c) Generally, in the pseudo-Finsler approach, for $\varphi = \mathcal{L}$, we have $p_i = FF_i$, $[\varphi_*] = [g]$, and hence for our case the Legendre mapping is obviously locally invertible.

Moreover, $\mathcal{L}$ is bijective, as stated in the following result:

Theorem 2.4. *Consider the subsets*

$$\widehat{TM} = \bigcup_{x \in M} \hat{T}_x M, \quad \widehat{T^*M} = \bigcup_{x \in M} \hat{T}^*_x M,$$

where

$$\begin{cases} \hat{T}_x M = \{ y \in \mathbb{R}^n \equiv T_x M | y^1 \cdot \ldots \cdot y^n \neq 0 \} \subset \widetilde{T_x M} = T_x M \backslash \{0_x\} \subset T_x M, \\ \hat{T}^*_x M = \{ p \in \mathbb{R}^{n*} \equiv T^*_x M | p_1 \cdot \ldots \cdot p_n \neq 0 \} \subset \widetilde{T^*_x M} = T^*_x M \backslash \{0_x\} \subset T^*_x M, \end{cases}$$

Then the (canonically defined) Legendre transform of $\mathcal{H}_n$*,*

$$\mathcal{L} : \widehat{TM} \to \widehat{T^*M}, \quad \mathcal{L}(x, y) = (x, p), \; p_i = FF_i$$

has the following properties:

a) it is characterized by $p_i = \dfrac{F^2}{ny^i} = \dfrac{(y^1 \cdot \ldots \cdot y^n)^{2/n}}{ny^i}$;

b) it is bijective and its inverse is

$$\mathcal{L} : \widehat{T^*M} \to \widehat{TM}, \ \ \mathcal{L}^{-1}(x,p) = (x,y), \ \ \textit{where } y^i = \frac{n\rho^{*2/n}}{p_i},$$

with $\rho^* = p_1 \cdot \ldots \cdot p_n$;

c) it satisfies the relations

$$g^{ij}(\mathcal{L}^{-1}(p)) = g^{*ij}(p), \quad g_{ij}(\mathcal{L}^{-1}(p)) = {g^*}_{ij}(p),$$

where $g^{*ij}(p) = \dfrac{1}{2}\dfrac{\partial F^{*2}(p)}{\partial p_i \partial p_j}$, $F^*(p) = F(\mathcal{L}^{-1}(p))$, *and* ${g^*}_{ij}$ *are the coefficients of the dual metric to* g^{*ij} *on* $\widehat{T^*M}$.

Corollary 2.5. *a) The fiber-preserving mapping* $F_* : T^*M \to TM$ *defined on fibers by* $F_*(p) = F \circ \mathcal{L}^{-1}(p)$ *defines a pseudo-Finsler structure on* T^*M*, of the same signature as* F.

b) The dual $(2,0)$ *Finsler tensor field* $g^{ij}(y)$ *given by* $g^{is}g_{sj} = \delta^i_j$ *satisfies the relation* $g^{ij}(\mathcal{L}^{-1}(p)) = \frac{1}{2}\frac{\partial^2 F_*^2}{\partial p_i \partial p_j}$, *i.e., it is the fundamental Finsler tensor field associated to* F_*.

3 The Berwald-Moor dual structure

3.1 Canonic geometric objects on T^*M

Each pseudo-Finsler space (M,F) is endowed with the following objects:

a) *the Hilbert* form $\theta = -y^i dx^i$, which is the pull-back via $\mathcal{L}$ of *the Poincare form* $\eta = -p_i dx^i$ $(\mathcal{L}_*\eta = \theta)$;

b) *the symplectic form* $\omega = d\theta = dx^i \wedge dp_i$, where $p_i = g_{i0}$;

c) *the Lagrangian* $L = \frac{1}{2}F^2 \in \mathcal{F}(TM)$;

d) *the Hamiltonian* $H = y^k p_k - L \in \mathcal{F}(T^*M)$. We note that $H = L \circ \mathcal{L}^{-1}$.

d) *the geodesic spray* of the space, given by the field $G = y^i \dfrac{\partial}{\partial x^i} - 2G^i \dfrac{\partial}{\partial y^i}$, where $G^i = \frac{1}{2}\gamma^i_{00}$.

It is known that in $\mathcal{H}_n$ (and generally, in locally Minkowski Finsler spaces), since F depends on y only, we have vanishing Christoffel symbols γ^i_{jk}, Cartan nonlinear connection coefficients N^i_j, and the associated to it vertical geodesic spray coefficients G^i.

3.2 Hamiltonian vector fields

Like in the Riemannian case, one can define a Hamiltonian-type vector field X_φ for any smooth function $\varphi : T^*M \to \mathbb{R}$ via the equality

$$i_{X_\varphi}\omega = d\varphi \Leftrightarrow \omega(X_\varphi, \cdot\,) = d\varphi. \tag{3.1}$$

Then one immediately gets

$$X_\varphi = \frac{\partial\varphi}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial\varphi}{\partial x^i}\frac{\partial}{\partial p_i}.$$

Particular examples.

a) The Hamiltonian vector field associated to the Legendre-induced Hamiltonian $H = \frac{1}{2}F_*^2 = L \circ \mathcal{L}^{-1}$ has the form:

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i}.$$

This satisfies the equality $d\eta(X_H, \cdot\,) = dH$, and its integral curves are given by the Hamiltonian-Jacobi system

$$\begin{cases} \dfrac{dx^i}{dt} = \dfrac{\partial H}{\partial p_i} \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial x^i}. \end{cases} \tag{3.2}$$

In particular, transporting the dual objects to the corresponding ones via the Legendre transform, it is known that ([4]):

$$X_L = G,$$

where $L = \frac{1}{2}F_*^2$, hence the field lines of the Hamiltonian field associated to F energy Lagrangian, are exactly the geodesics of the Finsler space (M, F).

b) The Hamiltonian field, which is basically induced by the Finslerian norm ([5]) is $X_F = G/F$.

However, in the $\mathcal{H}_n$ case, the geodesics are - like in the Minkowski E_1^n flat case, straight lines, and the Hamiltonian field is horizontal:

$$X_L = X_F = y^i\frac{\partial}{\partial x^i}.$$

4 Hamilton vs. Lagrange equations

A Hamilton space (M, F_*) is defined by a function $F_* : T^*M \to \mathbb{R}$ (called *fundamental Hamilton function*, which is differentiable on T^*M and continuous on the null section of the cotangent bundle of M, such that the fundamental tensor field $g^{ij} = \frac{1}{2}\frac{\partial^2 F_*^2}{\partial p_i \partial p_j}$ is non-degenerate and of constant signature on T^*M.

Theorem 4.1. *Denoting $L = \frac{1}{2}F^2$, the Euler-Lagrange equations,*

$$\frac{d}{dt}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = 0,$$

are equivalent to the Hamilton-Jacobi equations (3.2), and to the nonlinear geodesic (semispray) equations

$$\frac{d^2 x^i}{dt^2} + 2G^i(x, y) = 0,$$

where

$$G^i = \frac{1}{2}g^{ij}\left(\frac{\partial^2 L}{\partial y^j \partial x^k}y^k - \frac{\partial L}{\partial x^j}\right).$$

Corollary 4.2. *The HamiltonJacobi equations are equivalent to*

$$\begin{cases} \dfrac{dx^i}{dt} = \{H, x^i\} \\ \dfrac{dp_a}{dt} = \{H, p_a\}, \end{cases}$$

where the Poisson structure is defined by the brackets $\{f, g\} = \omega(X_f, X_g)$ for any functions $f, g : T^*M \to \mathbb{R}$.

Theorem 4.3. *There exists a canonical nonlinear connection defined respectively by the dual Finsler metric F_*, given by[1]:*

$$N_{ij} = \{g_{ij}, H\} - \left(\frac{\partial^2 H}{\partial p_k \partial x^i}g_{jk} + (i/j)\right),$$

where $H = \frac{1}{2}F_*^2$.

5 Geometric structures on $T^*\mathcal{H}_n$

Several geometric structures can be defined, and shown to coincide on $\mathcal{H}_n$ with the classical ones, as follows:

Definition 5.1. a) The almost complex structure $J : T^*M \to T^*M$,

$$J(\delta_{x^i}) = -g_{is}\partial_{p_s}, J(\partial_{p_i}) = g^{is}\delta_{x^s};$$

b) The almost product operator is given by $P(\delta_{x^i}) = \delta_{x^i}$, $P(\partial_{p_i}) = -\partial_{p_i}$;
c) The almost tangent structure is provided by $\Pi(\delta_{x^i}) = g_{is}\partial_{p_s}$, $\Pi(\partial_{p_s}) = 0$.

Proposition 5.2 ([1]). *a) The components of the Nijenhuis tensor field associated to J,*

$$\nu(X, Y) = -[X, Y] + [JX, JY] - J[JX, Y] - J[X, JY], \forall X, Y \in \Gamma(TTM)$$

[1] We denote by e.g., (i/j) the expression to the left of the symbol, with the indices i and j interchanged.

are explicitly given by

$$\nu_{ija} = \frac{\partial N_{ia}}{\partial x^j} + N_{ib}\frac{\partial N_{ja}}{\partial p_b} - (i/j).$$

These components trivially vanish for (H_n, F_*).

b) If the nonlinear connection on T^*M *is related to the similar (Kern) nonlinear connection of* TM*, then* J, P *and* Π *are* $\mathcal{L}-$*related with the similar tensor fields of* TM*. This trivially happens in the* H_n *case.*

Theorem 5.3. *a) The coefficients of the dual nonlinear connection on* $T^*\mathcal{H}_n$ *related to the trivial nonlinear connection on* TH_n *has vanishing coefficients. Then the adapted basis of horizontal and vertical sections of* $\Gamma(TTH_n)$ *reduces to the canonic one. As well, the tensors from the definition 5.1 act exactly as in the Euclidean case.*

b) The coefficients of the adapted $N-$*connection on* T^*H_n *are* (L^i_{jk}, C^{jk}_i)*, where ([1]):*

$$L^i_{jk} \equiv 0, \;\; C^{jk}_i = -\frac{1}{2}g_{is}(\dot\partial^j g^{sk} + (j/k) - \dot\partial^s g^{jk}) - \frac{1}{2}g_{is}(B^{jsk} + (j/k) - B^{sjk}),$$

where $B^{ijk} = g^{is}(g^{jt}\partial_{p_k}N_{st} - (j/k))$*. In the* $\mathcal{H}_n$ *case, we have*

$$C^{ij}_k = (C_{skt}g^{is}g^{kt}) \circ \mathcal{L}^{-1},$$

where the $v-$*coefficients of the Cartan linear connection are*

$$C^i_{jk} = g^i s C_{jsk} = \frac{y^i}{y^j y^k}\left(-\frac{2}{n^2} + \frac{\delta_{jk} + \delta^i_j + \delta^i_k}{n} - \delta^i_j\delta^i_k\right),$$

and the dual metric tensor has the coefficients given in (2.3).

c) The $v-$*curvature of the space* $\mathcal{H}_n$ *has the coefficients given by ([7])*

$$S_{ijkl} = -(g_{ik}g_{jl} - g_{il}g_{jk})/F^2.$$

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