

Homogeneous Lorentzian structures on the generalized Heisenberg group

W. Batat and S. Rahmani

Abstract. In [8], all the homogeneous Riemannian structures corresponding to the left-invariant Riemannian metric on the three-dimensional Heisenberg group have been obtained. In [7], N. Rahmani and S. Rahmani have extended this result to the Lorentzian case. In this paper, we consider the generalized Heisenberg group equipped with the left-invariant Lorentzian metric given by (2.1) and we determine all the homogeneous Lorentzian structures on it. This leads to the generalization of the Rahmani's result see [7].

M.S.C. 2000: 53C30, 53C50.

Key words: Homogeneous structures; Lorentzian metrics; generalized Heisenberg group.

1 Introduction and preliminaries

The characterization by E. Cartan of Riemannian locally symmetric spaces as those Riemannian manifolds whose curvature tensor is parallel was extended by Ambrose and Singer in [1]. They proved that a connected, simply connected and complete Riemannian manifold is homogeneous if and only if there exists a $(1, 2)$ tensor field T satisfying certain equations, see (1.1). In [5] Gadea and Oubiña have extended that characterization to pseudo-Riemannian manifolds. Specifically, let (M, g) be a connected C^∞ pseudo-Riemannian manifold of dimension n and signature $(k, n - k)$. Let ∇ be the Levi-Civita connection of g and R its curvature tensor field. A homogeneous pseudo-Riemannian structure on (M, g) is a tensor field T of type $(1, 2)$ on M such that the connection $\tilde{\nabla} = \nabla - T$ satisfies

$$(1.1) \quad \tilde{\nabla}g = 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}T = 0.$$

More explicitly, T is the solution of the following system of equations (known as Ambrose-Singer equations)

$$\begin{aligned} g(T_X Y, Z) + g(Y, T_X Z) &= 0, \\ (\nabla_X R)_{YZ} &= [T_X, R_{YZ}] - R_{T_X Y Z} - R_{Y T_X Z}, \\ (\nabla_X T)_Y &= [T_X, T_Y] - T_{T_X Y}, \end{aligned}$$

for all vector fields X, Y, Z . If g is Lorentzian metric ($k = 1$), we say that T is a homogeneous Lorentzian structure. The geometric meaning of the existence of a homogeneous pseudo-Riemannian structure is explained by the following:

Theorem 1.1. [5] *Let (M, g) be a connected, simply connected and complete pseudo-Riemannian manifold. Then, (M, g) admits a pseudo-Riemannian structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold.*

This means that $M = G/H$, where G is a connected Lie group acting transitively and effectively on M as a group of isometries, H is the isotropy group at a point $o \in M$, and the Lie algebra \mathfrak{g} of G may be decomposed into a vector space direct sum of the Lie algebra \mathfrak{h} of H and an $\text{Ad}(H)$ -invariant subspace \mathfrak{m} , that is $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\text{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$. It must be noted that any homogeneous Riemannian manifold is reductive, while a homogeneous pseudo-Riemannian manifold need not be reductive. Homogeneous Lorentzian structures have been investigated for several classes of Lorentzian manifolds. Some examples are: Lorentzian manifolds with large isometry groups [2]; three-dimensional Lorentzian manifolds admitting a parallel null vector field [3]; Oscillator Lie groups [4]; Godel-Levichev's spacetimes [6]. In [8], Tricerri and Vanhecke obtained all the homogeneous Riemannian structures on the three-dimensional Heisenberg group. A corresponding result had been obtained for the three-dimensional Lorentzian Heisenberg group [7]. In this work we start given in Section 2 the Levi-Civita connection and the curvature tensor of the generalized Heisenberg group endowed with the left-invariant Lorentzian metric (2.1). Then, in Section 3 we shall write down and solve the system of partial differential equations determining a homogeneous Lorentzian structures on generalized Heisenberg group. As an application, we deduce the explicit expression of homogeneous Lorentzian structure on the three-dimensional Heisenberg group.

2 Connection and Curvature of generalized Heisenberg group

Recall that the generalized Heisenberg group of dimension $2p + 1$ is the vector space $H_{2p+1} = \mathbb{R} \times \mathbb{C}^p$ endowed with the group law

$$(z, v) \cdot (z', v') = \left(z + z' + \frac{1}{2}B(v, v'), v + v' \right),$$

where B is the nondegenerate alternate \mathbb{R} -bilinear form

$$B(v, v') = \sum_{i=1}^p x_i y'_i - y_i x'_i,$$

with

$$v = (x_i + \sqrt{-1}y_i)_{1 \leq i \leq p}, \quad v' = (x'_i + \sqrt{-1}y'_i)_{1 \leq i \leq p} \quad \text{and} \quad z, z' \in \mathbb{R}.$$

Its Lie algebra \mathfrak{h}_{2p+1} has a basis $\{Z, X_1, \dots, X_p, Y_1, \dots, Y_p\}$ for which all brackets are zero except $[X_i, Y_i] = Z$ for $1 \leq i \leq p$. This means that the center of \mathfrak{h}_{2p+1} is one-dimensional. We consider on H_{2p+1} the left-invariant Lorentzian metric g , with non-vanishing inner products $\langle \cdot, \cdot \rangle$ on \mathfrak{h}_{2p+1} given by

$$(2.1) \quad \langle Z, Z \rangle = -1, \quad \langle X_i, X_i \rangle = \langle Y_i, Y_i \rangle = 1 \quad \text{for} \quad 1 \leq i \leq p.$$

We denote by ∇ the Levi-Civita connection of (H_{2p+1}, g) and by R its curvature tensor, taken with the sign convention:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}.$$

On account of Koszul's formula for the Levi-Civita connection for left-invariant metric on a Lie group,

$$2\langle \nabla_U V, W \rangle = \langle [U, V], W \rangle - \langle [V, W], U \rangle + \langle [W, U], V \rangle \text{ for all } U, V, W \in \mathfrak{h}_{2p+1},$$

we obtain that the possibly non-vanishing covariant derivatives between generators are

$$\begin{aligned} \nabla_Z X_j &= \frac{1}{2} Y_j = \nabla_{X_j} Z, \\ \nabla_Z Y_j &= -\frac{1}{2} X_j = \nabla_{Y_j} Z, \\ \nabla_{X_j} Y_k &= \frac{1}{2} \delta_{jk} Z, \\ \nabla_{Y_k} X_j &= -\frac{1}{2} \delta_{jk} Z, \end{aligned}$$

and the not always null components of the curvature tensor field are given by

$$\begin{aligned} R_{X_i X_j Y_k} &= \frac{1}{4} (\delta_{jk} Y_i - \delta_{ik} Y_j), \\ R_{Y_i Y_j X_k} &= \frac{1}{4} (\delta_{jk} X_i - \delta_{ik} X_j), \\ R_{X_j Y_k X_i} &= -\frac{1}{2} \left(\delta_{jk} Y_i + \frac{1}{2} \delta_{ik} Y_j \right), \\ R_{X_j Y_k Y_i} &= \frac{1}{2} \left(\delta_{jk} X_i + \frac{1}{2} \delta_{ij} X_k \right), \\ R_{Z X_j} Z &= -\frac{1}{4} X_j, \\ R_{Z X_j} X_k &= -\frac{1}{4} \delta_{jk} Z, \\ R_{Z Y_j} Z &= -\frac{1}{4} Y_j, \\ R_{Z Y_j} Y_k &= -\frac{1}{4} \delta_{jk} Z, \end{aligned}$$

and the ones obtained from them using the symmetries of the curvature tensor.

3 Homogeneous Lorentzian structures on (H_{2p+1}, g)

We shall now determine the homogeneous Lorentzian structures on the generalized Heisenberg group (H_{2p+1}, g) , in terms of the basis $\{\eta, \alpha^1, \dots, \alpha^p, \beta^1, \dots, \beta^p\}$ dual to $\{Z, X_1, \dots, X_p, Y_1, \dots, Y_p\}$, by solving the Ambrose-Singer equations (1.1). For convenience we will consider the $(0, 3)$ type tensor field, also denoted by T , instead of the

(1, 2) tensor field already introduced, via the isomorphism $T_{UVW} = g(T(U, V), W)$. If T is a homogeneous Lorentzian structure on (H_{2p+1}, g) and $\tilde{\nabla} = \nabla - T$, then the first condition in (1.1) is equivalent to $T_{UVW} = -T_{UWV}$ for all $U, V, W \in \mathfrak{h}_{2p+1}$. One can write the second Ambrose-Singer equation $\tilde{\nabla}R = 0$ as

$$(3.1) \quad (\nabla_W R)(V_1, V_2, V_3, V_4) = -R(T_W V_1, V_2, V_3, V_4) - R(V_1, T_W V_2, V_3, V_4) \\ - R(V_1, V_2, T_W V_3, V_4) - R(V_1, V_2, V_3, T_W V_4).$$

for all $W, V_1, V_2, V_3, V_4 \in \mathfrak{h}_{2p+1}$. Substituting (V_1, V_2, V_3, V_4) by (Z, X_j, X_i, X_j) , $(Z, X_j, X_i, Y_j)_{(i \neq j)}$, $(X_j, X_k, X_j, Y_j)_{(j \neq k)}$ and $(X_i, X_j, Y_i, Y_k)_{(i \neq j \neq k, i \neq k)}$, we obtain, respectively

$$(3.2) \quad T_{WZ X_i} = -\frac{1}{2}\beta^i(W), \quad T_{WZ Y_i} = \frac{1}{2}\alpha^i(W), \quad T_{WX_j Y_k} = T_{WX_k Y_j}, \quad T_{WX_j X_k} = T_{WY_j Y_k}.$$

for all $W \in \mathfrak{h}_{2p+1}$.

Next, routine but long calculations can show that the condition (3.1) is satisfied if and only if (3.2) holds. Put

$$\theta_{jk}(W) = T_{WX_j Y_k} \quad \text{and} \quad \mu_{jk}(W) = T_{WX_j X_k},$$

for all $1 \leq j, k \leq p$. It follows that $\theta_{jk} = \theta_{kj}$ and $\mu_{jk} = -\mu_{kj}$. We can now calculate the components of the connection $\tilde{\nabla} = \nabla - T$ with respect to $\{Z, X_1, \dots, X_p, Y_1, \dots, Y_p\}$. By (3.1) and (3.2), we get

$$(3.3) \quad \begin{aligned} \tilde{\nabla}_V Z &= 0, \\ \tilde{\nabla}_V X_j &= \sum_{i=1}^p \mu_{ij}(V) X_i - \sum_{i=1}^p \theta_{ji}(V) Y_i + \frac{1}{2} \eta(V) Y_j, \\ \tilde{\nabla}_V Y_j &= \sum_{i=1}^p \theta_{ij}(V) X_i + \sum_{i=1}^p \mu_{ij}(V) Y_i - \frac{1}{2} \eta(V) X_j, \end{aligned}$$

for all $V \in \mathfrak{h}_{2p+1}$. By the last equation of (1.1) $(\tilde{\nabla}_V T)(W, V_1, V_2) = 0$ and replacing (V_1, V_2) by (X_j, Y_k) , (X_j, X_k) and (Y_j, Y_k) , we obtain respectively

$$(3.4) \quad \tilde{\nabla} \theta_{jk} = \sum_{i=1}^p (\theta_{ik} \wedge \mu_{ji} + \mu_{ik} \wedge \theta_{ji}), \quad \tilde{\nabla} \mu_{jk} = \sum_{i=1}^p (\mu_{ij} \wedge \mu_{ik} + \theta_{ji} \wedge \theta_{ki}).$$

Furthermore, it follows that (3.4) is equivalent to the condition $\tilde{\nabla}T = 0$. Hence, we proved the following:

Theorem 3.1. *All the homogeneous Lorentzian structures on the generalized Heisenberg group H_{2p+1} with the left-invariant Lorentzian metric g are given by*

$$T = -\frac{1}{2} \sum_{i=1}^p [\beta^i \otimes (\eta \wedge \alpha^i) - \alpha^i \otimes (\eta \wedge \beta^i)] + \sum_{j,k=1}^p \theta_{jk} \otimes (\alpha^j \wedge \beta^k) \\ + \sum_{j < k} \mu_{jk} \otimes (\alpha^j \wedge \alpha^k + \beta^j \wedge \beta^k),$$

where θ_{jk} and μ_{jk} ($1 \leq j, k \leq p$) are left-invariant 1-forms on H_{2p+1} satisfying $\theta_{jk} = \theta_{kj}$ and $\mu_{jk} = -\mu_{kj}$ and the equations (3.4).

In particular, for $p = 1$, we can consider the Lie algebra \mathfrak{h}_3 with generators $\{Z, X, Y\}$ and nonzero structure equations

$$[X, Y] = Z, \quad [X, Z] = [Y, Z] = 0.$$

The simply connected Lie group H_3 with the Lie algebra \mathfrak{h}_3 is the three-dimensional Heisenberg group. Now, we endow H_3 with the left-invariant Lorentzian metric defined by the inner product $\langle \cdot, \cdot \rangle$ for which $\{Z, X, Y\}$ is the orthonormal basis such that

$$\langle Z, Z \rangle = -1, \quad \langle X, X \rangle = \langle Y, Y \rangle = 1.$$

By Theorem 3.1, all the homogeneous Lorentzian structures on H_3 are given by

$$T = \theta \otimes (\alpha \wedge \beta) + \frac{1}{2} \alpha \otimes (\eta \wedge \beta) - \frac{1}{2} \beta \otimes (\eta \wedge \alpha),$$

where $\{\eta, \alpha, \beta\}$ is the basis dual to $\{Z, X, Y\}$ and θ is a 1-form on H_3 satisfying $\tilde{\nabla} \theta = 0$. Hence, from (3.3) we obtain

$$\tilde{\nabla}_V \eta = 0, \quad \tilde{\nabla}_V \alpha = - \left[\left(\theta - \frac{1}{2} \eta \right) (V) \right] \beta, \quad \tilde{\nabla}_V \beta = \left[\left(\theta - \frac{1}{2} \eta \right) (V) \right] \alpha,$$

thus

$$\tilde{\nabla}_V \theta = \left[\left(\theta - \frac{1}{2} \eta \right) (V) \right] \theta(Y) \alpha - \left[\left(\theta - \frac{1}{2} \eta \right) (V) \right] \theta(X) \beta.$$

Replacing V by Z, X and Y , the condition $\tilde{\nabla} \theta = 0$ implies $\theta(Z)$ is constant $\theta(X) = \theta(Y) = 0$. Then, one can write $\theta = a\eta$, $a \in \mathbb{R}$. Conversely, if $\theta = a\eta$ then $\tilde{\nabla} \theta = 0$. Therefore, we have the following (compare with [7] Theorem 2.1):

Theorem 3.2. *All the homogeneous Lorentzian structures on the 3-dimensional Heisenberg group H_3 with the left-invariant Lorentzian metric (2.1) with $p = 1$, are given by*

$$T = a\eta \otimes (\alpha \wedge \beta) + \frac{1}{2} \alpha \otimes (\eta \wedge \beta) - \frac{1}{2} \beta \otimes (\eta \wedge \alpha),$$

where a is an arbitrary real constant.

Acknowledgement. The first author was supported by ENSET d'Oran.

References

- [1] W. Ambrose and I.M. Singer, *On homogeneous Riemannian manifolds*, Duke Math. J. 25 (1958), 647–669.
- [2] W. Batat, G. Calvaruso and B. De Leo, *Curvature Properties of Lorentzian Manifolds with Large Isometry Groups*, Math. Phys. Anal. Geom. 12 (2009), 201–217
- [3] W. Batat, G. Calvaruso and B. De Leo, *Homogeneous Lorentzian 3-manifolds with a parallel null vector field*, Balkan J. Geom. Appl. 14, 1 (2009), 11-20.

- [4] P.M. Gadea and J.A. Oubiña, *Homogeneous Lorentzian structures on the oscillator groups*, Archiv der Math, 73 (1999), 311-320.
- [5] P.M. Gadea and J.A. Oubiña, *Homogeneous pseudo-Riemannian structures and homogeneous almost para-Hermitian structures*, Houston J. Math., (3) 18 (1992), 449-465.
- [6] P.M. Gadea and A.P. Ramos, *Homogeneous Lorentzian structures on some Godel-Levichev's spacetimes, and associated reductive decomposition*, Balkan J. Geom. Appl. 8, 1 (2003), 45-51.
- [7] N. Rahmani and S. Rahmani, *Structures homogènes Lorentziennes sur le groupe de Heisenberg I*, J. Geom. Phys. 13 (1994), 254-258.
- [8] F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lect. Notes 83, Cambridge Univ. Press, 1983.

Authors' addresses:

Wafaa Batat

Ecole Normale Supérieure de l'Enseignement Technologique d'Oran,
Département de Mathématiques et Informatique,
B.P. 1523, El M'Naouar, Oran, Algérie.
E-mail: wafa.batat@enset-oran.dz, wafa.batat@usc.es

Salima Rahmani

Universite de Haute Alsace,
Laboratoire de Mathematiques-Informatique et Applications,
68093 Mulhouse, Cedex, France;
Universite des Sciences et de la Technologie d'Oran,
Faculte des Sciences. Departement de Mathematiques,
Ecole Doctorale de Systemes Dynamiques et Geometrie,
B.P. 1505 Oran El-M'Naouer, Algeria.
E-mail: ramses161616@yahoo.fr