On the minimal orbits of an abelian linear action

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Abstract. In this paper we prove that, for an abelian linear action, every orbit γ is contained in a locally closed submanifold $V(\gamma)$ such that $\gamma \subset V(\gamma) \subset \overline{\gamma}$ where $\overline{\gamma}$ is the closure of γ and γ is minimal if and only if $V(\gamma)$ is closed. Next, using this result and a classical classification of abelian Lie groups, we describe the minimal sets. After, we prove that there exist a decreasing finite sequence $F_0, F_1, ..., F_k$ of invariant closed sets such that $\Omega_j = F_j \setminus F_{j-1}$ is a dense open set(for the relative topology)of F_j and every orbit in Ω_j is minimal in it.

M.S.C. 2000: 37C85.
Key words: Group; linear action; orbit; minimal orbit; minimal set; commutative algebra.

1 Introduction

A *group action* is a pair (E, G) where E is a topological space and G is a group acting continuously on E. An *abelian linear action* is a pair (E, G) where E is a finite dimensional vector space over $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$) and G is an abelian subgroup of the general linear group $GL(E)$ acting naturally on E by $(g, x) \to gx = g(x)$.

In [1], the authors studied the dynamic of an abelian subgroup of $GL(\mathbb{K}^n)$, and proved a structure's Theorem deducing the existence of minimal sets. In this paper we introduce the notion of *rank of an orbit* and, using it, we prove by an independent manner, the existence of minimal sets giving a complete classification of them. We will start by studying the subspaces of E stable by a commutative subalgebra of $\mathcal{L}(E)$, where $\mathcal{L}(E)$ is the $\mathbb{K}$-algebra of the linear transformations of E. We consider here only the real case and suppose that a inner product is given on E.

Recall now some notions concerning a group action. We will use the definitions given in [1]. Let (E, G) be a group action and $x \in E$. The *orbit* of x is the set

$$Gx = \{gx, \ g \in G\}.$$

A subset $A \subset E$ is called *G-invariant* if $Gx \subset A$, for all $x \in A$. A G-invariant set A is said *minimal set* of (E, G), if

$$\overline{Gx} = A, \quad \text{for all} \quad x \in A,$$

Differential Geometry - Dynamical Systems, Vol.12, 2010, pp. 61-72.

that is A is closed and every orbit in A is dense in A. Let X be a G-invariant subset of E. A minimal set in X is a minimal set of (G_X, X) where G_X is the group G restricted to X. An orbit $\mathcal{O}$ is said *minimal* if $\overline{\mathcal{O}}$ is a minimal set.

We call *class* of an orbit $\mathcal{O}$ the set $cl(\mathcal{O})$ of orbits γ such that $\overline{\mathcal{O}} = \overline{\gamma}$. If $\mathcal{O}$ is a minimal orbit in a G-invariant open set U then $cl(\mathcal{O}) = \overline{\mathcal{O}} \cap U$.

This paper is organized as follows. In section 1, we present some results concerning subspaces of E stable by a commutative subalgebra of $\mathcal{L}(E)$ introducing the notions of *monogenous subspaces* and *rank of an element* of E. A correspondence between ideals of $\mathcal{A}$ and stable subspaces of E is given. In section 2, we introduce the *rank of an orbit* and using this notion we develop some result concerning minimal orbits and existence of minimal sets proving in particular the existence of subspaces of dimension 1 or 2 saturated by minimal sets. Next we give a topological characterization of the minimal sets and we prove the existence of an increasing finite sequence of invariant closed sets $F_0, F_1, ..., F_k$ $(k \leq \dim E)$ with the following properties.

1. $\Omega_j = F_j \setminus F_{j-1}$ is a dense open set of F_j for the relative topology $(1 \leq j \leq k)$.
2. Every orbit in Ω_j is minimal in Ω_j.
3. $F_0 = \varnothing$ and $F_k = E \setminus \{0\}$

All vectorial space or algebra introduced in this paper will be, implicitly, provided with his natural differential structure as a finite dimensional real vector space.

2 Subspaces stable by a commutative subalgebra of $\mathcal{L}(E)$

In this section, E is a finite dimensional real vector space provided with a scalar product $\langle . \mid . \rangle$ and $\dot{E}$ is the pointed space $E \setminus \{0\}$. We denote by $\mathcal{A}$ a commutative unitary subalgebra of $\mathcal{L}(E)$. A subspace F of E is said *stable* by $\mathcal{A}$ or simply *stable* if

$$u(F) \subset F, \quad \text{for all} \quad u \in \mathcal{A}.$$

Since $\mathcal{A}$ is commutative, we have for all $u \in \mathcal{A}$, the image, $\mathrm{Im}(u)$, and the kernel, $\ker(u)$, of u are stable subspaces of E.

For every stable subspace F of E, we denote by $\mathcal{A}_F$ the subalgebra of $\mathcal{L}(F)$ consisting of elements of $\mathcal{A}$ restricted to F. For every $a \in E$, we consider the linear map $\Phi_a : \ \mathcal{A} \longrightarrow E$ defined by

$$\Phi_a(u) = u(a), \quad u \in \mathcal{A}.$$

It's kernel is denoted by K_a

$$K_a = \ker \Phi_a = \{u \in \mathcal{A} \ / \ u(a) = 0\}.$$

Lemma 2.1. *If Φ_a is surjective then it is also injective and so it is a linear isomorphism.*

Proof. If Φ_a is surjective then every $x \in E$ can be written in the form

$$x = u(a)$$

for some $u \in \mathcal{A}$. Let $\varphi \in \ker \Phi_a$ (*i.e.* $\varphi(a) = 0$) then for all $x \in E$, we have

$$\varphi(x) = \varphi(u(a)) = u(\varphi(a)) = u(0) = 0$$

whence $\varphi = 0$ and so $\ker \Phi_a = 0$ which gives that Φ_a is injective. ☐

The vector space E is called *monogenous* (with respect to $\mathcal{A}$) if there exist a vector $a \in E$ such that the map Φ_a is surjective. Every such vector is called a *generator* of E. A stable subspace $F \subset E$ is called *monogenous subspace* if it is monogenous with respect to $\mathcal{A}_F$. Every vector $a \in E$ determines a monogenous subspace, namely the subspace

$$E_a = \operatorname{Im} \Phi_a = \{u(a),\ u \in \mathcal{A}\}.$$

We call *rank* of an element a of E, the integer

$$r(a) = \dim E_a.$$

It is easy to see that if F is a stable subspace then for all $a \in F$, $E_a \subset F$, and so $r(a) \leq \dim F$. In particular, we have

$$r(a) \leq \dim E, \quad a \in E$$

and by Lemma 2.1, $r(a) = \dim E$ if and only if Φ_a is a linear isomorphism.
We can easily check from the commutativity of $\mathcal{A}$ that $E_a = \Phi_a(\mathcal{A})$ is a stable subspace of E and $K_a = \Phi_a^{-1}(0)$ is an ideal in $\mathcal{A}$. More generally we have:

Lemma 2.2. *For all $a \in E$ we have:*

1. *If F is a stable subspace of E, $\Phi_a^{-1}(F)$ is an ideal in $\mathcal{A}$ containing K_a.*
2. *If $\boldsymbol{I}$ is an ideal in $\mathcal{A}$ then $\Phi_a(\boldsymbol{I})$ is a stable subspace of E contained in E_a.*

Proof. Let F be a stable subspace and $\boldsymbol{J} = \Phi_a^{-1}(F)$. Then for all u ,$v \in \boldsymbol{J}$ we have

$$u(a) \in F \text{ and } v(a) \in F$$

so

$$(u - v)(a) \in F$$

that is

$$u - v \in \boldsymbol{J}$$

this means that $\boldsymbol{J}$ is an additive subgroup of $\mathcal{A}$.

On the other hand, let $f \in \mathcal{A}$ be any element and $u \in \boldsymbol{J}$ arbitrary chosen , then

$$u(a) \in F$$

and since F is stable by $\mathcal{A}$, we have

$$fu(a) \in F$$

that is

$$fu \in \boldsymbol{J}$$

whence $\boldsymbol{J}$ is an ideal of $\mathcal{A}$. Since $0 \in F$, we have $\Phi_a^{-1}(0) \subset \Phi_a^{-1}(F)$. That is $K_a \subset \boldsymbol{J}$.

Let $\boldsymbol{I}$ be an ideal of $\mathcal{A}$, and $F = \Phi_a(\boldsymbol{I})$. It is clear that F is a subspace of E. Moreover if $x \in F$, then there exist $u \in \boldsymbol{I}$ such that $x = u(a)$. Since $\boldsymbol{I}$ is an ideal we have:

$$fu \in \boldsymbol{I}, \quad \text{for all } f \in \mathcal{A}$$

and therefore $f(x) = fu(a) \in F$ for all $f \in \mathcal{A}$ that is F is stable by $\mathcal{A}$. On the other hand $\boldsymbol{I} \subset \mathcal{A}$ implies $\Phi_a(\boldsymbol{I}) \subset \Phi_a(\mathcal{A})$. That is $F \subset E_a$ which completes the proof. □

Proposition 2.3. *If $r(x)$ is a constant, k, independent from $x \in \dot{E}$ then:*

i) $\dot{E}$ is a disjoint union of k-dimensional monogenous (pointed) subspaces.

ii) For every $a \in \dot{E}$, the ring $\mathcal{A}_{E_a}$ is a commutative field and $k = 1$ or $k = 2$.

iii) If $k = 1$, then $\mathcal{A} = \{\lambda Id_E,\ \lambda \in \mathbb{R}\}$.

iv) If $k = 2$, then E is isomorphic to $\mathbb{C}^p$ and, for some identification $E \simeq \mathbb{C}^p$,

$$\mathcal{A} = \{\lambda Id_E,\ \lambda \in \mathbb{C}\}.$$

Proof. i) For arbitrary vectors $a, b \in \dot{E}$, the subspace $E_a \cap E_b$ is stable, so if $E_a \cap E_b \neq \{0\}$ then it contains a non zero vector c and thus we have:

$$k = r(c) \leq dim(E_a \cap E_b) \leq \dim E_a = k.$$

It follows that $\dim(E_a \cap E_b) = k$ and therefore $E_a \cap E_b = E_a = E_b$. Consequently

$$E_a \cap E_b = \{0\} \text{ or } E_a = E_b$$

this means that $\dot{E}$ is a disjoint union of monogenous pointed subspaces of dimension k. That is, for some subset $A \subset E$,

$$E = \bigcup_{a \in A} E_a$$

with $\dim E_a = k$, for all $a \in A$ and $E_a \cap E_b = \{0\}$, for $a \neq b \in A$.
It is obvious that, in this case, we can write, for some $p \in \mathbb{N}$,

$$E = \bigoplus_{i=1}^{p} E_{a_i}.$$

ii) Suppose at first that $k = \dim E$ (that is for all $x \in \dot{E}$, $r(x) = \dim E$) and let $u \in \mathcal{A} \setminus \{0\}$ be an arbitrary non zero element. If $\ker(u) \neq \{0\}$, then it contains a non zero vector x. Since it is a stable subspace, we have $r(x) \leq \dim \ker(u) < \dim E$. Which contradicts the hypothesis that $r(x) = \dim E$, so $\ker(u) = \{0\}$ and therefore u is invertible. This implies that every non zero element of the ring $\mathcal{A}$ is invertible and so $\mathcal{A}$ is a field. Now $\mathcal{A}$ is a commutative $\mathbb{R}$-algebra and so $\mathcal{A}$ is isomorphic to $\mathbb{R}$ or isomorphic to $\mathbb{C}$. In particular we have

$$\dim \mathcal{A} = 1 \quad \text{or} \quad \dim \mathcal{A} = 2.$$

On the other hand, let $a \in \dot{E}$, then

$$r(a) = \dim(\mathrm{Im}\Phi_a) = \dim E.$$

It follows that Φ_a is surjective and by Lemma 2.1, Φ_a is an isomorphism which gives

$$\dim E = \dim \mathcal{A}$$

whence $k = \dim E = 1$ or $k = \dim E = 2$. If $k < \dim E$ we have, for all $a \in \dot{E}$ and all $x \in \dot{E}_a$,

$$r(x) = k = \dim E_a.$$

So, by considering the pair $(E_a, \mathcal{A}_{E_a})$, we obtain as above $\mathcal{A}_{E_a}$ is a commutative field isomorphic to $\mathbb{R}$ or $\mathbb{C}$ and $k = 1$ or $k = 2$.

iii) Suppose that $k = 1$, then for every $x \in \dot{E}$, the subspace generated by x is stable by $\mathcal{A}$. So for every $u \in \mathcal{A}$, the minimal polynomial P_u of u is of degree 1 and therefore $u(x) = \lambda_u.x$ for all $x \in E$ which proves the part iii).

iv) Suppose that $k = 2$. Then, by part i), we have, for some $p \in \mathbb{N}$,

$$E = \bigoplus_{i=1}^{p} E_{a_i}$$

with $\dim E_{a_i} = 2$, for all $i \in \{1, 2, ..., p\}$. It follows that $\dim E = 2p$. So we can identify E to $\mathbb{C}^p$ and obtain, as previously, that the elements of $\mathcal{A}$ are all of the form

$$u(z) = \lambda_u z, \quad z \in \mathbb{C}^p$$

where $\lambda_u \in \mathbb{C}$. □

Definition 2.4. Let L be a stable subspace. We say that L is *minimal* (with respect to $\mathcal{A}$) if L does not contain a non trivial stable subspace and we say that L is *maximal* if L is not contained in a proper stable subspace.

Clearly if L is minimal, then for all stable subspace F we have

$$L \subset F \quad \text{or} \quad L \cap F = \{0\}$$

Proposition 2.5. *Let L be a stable subspace. If L is minimal, then the ring $\mathcal{A}_L$ is a commutative field isomorphic to $\mathbb{R}$ or $\mathbb{C}$ and $\dim L = 1$ or $\dim L = 2$. If L is maximal, then $\dim L = n - 1$ or $\dim L = n - 2$.*

Proof. If L is minimal, then $\forall x \in L \setminus \{0\}$, $E_x = L$ and therefore $r(x) = \dim L$. The conclusion is, now, an immediate consequence of Proposition 2.3. If L is maximal, we consider the subalgebra $\mathcal{A}^* = \{u^*;\ u \in \mathcal{A}\}$ where u^* is the adjoint endomorphism of u. It is easy to see that a subspace F is stable by $\mathcal{A}$ if and only if $F^{\perp}$ is stable by $\mathcal{A}^*$ and F is minimal(resp. maximal) with respect to $\mathcal{A}$ if and only if $F^{\perp}$ is maximal(resp. minimal) with respect to $\mathcal{A}^*$. So if L is maximal with respect to $\mathcal{A}$, then $L^{\perp}$ is minimal with respect to $\mathcal{A}^*$ and by the first part $\dim L^{\perp} = 1$ or $\dim L^{\perp} = 2$. This completes the proof. □

Proposition 2.6. *Let L be a stable subspace and $a \in L \setminus \{0\}$. Then L is minimal if and only if K_a is a maximal ideal in $\mathcal{A}$.*

Proof. Suppose that L is minimal and consider the linear map $\sigma : \mathcal{A} \longrightarrow \mathcal{A}_L$ defined by

$$\sigma(u) = u_L, \quad u \in \mathcal{A}$$

where u_L is the restriction of u to L. Clearly σ is a surjective homomorphism between the two rings $\mathcal{A}$ and $\mathcal{A}_L$ and consequently $\mathcal{A}/\ker(\sigma)$ is isomorphic to $\mathcal{A}_L$. Now, by Proposition 2.6, $\mathcal{A}_L$ is a field. It follows that $\mathcal{A}/\ker(\sigma)$ is also a field and so $\ker(\sigma)$ is a maximal ideal in $\mathcal{A}$. So it is sufficient to prove that

$$K_a = \ker(\sigma)$$

Let $u \in \ker(\sigma)$ then $u_L = 0$, that is $u(x) = 0$, for all $x \in L$. In particular $u(a) = 0$ and therefore $u \in K_a$, so $\ker(\sigma) \subset K_a$. Conversely let $\varphi \in K_a$(i.e: $\varphi(a) = 0$). Since $\ker\varphi$ is stable by $\mathcal{A}$, we have by minimality of L, $\ker\varphi \supset L$ or $\ker\varphi \cap L = \{0\}$. Yet $a \in \ker\varphi \cap L$ therefore $\ker\varphi \cap L \neq \{0\}$ so that $\ker\varphi \supset L$ and therefore $\varphi_L = 0$ that is $\sigma(\varphi) = 0$ hence $K_a \subset \ker(\sigma)$ and finally $K_a = \ker(\sigma)$. To prove the converse, suppose that K_a is a maximal ideal in $\mathcal{A}$ and let F be a stable subspace of L. By Lemma 2.2, $\Phi_a^{-1}(F)$ is an ideal in $\mathcal{A}$ containing K_a. Since K_a is maximal it follows that $\Phi_a^{-1}(F) = \mathcal{A}$ or $\Phi_a^{-1}(F) = K_a$, hence

$$F = \Phi_a(\mathcal{A}) = L \ \text{ or } \ F = \Phi_a(K_a) = \{0\}.$$

This means that L is minimal. □

3 Minimal orbits and minimal sets

In this section, we consider an abelian linear action (E, G) and we denote by $\mathcal{G}$ the subalgebra of $\mathcal{L}(E)$ generated by G. It is easy to see that a subspace F is stable by $\mathcal{G}$ if and only if it is G-invariant. If F is a such space, we denote by G_F (resp. $\mathcal{G}_F$) the group (resp. algebra) consisting of elements of G (resp. $\mathcal{G}$) restricted to F.

For every G-invariant subset S of E, we denote by $E(S)$ the subspace of E generated by S, that is the subspace with smallest dimension containing S. When S is an orbit γ, $E(\gamma)$ is a monogeneous subspace. In fact it is easy to check that $E(\gamma) = E_a$ for any $a \in \gamma$. This means that the *rank* is constant on the orbits of G. We define the *rank* of an orbit γ by

$$r(\gamma) = r(a), \ a \in \gamma$$

and the *rank* of the group G by

$$r(G) = \max\{r(a), \ a \in E\}.$$

Tow orbits γ_1 and γ_2 of G are said isomorphic if there exist an isomorphism $\varphi : E(\gamma_1) \to E(\gamma_2)$ such that

$$\varphi(\gamma_1) = \gamma_2.$$

Proposition 3.1. *Let γ be an orbit of G. Then for every orbit α in $E(\gamma)$, there exist $u \in \mathcal{G}$ such that $u(\gamma) = \alpha$. In particular if $E(\alpha) = E(\gamma)$, then α and γ are isomorphic.*

Proof. Let $a \in \gamma$ and $b \in \alpha$ be arbitrary points. Then $E(\gamma) = E_a$ and so

$$E(\gamma) = \mathrm{Im}\Phi_a = \{u(a), u \in \mathcal{G}\}.$$

Since $b \in \alpha \subset E_a$, there exist $u \in \mathcal{G}$ such that $b = u(a)$. Let $x \in \gamma$ then there exist $g \in G$, such that $x = g(a)$ so that

$$u(x) = u(g(a)) = g(u(a)) = g(b) \in \alpha$$

whence

$$u(\gamma) \subset \alpha. \tag{3.1}$$

Conversely, let $y \in \alpha$ then there exist $h \in G$ such that $y = h(b)$ hence

$$y = h(u(a)) = u(h(a))$$

now $h(a) \in \gamma$ and so $y \in u(\gamma)$, consequently

$$\alpha \subset u(\gamma). \tag{3.2}$$

By (3.1) and (3.2) we have

$$\alpha = u(\gamma).$$

Clearly we have at the same time $u(E(\gamma)) = E(\alpha)$. If $E(\alpha) = E(\gamma)$ then the linear map $u_{E(\gamma)} : E(\gamma) \to E(\alpha)$ is an isomorphism. □

Corollary 3.2. *Let γ_1 and γ_2 are orbits of G such that $\overline{\gamma}_1 = \overline{\gamma}_2$. Then γ_1 and γ_2 have the same rank and are isomorphic.*

Proof. Remark at first that for any orbit γ we have $E(\gamma) = E(\overline{\gamma})$. In fact, since $\gamma \subset \overline{\gamma}$ then $E(\gamma) \subset E(\overline{\gamma})$ and since $E(\gamma)$ is a closed set containing γ then $\overline{\gamma} \subset E(\gamma)$ and so $E(\overline{\gamma}) \subset E(\gamma)$ whence $E(\overline{\gamma}) = E(\gamma)$. If $\overline{\gamma}_1 = \overline{\gamma}_2$, then $E(\gamma_1) = E(\gamma_2)$ and the conclusion is an immediate consequence of Proposition 3.1. □

Theorem 3.3. *For every orbit γ of G in $\dot{E}$, there exist a locally closed submanifold $V(\gamma)$ of $\dot{E}$ having the following properties.*

- *i) $V(\gamma)$ is G-invariant and $\gamma \subset V(\gamma) \subset \overline{\gamma}$.*
- *ii) For any orbit α, $\overline{\alpha} = \overline{\gamma}$ if and only if α lies in $V(\gamma)$.*
- *iii) If $\overline{\gamma} \setminus V(\gamma) \neq \emptyset$, then for every orbit $\alpha \subset \overline{\gamma} \setminus V(\gamma)$, we have $r(\alpha) < r(\gamma)$.*
- *iv) γ is minimal in $\dot{E}$ if and only if $V(\gamma)$ is closed in $\dot{E}$.*

Proof. Suppose at first, that $r(\gamma) = \dim(E)$, and let $\overline{G}$ be the closure of G in $\mathcal{G}$ and $\widetilde{G}$ be the closure of G in $GL(E)$ which is an abelian subgroup of $GL(E)$. Clearly we have

$$\widetilde{G} = \overline{G} \cap GL(E),$$

and so $G \subset \widetilde{G} \subset \overline{G}$. Then $\widetilde{G}$ is a closed subgroup of the Lie-group $GL(E)$ and therefore, by the Elie Cartan's Theorem (cf. [3]), $\widetilde{G}$ is a Lie-subgroup of the Lie-group

$GL(E)$ so it is a locally closed submanifold of $\mathcal{G}$. Fix a point $a \in \gamma$ and consider the map $\Phi_a : \mathcal{G} \to E$ defined as above by

$$\Phi_a(u) = u(a), \quad u \in \mathcal{G}.$$

Since $r(a) = r(\gamma) = \dim E$, the map Φ_a is surjective and by Lemma 2.1 it is an isomorphism. Consequently $\Phi_a(\widetilde{G})$ is a locally closed submanifold of E. We denote it by $V(\gamma)$. Let $x \in V(\gamma) = \Phi_a(\widetilde{G})$ then

$$x = \tilde{g}(a), \quad \text{for some} \quad \tilde{g} \in \widetilde{G}$$

and for any $g \in G$, we have

$$g(x) = g\tilde{g}(a) = \Phi_a(g\tilde{g})$$

now $g\tilde{g} \in \widetilde{G}$ and therefore $g(x) \in \Phi_a(\widetilde{G})$. It follows that $\Phi_a(\widetilde{G})$ is G-invariant. On the other hand, since $\gamma = \Phi_a(G)$ and $\overline{\gamma} = \Phi_a(\overline{G})$, they have

$$\gamma \subset V(\gamma) \subset \overline{\gamma}.$$

This proves the property i).

ii)Remark at first that $\widetilde{G}$ is a dense open subset of $\overline{G}$ (for the relative topology) and hence $V(\gamma)$ is a dense open subset of $\overline{\gamma}$ which gives that $\overline{\gamma} \setminus V(\gamma)$ is a closed G-invariant subset. Let α an orbit such that $\overline{\alpha} = \overline{\gamma}$ then α lies entirely in $V(\gamma)$ or lies entirely in $\overline{\gamma} \setminus V(\gamma)$ now $\overline{\gamma} \setminus V(\gamma)$ is closed, so if $\alpha \subset \overline{\gamma} \setminus V(\gamma)$ then $\overline{\alpha} \subset \overline{\gamma} \setminus V(\gamma)$ whence $\overline{\gamma} \subset \overline{\gamma} \setminus V(\gamma)$ which is evidently false. Consequently $\alpha \subset V(\gamma)$.

Conversely, let α an orbit in $V(\gamma)$ and $b \in \alpha$ an arbitrary point, then there exist $\varphi \in \widetilde{G}$ such that $b = \varphi(a)$. It follows that $\Phi_b = \varphi \circ \Phi_a$. Since φ is an isomorphism, Φ_b is also an isomorphism and therefore

$$\Phi_b(G) \subset \Phi_b(\widetilde{G}) \subset \Phi_b(\overline{G})$$

since $\Phi_b(G) = \alpha$ and $\Phi_b(\overline{G}) = \overline{\alpha}$, it remains to prove that $\Phi_b(\widetilde{G}) = V(\gamma)$. In fact we have for any $\tilde{g} \in \widetilde{G}$

$$\Phi_b(\tilde{g}) = \tilde{g}(b) = \tilde{g}\varphi^{-1}(a) = \Phi_a(\tilde{g}\varphi^{-1})$$

but $\tilde{g}\varphi^{-1} \in \widetilde{G}$ it follows that

$$\Phi_b(\widetilde{G}) \subset \Phi_a(\widetilde{G}) = V(\gamma).$$

likewise they show that

$$V(\gamma) = \Phi_a(\widetilde{G}) \subset \Phi_b(\widetilde{G}).$$

Whence $V(\gamma) = \Phi_b(\widetilde{G})$ and so

$$\alpha \subset V(\gamma) \subset \overline{\alpha}.$$

This implies that

$$\overline{\alpha} = \overline{V(\gamma)} = \overline{\gamma}.$$

iii) It is clear that the elements of $\overline{G} \setminus \widetilde{G}$ are not invertible. Let α an orbit of G such that $\alpha \subset \overline{\gamma} \setminus V(\gamma)$. Then there exist $u \in \overline{G} \setminus \widetilde{G}$ such that $\alpha = u(\gamma)$. Since u is not invertible and $u(E(\gamma)) = E(\alpha) \subset E(\gamma)$ it follows that

$$\dim E(\alpha) < \dim E(\gamma).$$

That is

$$r(\alpha) < r(\gamma).$$

iv) Suppose that $V(\gamma)$ is closed. Then, for all orbit $\alpha \subset V(\gamma)$, we have by the part *ii)*

$$\overline{\alpha} = \overline{\gamma} = V(\gamma).$$

Therefore $\overline{\alpha} = V(\gamma)$. Hence $V(\gamma) = \overline{\gamma}$ is a minimal set. Conversely suppose that $\overline{\gamma}$ is minimal. Since $\overline{\gamma} \setminus V(\gamma)$ is an invariant closed set contained in $\overline{\gamma}$, we have necessarily

$$\overline{\gamma} \setminus V(\gamma) = \varnothing\ ,$$

Thus, seeing that $V(\gamma) \subset \overline{\gamma}$, we deduce that $V(\gamma) = \overline{\gamma}$ and so $V(\gamma)$ is closed which gives that γ is minimal. This complete the proof in the case where $r(\gamma) = \dim E$. If $r(\gamma) < \dim E$, we replace E by $E(\gamma)$ and G by $G_{E(\gamma)}$. □

Corollary 3.4. *Let $F \subset E$ be a stable subspace and $\gamma \subset \dot{F}$ be an orbit of G. Suppose that for every orbit $\mathcal{O} \subset \dot{F}$ we have*

$$r(\gamma) \leq r(\mathcal{O})$$

then γ is a minimal orbit in $\dot{E}$.

Proof. If $r(\gamma)$ is minimal in $\dot{F}$ then, by the part *iii)* of Theorem 3.3, we have

$$\overline{\gamma} \setminus V(\gamma) = \varnothing$$

and consequently $V(\gamma)$ is closed, hence γ is minimal. □

Corollary 3.5. *Let L be a minimal subspace of E with respect to $\mathcal{G}$. Then every orbit in $\dot{L}$ is minimal in $\dot{E}$.*

Proof. In fact, since L is minimal, then $r(\gamma) = \dim L$ for all orbit γ in $\dot{L}$. Thus, by Corollary 3.4 every orbit in $\dot{L}$ is minimal in $\dot{E}$. □

Now we are going to describe the minimal sets. We need the following definition.

Definition 3.6. We call *dimension* of an orbit γ the dimension of the submanifold $V(\gamma)$. We denote it by $\dim(\gamma)$. If $\mathfrak{M}$ is a minimal set we put $\dim \mathfrak{M} = \dim(\gamma)$ where γ is any orbit in $\mathfrak{M}$.

We need in the sequel the classical Theorem (cf. [3]).

Theorem 3.7. *Let G be a connected abelian Lie-group of dimension k. Then G is isomorphic to a Lie group of the form*

$$\mathbb{R}^p \times \mathbb{T}^{k-p}\ (0 \leq p \leq k)$$

where $\mathbb{T} = \{z \in \mathbb{C} / \ |z| = 1\}$ is the unit circle.

Remark 3.8. If the Lie-group G is not connected, then the connected component of G containing the unit element is a Lie-subgroup of G and, if we denote it by G_0, one have the quotient G/G_0 is a discrete group and thus it is countable (because G must have a topology with countable basis). The cosets are precisely the connected components of G.

The following theorem describe the minimal sets of G. It is an easy consequence of theorems 3.3 and 3.7.

Theorem 3.9. *Let $\mathfrak{M}$ be a minimal set of G. Suppose that $\dim \mathfrak{M} = k$. Then there exist an integer p $(0 \leq p \leq k)$ such that $\mathfrak{M}$ is a countable union of k-dimensional closed submanifolds of E each of which is homeomorphic to $\mathbb{R}^p \times \mathbb{T}^{k-p}$.*

Proof. Let $\mathfrak{M}$ be a minimal set of dimension k and $\gamma \subset \mathfrak{M}$ be an orbit. Then $\mathfrak{M} = \overline{\gamma} = V(\gamma)$. Without loss of generality, we can suppose that $E(\gamma) = E$. Now in view of Theorem 3.3 and its proof, $V(\gamma)$ is a closed k-dimensional submanifold of E diffeomorphic to the Lie-group $\widetilde{G}$. Since $\widetilde{G}$ is abelian (not necessarily connected), the conclusion is a consequence of Theorem 3.7 and Remark 3.8. ☐

Theorem 3.10. *Let (E, G) be an abelian linear action then there exist a finite increasing sequence of G-invariant closed subsets of $\dot{E}$*

$$F_0 \subset F_1 \subset ... \subset F_{k-1} \subset F_k, \quad k \leq \dim E$$

with the following properties:

1. *$F_0 = \emptyset$ and $F_k = \dot{E}$.*
2. *$\Omega_j = F_j \setminus F_{j-1}$ $(1 \leq j \leq k)$ is a dense G-invariant open set of F_j.*
3. *Every orbit in Ω_j is minimal in Ω_j.*

To prove this theorem we need the next lemma.

Lemma 3.11. *Let $t < r(G)$ be a real number. Then the set*

$$U_t = \{x \in E \ / r(x) > t\}$$

is a dense G-invariant open subset of E.

Proof. Let $x \in \dot{E}$ such that $r(x) = r(G)$ then $x \in U_t$ and therefore $U_t \neq \emptyset$. On the other hand, since the rank is constant on the orbits, U_t is G-invariant. Let $a \in U_t$ and $p = r(a) = \dim E_a$. Then there exist $u_1, u_2, ..., u_p$ in the algebra $\mathcal{G}$ such that the p vectors $u_1(a), u_2(a), ..., u_p(a)$ are linearly independent in E. Fix these linear transformations $u_1, ..., u_p \in \mathcal{G}$ and consider, for all $x \in E$, the Gram's determinant

$$\Delta(x) = det(\langle u_i(x) \mid u_j(x)\rangle)_{1 \leq i,j \leq p}$$

of the vectors $u_1(x), ..., u_p(x)$ where $\langle . \mid . \rangle$ denotes the scalar product in E. It is well known that these vectors are independent if and only if $\Delta(x) \neq 0$, in particular $\Delta(a) \neq 0$. Let

$$V_a = \{x \in E \ / \Delta(x) \neq 0\}$$

Since the map $x \to \Delta(x)$ is continuous, V_a is an open set of E. Now $\Delta(a) \neq 0$, and so $a \in V_a$. On the other hand we have

$$V_a \subset U_t.$$

In fact, for every $x \in V_a$, $\Delta(x) \neq 0$ and so the p vectors $u_1(x), u_2(x), ..., u_p(x)$ are linearly independent. Consequently $r(x) \geq p > t$ which gives $x \in U_t$, thus $V_a \subset U_t$. This chows that U_t is a neighborhood of the arbitrary point $a \in U_t$. Hence U_t is an open set of E. It remains to prove that U_t is dense in E.

Now using the coordinates $(\xi_1, \xi_2, ..., \xi_n)$ of x in a fixed basis of E, we obtain $\Delta(x)$ as a polynomial $P(\xi_1, \xi_2, ..., \xi_n)$ (not identically zero) of the n variables $(\xi_1, \xi_2, ..., \xi_n)$. But we know that the set of the zeros of an analytical function not identically zero is a closed set without interior. Yet V_a is precisely the complementary of the zeros of P, so V_a is dense in E and consequently (since $V_a \subset U_t$) U_t is also dense in E. This completes the proof of the Lemma 3.11. ☐

To prove Theorem 3.10, consider the distinct values $r_0 = 0 < r_1 < .. < r_k = r(G)$ taken by the map $r : E \to \mathbb{N}$ given by

$$x \to r(x), \quad x \in E$$

and let

$$F_j = \{x \in E / r(x) \leq r_j\} \quad j = 0, 1, ..., k.$$

Evidently F_j $(0 \leq j \leq k)$ is the complementary of the G-invariant open set U_{r_j} and so the sequence $F_0, ..., F_k$ is a increasing sequence of G-invariant closed subsets of E satisfying the properties *i*). On the other hand we have

$$\Omega_j = F_j \cap U_{r_{j-1}}.$$

Yet, by Lemma 3.11, $U_{r_{j-1}}$ is a dense open set of E so Ω_j is a dense open set of F_j for the relative topology. Whence the part *ii*) of the Theorem 2.10 is proved. To prove property *iii*), remark at first that Ω_j is precisely the set of vectors $x \in E$ such that $r(x) = r_j$. Let γ be an arbitrary orbit in $\Omega_j (1 \leq j \leq k)$ then we must prove that for every orbit $\alpha \subset \overline{\gamma} \cap \Omega_j$, we have

$$\overline{\alpha} \cap \Omega_j = \overline{\gamma} \cap \Omega_j.$$

Now, since $\gamma \subset \Omega_j$, then $r(\gamma) = r_j$. Let $\alpha \subset \overline{\gamma} \cap \Omega_j$ be any orbit. Then

$$\alpha \subset \overline{\gamma} \text{ and } r(\alpha) = r_j.$$

So, by Theorem 3.3 part *iii*) they has

$$\alpha \subset V(\gamma)$$

and by part *ii*) they has $\overline{\alpha} = \overline{\gamma}$. Whence

$$\overline{\alpha} \cap \Omega_j = \overline{\gamma} \cap \Omega_j.$$

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