On the m-projective curvature tensor of a Kenmotsu manifold

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Abstract. In this paper, we study the properties of the $M-$projective curvature tensor in Riemannian and Kenmotsu manifolds.

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Key words: Riemannian manifold; Kenmotsu manifold; $M-$projective curvature tensor; $\eta-$Einstein manifold and irrotational $M-$projective curvature tensor.

1 Introduction

Let M_n be an $n-$dimensional differentiable manifold of differentiability class C^{r+1} with a $(1,1)$ tensor field ϕ, the associated vector field ξ, a contact form η and the associated Riemannian metric g. In 1958, Boothby and Wong [1] studied odd dimensional manifolds with contact and almost contact structures from topological point of view. Sasaki and Hatakeyama [14] re-investigated them using tensor calculus in 1961. In 1972, K. Kenmotsu studied a class of contact Riemannian manifold and call them Kenmotsu manifold [7]. He proved that if a Kenmotsu manifold satisfies the condition $R(X,Y).R = 0$, then the manifold is of negative curvature -1, where R is the Riemannian curvature tensor of type $(1,3)$ and $R(X,Y)$ denotes the derivation of the tensor algebra at each point of the tangent space. In 1963, Kobayashi and Nomizu [8] shown that any two simply connected complete Riemannian manifolds of constant curvature k are isometric to each other. A space form (i.e., a complete simply connected Riemannian manifold of constant curvature) is said to be elliptic, hyperbolic or euclidean according as the sectional curvature tensor is positive, negative or zero [2]. The properties of Kenmotsu manifold have been studied by several authors such as De [3], Sinha and Srivastava [15], Jun, De and Pathak [6], De and Pathak [4], De, Yildiz and Yaliniz [5], C̈ihan and De [12] and many others.

In this paper, we studied the properties of the Kenmotsu manifold equipped with $M-$projective curvature tensor. Section 2 consist the basic definitions of the Kenmotsu and $\eta-$Einstein manifolds. Section 3 is the study of $M-$projective curvature tensor in the Riemannian manifold and obtain the relation between different curvature tensors. In section 4, we prove that an $n-$dimensional Kenmotsu manifold M_n is $M-$projectively flat if and only if it is either locally isometric to the hyperbolic space $H^n(-1)$ or M_n has constant scalar curvature $-n(n-1)$. Section 5 deals

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with $W^*(\xi, X).R = 0$ and obtain some interesting results. In section 6, we proved that the $M-$projective curvature tensor in an $\eta-$Einstein Kenmotsu manifold M_n is irrotational if and only if it is locally isometric to the hyperbolic space $H^n(-1)$.

2 Preliminaries

If on an odd dimensional differentiable manifold M_n, $n = 2m + 1$, of differentiability class C^{r+1}, there exist a vector valued real linear function ϕ, a $1-$form η, the associated vector field ξ and the Riemannian metric g satisfying

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for arbitrary vector fields X and Y, then (M_n, g) is said to be an almost contact metric manifold and the structure $\{\phi, \eta, \xi, g\}$ is called an almost contact metric structure to M_n [9].

In view of (2.1), (2.2) and (2.3), we find

$$\eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \quad \phi(\xi) = 0. \tag{2.4}$$

If moreover,

$$(D_X\phi)(Y) = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{2.5}$$

and

$$D_X\xi = X - \eta(X)\xi, \tag{2.6}$$

where D denotes the operator of covariant differentiation with respect to the Riemannian metric g, then $(M_n, \phi, \xi, \eta, g)$ is called a Kenmotsu manifold [7]. Also, the following relations hold in Kenmotsu manifold [5], [4], [6]

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.8}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.9}$$

$$\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z), \tag{2.10}$$

for arbitrary vector fields X, Y, Z.

A Kenmotsu manifold (M_n, g) is said to be $\eta-$Einstein if its Ricci-tensor S takes the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \tag{2.11}$$

for arbitrary vector fields X, Y; where a and b are functions on (M_n, g). If $b = 0$, then $\eta-$Einstein manifold becomes Einstein manifold. Kenmotsu [7] proved that if (M_n, g) is an $\eta-$Einstein manifold, then $a + b = -(n-1)$.

In view of (2.4) and (2.11), we have

$$QX = aX + b\eta(X)\xi, \tag{2.12}$$

where Q is the Ricci operator defined by

$$S(X,Y) \stackrel{\text{def}}{=} g(QX,Y). \tag{2.13}$$

Again, contracting (2.12) with respect to X and using (2.4), we have

$$r = na + b. \tag{2.14}$$

Now, substituting $X = \xi$ and $Y = \xi$ in (2.11) and then using (2.4) and (2.9), we obtain

$$a + b = -(n-1). \tag{2.15}$$

Equations (2.14) and (2.15) gives

$$a = \left(\frac{r}{n-1} + 1\right) \quad \textit{and} \quad b = -\left(\frac{r}{n-1} + n\right). \tag{2.16}$$

3 The M-projective curvature tensor

In 1971, G. P. Pokhariyal and R. S. Mishra [13] defined a tensor field W^* on a Riemannian manifold as

$$\begin{aligned} W^*(X,Y)Z = R(X,Y)Z - \tfrac{1}{2(n-1)}[S(Y,Z)X \\ -S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \end{aligned} \tag{3.1}$$

so that

$$'W^*(X,Y,Z,U) \stackrel{\text{def}}{=} g(W^*(X,Y)Z,U) =' W^*(Z,U,X,Y)$$

and

$$'W^*_{ijkl}w^{ij}w^{kl} =' W_{ijkl}w^{ij}w^{kl},$$

where $'W^*_{ijkl}$ and $'W_{ijkl}$ are components of $'W^*$ and $'W$ respectively and w^{kl} is a skew-symmetric tensor [10], [16]. Such a tensor field W^* is known as $M-$projective curvature tensor. Second author [11], [10] defined and studied the properties of $M-$projective curvature tensor in Sasakian and Kähler manifolds. He has also shown that it bridges the gap between conformal curvature tensor, con-harmonic curvature tensor and con-circular curvature tensor on one side and $H-$projective curvature tensor on the other.

The Weyl projective curvature tensor W, con-circular curvature tensor C and conformal curvature tensor V are given by [9]

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\} \tag{3.2}$$

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\} \tag{3.3}$$

$$\begin{aligned} V(X,Y)Z &= R(X,Y)Z - \tfrac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &-g(X,Z)QY] + \tfrac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{3.4}$$

Theorem 3.1. *The $M-$projective and Weyl projective curvature tensors of the Riemannian manifold M_n are linearly dependent if and only if M_n is an Einstein manifold.*

Proof. We consider,

$$W^*(X,Y)Z = \alpha W(X,Y)Z,$$

where α being any non-zero constant. In view of (3.1) and (3.2), above equation becomes

$$\begin{aligned} &(1-\alpha)R(X,Y)Z + \left(\tfrac{\alpha}{n-1} - \tfrac{1}{2(n-1)}\right)\{S(Y,Z)X - S(X,Z)Y\} \\ &-\tfrac{1}{2(n-1)}\{g(Y,Z)QX - g(X,Z)QY\} = 0 \end{aligned}$$

Contracting last result with respect to X, we get

$$S(Y,Z) = \frac{r}{n}g(Y,Z) \Longleftrightarrow QY = \frac{r}{n}Y, \tag{3.5}$$

which gives the first part of the theorem. In consequence of (3.1), (3.2) and (3.5), we obtain the converse part of the theorem. □

Theorem 3.2. *The necessary and sufficient condition for a Riemannian manifold to be an Einstein manifold is that the $M-$projective curvature tensor W^* and concircular curvature tensor C are linearly dependent.*

Proof. Let

$$W^*(X,Y,Z) = \alpha C(X,Y,Z),$$

where α is a non zero scalar. In consequence of (3.1) and (3.3), above equation becomes

$$\begin{aligned} &(1-\alpha)R(X,Y,Z) - \tfrac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &-g(X,Z)QY] + \tfrac{\alpha r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] = 0. \end{aligned}$$

Contracting above with respect to X, we obtain (3.5). Converse part is obvious from (3.1), (3.3) and (3.5). □

Theorem 3.3. *A Riemannian manifold becomes an Einstein manifold if and only if conformal and $M-$projective curvature tensors of the manifold are linearly dependent.*

The proof is straight forward as theorem (3.4).

Corollary 3.4. *In an $n-$dimensional Riemannian manifold M_n, the following are equivalent*

(i) M_n is an Einstein manifold,

(ii) $M-$projective and Weyl projective curvature tensors are linearly dependent.

(iii) $M-$projective and Con-circular curvature tensors are linearly dependent.

(iv) $M-$projective curvature and conformal curvature tensors are linearly dependent.

4 $M-$projectively flat Kenmotsu manifolds

In view of $W^* = 0$, (3.1) becomes

$$R(X,Y)Z = \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{4.1}$$

Replacing Z by ξ in (4.1) and then using (2.4), (2.7) and (2.9), we obtain

$$(n-1)\left(\eta(X)Y - \eta(Y)X\right) = \eta(Y)QX - \eta(X)QY.$$

Again putting $Y = \xi$ in the above relation and using (2.4) and (2.9), we have

$$QX = -(n-1)X \Longleftrightarrow S(X,Y) = -(n-1)g(X,Y) \tag{4.2}$$

and

$$r = -n(n-1). \tag{4.3}$$

In consequence of (4.2), (4.1) becomes

$$R(X,Y)Z = -\left\{g(Y,Z)X - g(X,Z)Y\right\}. \tag{4.4}$$

A space form is said to be hyperbolic if and only if the sectional curvature tensor is negative [2]. Thus, we can state

Theorem 4.1. *An $n-$dimensional Kenmotsu manifold M_n is $M-$projectively flat if and only if it is either locally isometric to the hyperbolic space $H^n(-1)$ or M_n has constant scalar curvature $-n(n-1)$.*

5 An $\eta-$Einstein Kenmotsu manifold satisfying a certain condition

In view of (2.4), (2.8), (2.11), (2.12) and (2.16), (3.1) becomes

$$W^*(\xi,X)Y = \frac{1}{2}\left\{1 + \frac{1}{n-1}\left(\frac{r}{n-1} + 1\right)\right\}\left\{\eta(Y)X - g(X,Y)\xi\right\}. \tag{5.1}$$

Now, we have

$$\begin{aligned}(W^*(\xi,X).R)(Y,Z)U &= W^*(\xi,X)R(Y,Z)U - R(W^*(\xi,X)Y,Z)U \\ &\quad - R(Y,W^*(\xi,X)Z)U - R(Y,Z)W^*(\xi,X)U.\end{aligned}$$

In consequence of $W^*(\xi,X).R = 0$, (5.2) becomes

$$\begin{aligned}&W^*(\xi,X)R(Y,Z)U - R(W^*(\xi,X)Y,Z)U \\ &\quad - R(Y,W^*(\xi,X)Z)U - R(Y,Z)W^*(\xi,X)U = 0.\end{aligned}$$

In view of (2.4), (2.7), (2.8), (2.10) and (5.1), last result becomes

$$\begin{aligned} & \frac{1}{2}\left\{1+\frac{1}{n-1}\left(\frac{r}{n-1}+1\right)\right\}[\eta(R(Y,Z)U)X -' R(Y,Z,U,X)\xi \\ - \quad & \eta(Y)R(X,Z)U + g(X,Y)R(\xi,Z)U - \eta(Z)R(Y,X)U \\ + \quad & g(X,Z)R(Y,\xi)U - \eta(U)R(Y,Z)X + g(X,U)R(Y,Z)\xi] = 0, \end{aligned}$$

where

$$'R(X,Y,Z,U) \stackrel{\text{def}}{=} g(R(X,Y)Z,U). \tag{5.2}$$

Taking inner-product of above with respect to the Riemannian metric g and then using (2.4) and (2.10), we have

$$\frac{1}{2}\left\{1+\frac{1}{n-1}\left(\frac{r}{n-1}+1\right)\right\}[-'R(Y,Z,U,X)-g(X,Y)g(Z,U)+g(X,Z)g(Y,U)] = 0$$

$$\Longrightarrow \ 'R(Y,Z,U,X) = g(X,Z)g(Y,U) - g(X,Y)g(Z,U).$$

Using (2.4) and (5.2) in the above equation, we obtain

$$R(Y,Z)U = g(Y,U)Z - g(Z,U)Y. \tag{5.3}$$

Contracting (5.3) with respect to the vector field Y, we find

$$S(Z,U) = -(n-1)g(Z,U), \tag{5.4}$$

which gives

$$QZ = -(n-1)Z \tag{5.5}$$

and

$$r = -n(n-1). \tag{5.6}$$

In view of (2.4), (5.1), (5.3), (5.4) and (5.5), (5.2) gives $W^*(\xi,X).R = 0$. Thus, consequently we state

Theorem 5.1. *An $n-$dimensional $\eta-$Einstein Kenmotsu manifold M_n satisfies the condition $W^*(\xi,X).R = 0$ if and only if either M_n is locally isometric to the hyperbolic space $H^n(-1)$ or M_n has constant scalar curvature tensor $-n(n-1)$.*

In the light of Corollary (3.4) and theorem (4.1), theorem (5.1) state

Corollary 5.2. *An $n-$dimensional $\eta-$Einstein Kenmotsu manifold M_n satisfies $W^*(\xi,X).R = 0$ if and only if it is Conformally flat.*

6 The irrotational $M-$projective curvature tensor

Definition 6.1. - *Let D be a Riemannian connection, then the rotation (Curl) of $M-$projective curvature tensor W^* on a Riemannian manifold M_n is defined as*

$$\begin{aligned} RotW^* &= (D_U W^*)(X,Y)Z + (D_X W^*)(U,Y)Z \\ &+ (D_Y W^*)(X,U)Z - (D_Z W^*)(X,Y)U. \end{aligned} \tag{6.1}$$

In consequence of Bianchi's second identity for Riemannian connection D, (6.1) becomes

$$RotW^* = -(D_Z W^*)(X,Y)U. \tag{6.2}$$

If the $M-$projective curvature tensor is irrotational, then curl W^*=0 and therefore

$$(D_Z W^*)(X,Y)U = 0,$$

which gives

$$D_Z(W^*(X,Y)U) = W^*(D_Z X, Y)U + W^*(X, D_Z Y)U + W^*(X,Y)D_Z U. \tag{6.3}$$

Replacing $U = \xi$ in (6.3), we have

$$D_Z(W^*(X,Y)\xi) = W^*(D_Z X, Y)\xi + W^*(X, D_Z Y)\xi + W^*(X,Y)D_Z \xi. \tag{6.4}$$

Now, substituting $Z = \xi$ in (3.1) and using (2.4), (2.7) and (2.9), we obtain

$$W^*(X,Y)\xi = k[\eta(X)Y - \eta(Y)X], \tag{6.5}$$

where

$$k = \frac{1}{2}\left\{1 + \frac{1}{n-1}\left(\frac{r}{n-1} + 1\right)\right\}. \tag{6.6}$$

Using (6.5) in (6.4), we obtain

$$W^*(X,Y)Z = k[g(X,Z)Y - g(Y,Z)X]. \tag{6.7}$$

Also equations (3.1) and (6.7) gives

$$\begin{aligned} k[g(X,Z)Y - g(Y,Z)X] &= R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X \\ &- S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \end{aligned}$$

Contracting above equation with respect to the vector X and then using (6.6), we find

$$S(Y,Z) = -(n-1)g(Y,Z) \Longleftrightarrow QY = -(n-1)Y, \tag{6.8}$$

which gives

$$r = -n(n-1). \tag{6.9}$$

In consequence of (3.1), (6.6), (6.7), (6.8) and (6.9), we can find

$$R(X,Y)Z = -[g(Y,Z)X - g(X,Z)Y]. \tag{6.10}$$

Thus, we can state

Theorem 6.2. *The $M-$projective curvature tensor in an $\eta-$Einstein Kenmotsu manifold M_n is irrotational if and only if it is locally isometric to the hyperbolic space $H^n(-1)$.*

Theorem 5.1 together with Theorem 6.2 lead to

Corollary 6.3. *An $n-$dimensional $\eta-$Einstein Kenmotsu manifold M_n satisfies $W^*(\xi, X).R = 0$ if and only if the $M-$projective curvature tensor is irrotational.*

Corollary 6.4. *The $M-$projective curvature tensor in an $\eta-$Einstein Kenmotsu manifold M_n is irrotational if and only if the manifold is conformally flat.*

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