Generic Jacobi manifolds

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In memoriam
A la mémoire de mon père
A la mémoire de mon beau-père

Abstract. We build a stratification on the 1-jets of pairs (vector fields, twice contravariant tensors) which allows to define the notion of finite-dimensional generic Jacobi manifold. We then describe the singularities of the associated characteristic field. We also give an example of such manifolds in thermodynamics.

M.S.C. 2000: 53B50, 57N80.
Key words: Jacobi manifold; stratifications; genericity; thermodynamics.

1 Introduction

The notion of Jacobi manifold was introduced independently by Kirillov (cf [4]) and A. Lichnerowicz ([7]) in the study of contact structures using a contravariant approach.

A Jacobi manifold can be seen as a manifold M endowed with a skew-symmetric bidifferential operator on the functions space $\mathcal{C}^{\infty}(M)$ of maximal order 1. $\mathcal{C}^{\infty}(M)$ is then endowed with a structure of local Lie algebra, the so-called Jacobi bracket (cf [4]).

The Jacobi manifolds are a possible generalization of the notion of Poisson manifolds (which are a natural framework for analytic and quantic mechanics) and the notion of contact manifolds ([2], [10], [14]). Such manifolds can be used in the representation of energy in thermodynamics [11].

Weinstein's work about the local study of Poisson manifolds is well known ([15]). One can find an interesting study of local structure of Jacobi manifolds in [2].

In this paper, after having given the definition of a Jacobi manifold and some examples (part 2), we define a stratification on a set of 1-jets linked with the dimension of the characteristic space (part 3). This stratification allows to get the main result obtained in part 4 (theorem 4.1) about stratifications on n-dimensional generic Jacobi manifolds. The last part is devoted to the development of an example of such manifolds in thermodynamics.

Differential Geometry - Dynamical Systems, Vol.12, 2010, pp. 41-51.

2 Jacobi manifolds

2.1 Definition

Let M be a paracompact C^{∞} manifold of dimension n.

A m-tensor corresponds to a contravariant skew-symmetric m-tensor.

For each pair (A,B) where A is a m-tensor et B is a p-tensor, one considers a $(m+p-1)$ tensor $[A,B]$, called Schouten bracket of A and B (cf [3]) defined, for all $(m+p-1)$ closed form α as

$$i\,[A,B]\,\alpha=(-1)^{mp+p}\,i\,(A)\,di\,(B)\,\alpha+(-1)^{p}\,i\,(B)\,di\,(A)\,\beta.$$

We then have:

$$[A,B]=(-1)^{mp}\,[B,A]\,.$$

If one considers a r-tensor C, one gets the Jacobi identity:

$$(-1)^{mp}\,[[B,C]\,,A]+(-1)^{pr}\,[[C,A]\,,B]+(-1)^{rm}\,[[A,B]\,,C]=0.$$

A. Lichnerowicz gives in [6] the local components $[A,B]^{K_2\cdots K_{m+p}}$ of the $m+p-1$-tensor $[A,B]$:

$$\frac{1}{(m-1)!p!}\varepsilon^{K_2\cdots K_{m+p}}_{I_2\cdots I_m J_1\cdots J_p}A^{LI_2\cdots I_m}\frac{\partial B^{J_1\cdots J_p}}{\partial x^L}+\frac{1}{m!\,(p-1)!}\varepsilon^{K_2\cdots K_{m+p}}_{I_1\cdots I_m J_2\cdots J_p}B^{LJ_2\cdots J_p}\frac{\partial A^{I_1\cdots I_m}}{\partial x^L}$$

where ε is the generalized Kronecker symbol and where the Einstein convention is used.

For all 1-tensor (i.e. vector field) X, one has:

$$[X,\alpha]=L_X\alpha$$

where $L_X\alpha$ is the Lie derivative of the closed p-form α.

If P is a 2-tensor, we have:

$$i_{[P,P]}\alpha=2i\,(P)\,di\,(P)\,\alpha.$$

A skew-symmetric operator

$$D\,:\,C^{\infty}\,(M)\times C^{\infty}\,(M)\longrightarrow C^{\infty}\,(M)$$

which is bidifferential and of maximal order 1 can be written as (cf [5]):

$$D\,(f,g)=i\,(P)\,(df\wedge dg)+f\;i\,(X)\;\;dg-g\;i\,(X)\;\;df$$

where P is a 2-tensor and X is a vectors field.

We the obtain the following formula ([7]):

$$\underset{f,g,h}{\sigma}D\,(D\,(f,g)\,,h)=i\left(\frac{1}{2}\,[P,P]-X\wedge P\right)(df\wedge dg\wedge dh)-i\,[X,P]\left(\underset{f,g,h}{\sigma}hdf\wedge dg\right),$$

So D defines a Lie algebra structure on $C^{\infty}\,(M)$ if and only if:

$$\begin{cases}[P,P]= & 2X\wedge P\\ [X,P]= & 0.\end{cases}$$

A *Jacobi structure* on a manifold M of dimension n is a pair (X,P) where X is a vector field and P is a 2-tensor such that: $\left\{\begin{array}{ll}[P,P]= & 2X\wedge P\\ [X,P]= & 0\end{array}\right.$.
The vector field X is called *Reeb field*. The Jacobi structure is said *regular* if there exists an integer m such that $X\wedge P^m$ vanishes nowhere and $P^{m+1}=0$. The integer $n-(2m+1)$ is called *codimension* of the Jacobi manifold

2.2 Characteristic field of a Jacobi manifold

Let (M,X,P) be a Jacobi manifold. The space C spanned by X and range P is called characteristic field of the structure. The Jacobi manifold is called *transitive* if for all x in M we have $C_x=T_xM$.

C is not in general a sub-bundle of TM because its rank is not necessarily constant.

A. Kirillov proved that this field is integrable and defines a Stefan generalized foliation of M (foliation with singularities). In this case, the leaves are not necessarily of the same dimension; the even dimensional leaves can be endowed with a structure of symplectic manifold and the odd dimensional leaves can be endowed with a pfaffian structure (a Pffaf form ω is such that $\omega\wedge(d\omega)^m$ is a volume form).

2.3 Hamiltonian vector field

Let (M,X,P) be a Jacobi manifold. The 2-tensor P defines a vector bundle homomorphism $P^{\#}:T^*M\to TM$ which maps a function $f\in C^\infty(M)$ to a vector field $V_f=P^{\#}df+fX$ called hamiltonian field associated with f.

The mapping $f\mapsto V_f$ is a homomorphism of the Lie algebra $(C^\infty(M),\{,\})$ (where $\{,\}$ is the Jacobi bracket) into the Lie algebra $(\mathcal{X}(M),[,])$:

$$\{f,g\}=i_{V_f}dg-(i_X df)\,g.$$

If $X=0$, we have a Poisson structure and the classical relation $\{f,g\}=i_{V_f}dg$ where V_f is the hamiltonian vector field associated with f.

2.4 Examples of Jacobi manifolds

One can find in [7], [1] and [8] different examples of Jacobi manifolds.

1. Poisson manifold

   In the following examples one gets a Poisson structure, i.e. a Jacobi manifold where $X=0$

   (a) Lie-Poisson structure

   M is the dual $\mathcal{G}^*$ of a Lie algebra $\mathcal{G}$.

   The Lie-Poisson bracket of two functions f and g in $C^\infty(M)$ is defined as follows:

   $\{f,g\}(\alpha)=\langle\alpha,[f_*(\alpha),g_*(\alpha)]\rangle$

   where $\alpha\in\mathcal{G}^*$ and where $f_*(\alpha)$ and $g_*(\alpha)$ are seen as elements of $\mathcal{G}$.

We then have the Lie Poisson or KKS (Kirillov, Konstant, Souriau) structure P, whose the expression in the basis associated with the global coordinates $\left(a^k\right)$ of $\mathcal{G}^*$ is:

$$P=\sum_{i<j}\sum_k c_{ij}^k \alpha_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$$

where c_{ij}^k are the constants of structure of the Lie algebra $\mathcal{G}$.

(b) Symplectic manifold

A symplectic manifold (M,Ω) is an even dimensional manifold M endowed with a closed 2-form of maximal rank Ω. We associate to this 2-form a 2-tensor P. The condition $d\Omega=0$ implies $[P,P]=0$. So (M,P) is a Poisson manifold.

2. Contact manifold

Let M be a manifold of dimension $2m+1$ and θ a 1-form on M. θ is a contact form if $\theta\wedge(d\theta)^m$ does not vanish. (M,θ) is called a contact manifold. One can find in a neighborhood of every point canonical coordinates $\left(t,q^1,\dots,q^m,p_1,\dots,p_m\right)$ such that the form θ can be written as:

$$\theta=dt-\sum_{i=1}^m p_i dq^i.$$

A contact manifold can be endowed with a structure of Jacobi manifold where the 2-tensor P is defined as: $P(\alpha,\beta)=d\theta\left(\flat^{-1}(\alpha),\flat^{-1}(\beta)\right)$ where $\flat:\chi(M)\to\chi^*(M)$ is the isomorphism of $C^\infty(M)$-modules defined by: $\flat(V)=i_V d\theta+(i_V\theta)\,\theta$.
The associated Reeb field X is characterized by the relations $i_X\theta=1$ and $i_X d\theta=0$.
We then obtain the following expressions of P and X:

$$P=\sum_{i=1}^m\left(\frac{\partial}{\partial q^i}+p_i\frac{\partial}{\partial t}\right)\wedge\frac{\partial}{\partial p^i}\qquad X=\frac{\partial}{\partial t}.$$

3. Let E be a smooth vectors field on a manifold M.

One can define a bracket on $C^\infty(M)$ as follows:

$$\{f,g\}=\langle f dg-g df,E\rangle$$

M endowed with this bracket is a Jacobi manifold.

3 Stratification of pairs of jets (1-tensor, 2-tensor) on a manifold

3.1 Stratification in the space of pairs (vector field, 2-tensor) in $\mathbb{R}^n$

A bivector w of $\mathbb{R}^n$ is a skew-symmetric bilinear form on $(\mathbb{R}^n)^*$ seen as a linear mapping : $w^\# : (\mathbb{R}^n)^* \to \mathbb{R}^n$ defined by $\left\langle \beta, w^\#(\alpha)\right\rangle = w(\alpha,\beta)$ where α and β are covectors of $\mathbb{R}^n$. The range of $w^\#$ is called support of the bivector w and is denoted by $\operatorname{supp} w$.

One introduces the following sets:

$\mathcal{W}$: space of bivectors of $\mathbb{R}^n$ (dimension $\dfrac{n(n-1)}{2}$)

$\mathcal{J} = \mathbb{R}^n \times \mathcal{W}$ (dimension $\dfrac{n(n+1)}{2}$)

$E(v,w) = \mathbb{R}v + \operatorname{supp} w$, called *characteristic space of* (v,w).

There exists an action of $GL(n,\mathbb{R})$ on $\mathcal{J}$ defined by $(A,(v,w)) \mapsto (A\bullet v, A\bullet w)$ where $A\bullet v = A(v)$ and where $A\bullet w(\alpha,\beta) = w({}^tA\alpha, {}^tA\beta)$.

We are going to define a stratification S on $\mathcal{J}$ invariant under the action of $GL(n,\mathbb{R})$ above.

If w is a bivector of $\mathbb{R}^n$ of rank r, we denote $c = n - r$ the corank of w.

It is well known that $\mathcal{W}$ can be stratified according to the rank. We then get, on $\mathcal{W}$, the finite stratification $\sigma = \{\sigma_c\}_{0\le c\le n}$ where

$$\sigma_c = \{w \in \mathcal{W},\ \operatorname{corang} w = c\}$$

is a submanifold of $\mathcal{W}$ of codimension $\dfrac{c(c-1)}{2}$ and where $\overline{\sigma_c}$ is an algebraic submanifold of $\mathcal{W}$ such that $\overline{\sigma_c} = \bigcup\limits_{c'\ge c} \sigma_{c'}$.

We then introduce the following subsets S^i, $i = 0,\dots,n$, of $\mathcal{J}$:

$$S^i = \{(v,w) \in \mathcal{J} : \dim E(v,w) = i\}.$$

The following lemma whose proof is obvious gives for even and odd dimensional manifolds some properties of these subsets:

Lemma 3.1. *1) We have the following characterizations:*
a) *if M is even dimensional $2m$:*
for $0 \le k \le m$, $S^{2k} = \{(v,w) \in \mathcal{J} : \dim(\operatorname{supp} w) = 2k,\ v \in \operatorname{supp} w\}$
for $1 \le k \le m$, $S^{2k-1} = \{(v,w) \in \mathcal{J} : \dim(\operatorname{supp} w) = 2k-2,\ v \notin \operatorname{supp} w\}$.
b) *if M is odd dimensional $2m+1$:*
for $0 \le k \le m$, $S^{2k+1} = \{(v,w) \in \mathcal{J} : \dim(\operatorname{supp} w) = 2k,\ v \notin \operatorname{supp} w\}$
$S^{2k} = \{(v,w) \in \mathcal{J} : \dim(\operatorname{supp} w) = 2k,\ v \in \operatorname{supp} w\}$.
2) If $(v,w) \in S^i$ then $A\bullet w \in S^i$ (for any A in $GL(n,\mathbb{R})$).

We then get the stratification $\mathcal{J}$:

Proposition 3.1. *1)* $S=\left\{S^i\right\}_{i=0,\ldots,n}$ *is a stratification of* $\mathcal{J}$ *where each stratum* S^i *(*$i=0,\ldots,n$*) is a submanifold of* $\mathcal{J}$ *invariant with respect to the action of* $GL(n,\mathbb{R})$ *; moreover we have:* a) *if* M *is even dimensional* $2m$*,* S^{2m} *is open*

for $0\leq k\leq m-1$*,* S^{2k+1} *is a submanifold of codimension* $(m-k)(2m-2k-1)$

S^{2k} *is a submanifold of codimension* $(m-k)(2m-2k+1)$.

b) *if* M *is odd dimensional* $2m+1$*,* S^{2m+1} *is open*

for $0\leq k\leq m$*,* S^{2k+1} *is a submanifold of codimension* $(m-k)(2m-2k+1)$

S^{2k} *is a submanifold of codimension* $(m-k)(2m-2k+3)$.

2) We also have the following topological properties : $\forall\ i\in\{0,\ldots,n\},\ \overline{S^i}=\bigcup\limits_{i'\leq i}S^{i'}$

Proof. 1) We indicated above that we had a stratification $\{\sigma_c\}_{0\leq c\leq n}$ de $\mathcal{W}$; the result is analogous to the result obtained in [9] on the stratification of the set of skew-symmetric bilinear forms of $\mathbb{R}^n$. Consider the projection $\begin{array}{cccc}\pi: & \mathcal{J} & \to & \mathcal{W}\\ & (v,w) & \mapsto & w\end{array}$ which is a submersion.

a) Assume the manifold M is even dimensional ($n=2m$). We then have for $0\leq k\leq m-1$ and $c=2m-2k$, $\pi^{-1}(\sigma_c)=\mathbb{R}^n\times\sigma_c=S^{2k+1}\cup S^{2k}$ which is a submanifold fibred on σ_c whose codimension is $\dfrac{c(c-1)}{2}=(m-k)(2m-2k-1)$. Consider w_{2k}^0 in σ_{2k} and find out the dimension of $\pi^{-1}\left(w_{2k}^0\right)$: there exists a basis $(e_1,e_2,\ldots,e_{2k-1},e_{2k})$ of $\mathbb{R}^{2m}$ such that: $w_{2k}^0=e_1\wedge e_2+\cdots+e_{2m-1}\wedge e_{2m}$; a vector $u(u_1,\ldots,u_{2m})$ belongs to the support of w_{2k}^0 iff we have $2m-2k$ independent equations: $u_{2k+1}=0,\ldots,u_{2m}=0$. Therefore, S^{2k+1} is an open subset of the submanifold $S^{2k+1}\cup S^{2k}$ and S^{2k} is of codimension $2m-2k$ in $S^{2k+1}\cup S^{2k}$.

b) For an odd dimensional manifold we have an analogous proof.

2) The topological property results from $\overline{\sigma_c}=\bigcup\limits_{c'\geq c}\sigma_{c'}$ □

3.2 Stratification generated on the 1-jets of pairs (1-tensor, 2-tensor)

Let $\mathcal{J}(M)$ be the bundle $TM\underset{M}{\times}\Lambda^2TM$. It is clear that $\mathcal{J}(M)\to M$ is a vector bundle whose fiber is $\mathcal{J}$ and we have an action of $GL(n,\mathbb{R})$ on this fiber.

$\mathcal{J}$ can be endowed with the stratification S (as defined above) and each stratum S^i is invariant under the action of the Lie group $GL(n,\mathbb{R})$. Using this stratification along the manifold M one generates a stratification $S(M)$ on the bundle $\mathcal{J}(M)$ (cf [9], [12], [13] for example).

Let $\underline{\mathcal{J}(M)}$ be the set of sections C^∞ of the bundle $\mathcal{J}(M)\to M$ endowed with the C^∞ Whitney topology. We then have a structure of Baire topological vector space.

The natural projection $\begin{array}{cccc}\pi_1: & J^1\left(\underline{\mathcal{J}(M)}\right) & \to & J^0\left(\underline{\mathcal{J}(M)}\right)\cong\mathcal{J}(M)\\ & j_x^1(X,P) & \mapsto & j_x^0(X,P)\end{array}$ is a submersion.

We have a stratification $\Sigma(M)$ on $J^1\left(\underline{\mathcal{J}(M)}\right)$ as a pull-back of $S(M)$ by the submersion π_1.

The subset $\widetilde{\mathcal{J}} = \left\{(X,P) \in \underline{\mathcal{J}(M)} : [P,P] = 2X \wedge P, [X,P] = 0\right\}$ is endowed with the induced topology.

Thanks to Thom transversality theorem we know that the set $\tau\widetilde{\mathcal{J}}$ of the elements of $\widetilde{\mathcal{J}}$ whose 1-jet is transverse to $\Sigma(M)$ is residual in $\widetilde{\mathcal{J}}$.

In the next part, we shall prove that $\tau\widetilde{\mathcal{J}}$ is non empty. So to every generic element J, i.e. belonging to $\tau\widetilde{\mathcal{J}}$, it will correspond a stratification $S(J)$ on the manifold M obtained by pull-back.

4 Stratification on a generic Jacobi manifold

The main result of this paper is the following theorem:

Theorem 4.1. *1) On the manifolds of dimension n, generically in the set of pairs $J = (X,P)$ where X is a vector field and P is a 2-tensor fulfilling the conditions:* $\left\{\begin{array}{ll} [P,P] = & 2X \wedge P \\ [X,P] = & 0 \end{array}\right.$ *, there exists a stratification of the manifold:* $S(J) = \left\{S^i(J)\right\}_{I(n) \leq i \leq n}$ *such that above the stratum $S^i(J)$, if it is non empty, for $i = I(n), \ldots, n$, the characteristic field is of dimension i.*
2) Moreover, one has the following properties:
a)*if the manifold M is even dimensional ($n = 2m$), $S^{2m}(J)$, is open and: for all i such that $(m-i)(2m-2i-1) \leq 2m$, $S^{2i+1}(J)$, if non empty, is a submanifold of codimension $(m-i)(2m-2i-1)$. For all i such that $(m-i)(2m-2i+1) \leq 2m$, $S^{2i}(J)$, if no empty, is a submanifold of codimension $(m-i)(2m-2i+1)$.* b) *if the manifold M is odd dimensional ($n = 2m+1$), $S^{2m+1}(J)$ is open and: for all i such that $(m-i)(2m-2i+1) \leq 2m$, $S^{2i+1}(J)$, if non empty, is a submanifold of codimension $(m-i)(2m-2i+1)$. for any i such that $(m-i)(2m-2i+3)$ $\leq 2m$, $S^{2i}(J)$, if non empty, is a submanifold of codimension$(m-i)(2m-2i+3)$.*

According to the results of the proposition 3.1, we have to prove that the set $\tau\widetilde{\mathcal{J}}$ is non empty.

Each following proposition gives an example of a pair whose 1-jet is transverse to the non open stratum of minimum codimension.

Proposition 4.1. *If M is a manifold of dimension $2m$,*

$$j_{2m-1} = \begin{pmatrix} \dfrac{\partial}{\partial x^{2m}}, \\ \sum\limits_{k=1}^{m-1} \left(\dfrac{\partial}{\partial x^{2k-1}} \wedge \dfrac{\partial}{\partial x^{2k}} + 2x^{2k} \dfrac{\partial}{\partial x^{2k}} \wedge \dfrac{\partial}{\partial x^{2m}} \right) \\ +x^{2m-1} \dfrac{\partial}{\partial x^{2m-1}} \wedge \dfrac{\partial}{\partial x^{2m}} \end{pmatrix}$$

is an element of $\widetilde{\mathcal{J}}$ whose 1-jet is transverse to the stratum $\Sigma^{2m-1}(M)$.

Proof. Consider a local chart centered at a point x_0 of the manifold M of dimension $2m$, whose coordinates $\left(x^1, \ldots, x^{2m}\right)$ are such that the vector field X is written as

$\dfrac{\partial}{\partial x^{2m}}$; the 2-tensor P can be written as:

$$P = \sum_{1 \leq i < j \leq 2m} P^{\,i\ j} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

The condition $[X, P] = 0$ corresponds to the $\left(\begin{array}{c} 2m \\ 2 \end{array}\right)$ equations :

$$\frac{\partial P^{\,i\ j}}{\partial x^{2m}} = 0, \quad 1 \leq i < j \leq 2m$$

and the condition $[P, P] = 2X \wedge P$ gives the $\left(\begin{array}{c} 2m \\ 3 \end{array}\right)$ equations:

$$\underset{i,j,k}{\sigma} \sum_{1 \leq r \leq 2m} P^{\,i\ r} \frac{\partial P^{\,j\ k}}{\partial x^r} \quad = \quad 2\phi^{i\ j\ k}\ P^{\,j\ k}, \quad 1 \leq i < j < k \leq 2m,$$

where $\phi^{i\ j\ k} = 0$ if $k = 2m$ and $\phi^{i\ j\ k} = 1$ if $k \neq 2m$.

An easy computation gives $j_{2m-1} \in \widetilde{\mathcal{J}}$.
Let us prove that $j^1\left(j_{2m-1}\right)$ is transverse to the stratum $\Sigma^{2m-1}(M)$ at x_0.
The local equation of $\Sigma^{2m-1}(M)$ is obtained from the condition $\det P = 0$, which gives :

$$\sum_{\substack{\sigma \in \sigma_{2m} \\ \sigma(1)<\sigma(2),\ldots,\sigma(2m-1)<\sigma(2m)}} y^{\sigma(1)\ \sigma(2)} y^{\sigma(3)\ \sigma(4)} \ldots y^{\sigma(2m-1)\ \sigma(2m)} = 0.$$

The tangent space at $j^1\left(j_{2m-1}\right)_{x_0}$ to $\Sigma^{2m-1}(M)$ has the following equation:

$$\left(dy^{2m-1\ 2m}\right)_{j^1\left(j_{2m-1}\right)_{x_0}} = 0.$$

$\left(j_{2m-1\ *}\right)_{x_0}$ maps the tangent vector $\left(\dfrac{\partial}{\partial x^{2m-1}}\right)_{x_0}$ to $\left(\dfrac{\partial}{\partial y^{2m-1\ 2m}}\right)_{j^1\left(j_{2m-1}\right)_{x_0}}$.
Because this vector does not belong to $T_{j^1\left(j_{2m-1}\right)_{x_0}} \Sigma^{2m-1}(M)$ and because $\Sigma^{2m-1}(M)$ is a hypersurface, the proof is completed. □

Proposition 4.2. *If M is a manifold of dimension $2m+1$,*

$$j_{2m} = \left(\begin{array}{c} \dfrac{\partial}{\partial x^{2m}}, \\ \displaystyle\sum_{k=0}^{m-2} \left(\frac{\partial}{\partial x^{2k+1}} \wedge \frac{\partial}{\partial x^{2k+2}} + 2x^{2k+2} \frac{\partial}{\partial x^{2k+2}} \wedge \frac{\partial}{\partial x^{2m}} \right) \\ + \left(1 + 2x^{2m-1}\right) \dfrac{\partial}{\partial x^{2m-1}} \wedge \dfrac{\partial}{\partial x^{2m}} + x^{2m+1} \dfrac{\partial}{\partial x^{2m-1}} \wedge \dfrac{\partial}{\partial x^{2m+1}} \end{array} \right)$$

is an element of $\widetilde{\mathcal{J}}$ whose 1-jet is transverse to the stratum $\Sigma^{2m}(M)$.

Proof. As in the previous proposition consider a local chart centered at a point x_0 of the manifold of dimension $2m+1$, whose coordinates $\left(x^1,\ldots,x^{2m},x^{2m+1}\right)$ are such that the vector field X is written as $\dfrac{\partial}{\partial x^{2m}}$ and the $2-$tensor P is written as:

$$P=\sum_{1\leq i<j\leq 2m+1} P^{\,i\,j}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}$$

where $P_{x_0}=\sum\limits_{1\leq i<j\leq 2m}\left(\dfrac{\partial}{\partial x^i}\right)_{x_0}\wedge\left(\dfrac{\partial}{\partial x^j}\right)_{x_0}$.
The skew-symmetric matrix $\widetilde{p}=\left(P^{i\,j}\right)$ where $1\leq i<j\leq 2m$ has, in a neighborhood of x_0, a determinant non equal to 0.
Consider, for $1\leq i\leq 2m$, the $2m\times 2m$ matrix $\widehat{p}_i$ obtained from the matrix of P by taking the i^{th} row and the i^{th} column off.
The range P is then spanned, in a neighborhood of x_0, by the $2m$ vector fields $V_1,\ldots,V_{2m}$ where

$$V_i=\frac{\partial}{\partial x^i}\pm\sqrt{\frac{\det\widehat{p}_i}{\det\widetilde{p}}}.\frac{\partial}{\partial x^{2m+1}}$$

$\left(\dfrac{\partial}{\partial x^{2m}}\right)_x\in\operatorname{im}P_x$ iff $\det\widehat{p}_{2m}=0$
The local equation of $\Sigma^{2m}(M)$ is:

$$\sum_{\substack{\sigma\in\sigma_{2m+1}\\ \sigma(1)<\sigma(2),\ldots,\sigma(2m-1)<\sigma(2m)\\ 2m\notin\operatorname{im}\sigma}} y^{\sigma(1)\ \sigma(2)}y^{\sigma(3)\ \sigma(4)}\ldots y^{\sigma(2m-1)\ \sigma(2m)}=0.$$

So an equation of the tangent space at $j^1\left(j_{2m}\right)_{x_0}$ to the stratum $\Sigma^{2m}(M)$ is

$$\left(dy^{2m-1\ 2m+1}\right)_{j^1\left(j_{2m}\right)_{x_0}}=0.$$

It is clear that $j^1\left(j_{2m}\right)_{x_0}$ maps the tangent vector $\left(\dfrac{\partial}{\partial x^{2m+1}}\right)_{x_0}$ to $\left(\dfrac{\partial}{\partial y^{2m-1\ 2m+1}}\right)_{j^1\left(j_{2m}\right)_{x_0}}$ which does not belong to $T_{j^1\left(j_{2m}\right)_{x_0}}\Sigma^{2m}(M)$. Because $\Sigma^{2m}(M)$ is a hypersurface, we get the transversality. It is easy to obtain: $j_{2m}\in\widetilde{\mathcal{J}}$. The proof is complete. □

5 Example in thermodynamics

Recall that the Jacobi manifolds used as a framework in thermodynamics are open subsets of the contact manifold $\mathbb{R}^{2m+1}$ (cf [11]).

In the representation of energy, we use the following coordinates:
$\left(t,q^1,\ldots,q^m,p_1,\ldots,p_m\right)=\left(U,S,V,N^1,\ldots,N^{m-2},-T,P,-\mu_1,\ldots,-\mu_{m-2}\right)$.
The contact form may be written as :
$\theta=dU-TdS+PdV-\sum\limits_{k=1}^{m-2}\mu_k dN^k$.

Using the contact form θ, seen as a connection form on a certain bundle, one can associate a vector field X_f to a function f defined by the equations $i_{X_f} d\theta = -Df$ and $i_{X_f}\theta = f$ where Df is the covariant derivative of f.

In the coordinates $\left(t, q^1, \ldots, q^m, p_1, \ldots, p_m\right)$, the flow of this vector field is given by the equations:

$$\begin{cases} \dot{t} = f - p_i \dfrac{\partial f}{\partial p_i} \\ \dot{p_i} = p_i \dfrac{\partial f}{\partial t} - \dfrac{\partial f}{\partial q^i} \\ \dot{q^i} = \dfrac{\partial f}{\partial p_i} \end{cases}$$

In the case of $f = PV - NRT$ the integral curves of the field

$X_f = -P\dfrac{\partial}{\partial P} - RT\dfrac{\partial}{\partial \mu} + NR\dfrac{\partial}{\partial S} + V\dfrac{\partial}{\partial V}$

are given by:

$U = U_0, \quad T = T_0, \quad N = N_0, \quad P = P_0 e^{-t}, \quad V = V_0 e^t,$

$S = S_0 + N_0 R t, \quad \mu = \mu_0 - RT_0 t.$

Let us prove that such Jacobi manifolds are generic, because lying in the open stratum S^{2m+1}.

The matrix U of the $2m+1$ vectors spanning the characteristic field C in the coordinates $\left(U, S, V, N^1, \ldots, N^{m-2}, -T, P, -\mu_1, \ldots, -\mu_{m-2}\right)$ is:

$$\begin{pmatrix}
0 & -1 & & & & & & & & 0 \\
1 & 0 & & & & & & & & 0 \\
 & & 0 & -1 & & & & & & 0 \\
 & & 1 & 0 & & & & & & 0 \\
 & & & & 0 & -1 & & & & 0 \\
 & & & & 1 & 0 & & & & 0 \\
 & & & & & & \ddots & & & \vdots \\
 & & & & & & & 0 & -1 & 0 \\
 & & & & & & & 1 & 0 & 0 \\
0 & T & 0 & -P & 0 & \mu_1 & \cdots & 0 & \mu_{m-2} & 1
\end{pmatrix}$$

It is obvious that $det(U) = 1$. So $dim(C) = 2m + 1$.

References

[1] D. Chinea, J. C. Marrero, M. de Leon, *Prequantizable Poisson manifolds and Jacobi structures*, J. Phys. A: Math. Gen. 29 (1996), 6313-6324

[2] P. Dazord, A. Lichnerowicz, C.-M. Marle, *Structure locale des variétés de Poisson*, J. Math. Pures et Appl., 70 (1991), 101-152

[3] A. Frolicher, A. Nijenhuis, *Theory of vector-valued Differential Forms*, I, Nedel. Akad. Wetinsch. Proc., 59 (1956), 338-359.

[4] A. Kirillov, *Local Lie algebras*, Russian Math. Surveys 31, 4 (1976), 55-75.

[5] P. Libermann, C.-M. Marle, *Géométrie symplectique, bases théoriques de la Mécanique*, Publications Mathématiques de l'Université Paris VII, 1986.

[6] A. Lichnerowicz, *Les variétés de Poisson et leurs algèbres de Lie associées*, J. of Differential Geometry 12 (1977), 253-300.

[7] A. Lichnerowicz, *Les variétés de Jacobi et leurs algèbres de Lie associées*, J. Math. Pures et Appliquées 51 (1978), 453-488.

[8] C.-M. Marle, *On Jacobi manifolds and Jacobi bundles*, In "Symplectic geometry, groupoids and integrable systems" (Eds: P. Dazord and A. Weinstein), MSRI Publications 20, Springer (1991), 227246.

[9] J. Martinet, *Sur les singularités des formes différentielles*, Annales de l'Institut Fourier, Grenoble, 1970.

[10] A.M. Mendes-Justino, *Quelques propriétés des variétés de Poisson et des variétés de Jacobi*, Thèse $3^{ème}$ cycle, Paris VI, 1984.

[11] R. Mrugala, *Lie, Jacobi, Poisson and quasi-Poisson structures in thermodynamics,* Tensor, N. S. 56 (1995), 37-45.

[12] F. Pelletier, *Singularités d'ordre supérieur de formes différentielles*, Thèse, Université de Dijon, 1980.

[13] F. Pelletier, P. Cabau, *Generalized PN manifolds and separation of variables*, Banach Center Publications, Geometry and Topology of Caustics 82 (2008), 163-181.

[14] M. Tarafdar, A. Bhattacharyya, T. De, *Some types of contact manifolds*, Differ. Geom. Dyn. Syst. 9, (2007), 149-155.

[15] A. Weinstein, *The local structure of Poisson manifolds*, Journal of Differential Geometry 18 (1983), 523-557.

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