

On quasi Einstein manifold and quasi Einstein spacetime

Pratyay Debnath and Arabinda Konar

Abstract. Some properties of conformally flat quasi Einstein manifold and quasi Einstein manifold are studied and it is shown that a conformally flat quasi Einstein manifold of dimension n is isometrically locally immersed in a Euclidean space of dimension $n + 1$. Also, a quasi Einstein spacetime is studied specially when the generator vector field is parallel.

M.S.C. 2000: 53C25, 53C50, 53B30.

Key words: quasi Einstein manifold; parallel vector field; Ricci-recurrent; conformally flat manifold; spacetime.

1 Introduction

The notion of quasi Einstein manifold has been first introduced by R.Deszcz et.al [8]. According to them, a non-flat n -dimensional Riemannian manifold (M_n, g) , $n \geq 3$ is said to be a quasi Einstein manifold if its Ricci tensor is not identically zero and satisfies the condition

$$(1.1) \quad Ric(X, Y) = ag(X, Y) + bw(X)w(Y)$$

where a, b are constants and w is a non-zero 1-form, metrically equivalent to the unit vector field U i.e. for all vector fields X

$$(1.2) \quad g(X, U) = w(X), g(U, U) = 1.$$

The notion of quasi Einstein manifold has been generalized by M.C.Chaki and R.K.Maity [4], considering a, b as scalars instead of a, b as constants.

A Riemannian or semi-Riemannian manifold is said to be Ricci-recurrent if the Ricci tensor 'Ric' is non-zero and satisfies the condition

$$(\nabla_Z Ric)(X, Y) = B(Z)Ric(X, Y)$$

where B is a non-zero 1-form and in particular when $B = 0$, the manifold is called Ricci-symmetric.

Further, a Riemannian manifold is conformally flat if it has vanishing conformal curvature tensor $C(X, Y)Z$.

Recently, the authors have proved, if the associated scalar a of a quasi Einstein hypersurface M_n of \mathbb{R}^{n+1} , ($n \geq 3$), is positive then the associated scalar b is also positive and the manifold is conformally flat [7]. Now, naturally the question arises about the isometric immersion of a conformally flat quasi Einstein manifold M_n , $n \geq 3$ in \mathbb{R}^{n+1} . The second section of this paper gives the answer of this question and in the third section some geometric properties of quasi Einstein manifold have been discussed. The final section deals with the quasi Einstein spacetime.

2 Conformally flat quasi Einstein manifold

R. N. Sen [11] proved that a conformally flat manifold (M_n, g) , $n \geq 3$ is isometrically locally immersed in a Euclidean space \mathbb{R}^{n+1} if the (0,4) type curvature tensor R of the manifold satisfies the following conditions

$$R(X, Y, Z, W) = E(Ric(Y, W)Ric(X, Z) - Ric(X, W)Ric(Y, Z)) \\ + F(g(Y, Z)g(X, W) - g(X, Z)g(Y, W))$$

where $E(\neq 0)$ and F are scalars and the fundamental form $L(X, Y) = g(L(X), Y)$ of the manifold is given by

$$L(X, Y) = \frac{1}{n-2} \left[\frac{1}{p} Ric(X, Y) - qg(X, Y) \right]$$

where $p(\neq 0)$ and q are determined by the relation

$$\frac{1}{E} = (n-2)[(n-2)p^2 - pq], \quad F = -\frac{pq}{n-2}.$$

Now, in a conformally flat Riemannian manifold the (0, 4) type Riemann curvature tensor R can be expressed as

$$R(X, Y, Z, W) = \frac{1}{n-2} [g(Y, Z)Ric(X, W) - g(X, Z)Ric(Y, W) \\ + Ric(Y, Z)g(X, W) - Ric(X, Z)g(Y, W)] \\ + \frac{r}{(n-1)(n-2)} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]$$

where r is the scalar curvature of the manifold. If the manifold is quasi Einstein, then using equation (1.1), we have

$$R(X, Y, Z, W) = \frac{1}{a(n-2)} [a^2 g(Y, Z)g(X, W) + abg(Y, Z)w(X)w(W) \\ + abg(X, W)w(Y)w(Z) - g(X, Z)g(Y, W) - abg(Y, W)w(X)w(Z) \\ - abg(X, Z)w(Y)w(W)] - \left(\frac{r}{(n-1)(n-2)} - \frac{a}{n-2} \right) [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]$$

$$\begin{aligned} \text{i.e. } R(X, Y, Z, W) &= E[\text{Ric}(Y, W)\text{Ric}(X, Z) - \text{Ric}(X, W)\text{Ric}(Y, Z)] \\ &\quad + F[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \end{aligned}$$

where $E = -\frac{1}{a(n-2)}$, $a \neq 0$, $F = -\frac{a+b}{(n-1)(n-2)}$

and $p = \pm[\frac{b-(n-2)a}{(n-1)(n-2)}]^{1/2}$ $q = \pm\frac{a+b}{n-1}[\frac{b-(n-2)a}{(n-1)(n-2)}]^{-1/2}$, $b > (n-2)a$.

Therefore we can state the following theorem

Theorem 2.1. *A conformally flat quasi Einstein manifold of dimension n is isometrically locally immersed in a Euclidean space of dimension $n+1$ provided $a \neq 0$ and $b > (n-2)a$.*

Again the dimension of a quasi Einstein hypersurface of \mathbb{R}^{n+1} is odd if $a+b=0$ [7]. Therefore, we have

Corollary 2.1.1. *Conformally flat quasi Einstein manifold satisfying $a+b=0$ and $b > (n-2)a$ is odd dimensional.*

Now in a Riemannian manifold we know that the conformal curvature tensor $C(X, Y)Z$ is given by

$$\begin{aligned} (2.1) \quad C(X, Y)Z &= R(X, Y)Z - \frac{1}{n-2}[g(Y, Z)Q(X) - g(X, Z)Q(Y)] \\ &\quad + \text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y] \end{aligned}$$

Where $g(Q(X), Y) = \text{Ric}(X, Y)$

Now, taking the inner product of equation (2.1) with U , we get

$$\begin{aligned} C(X, Y, Z, U) &= R(X, Y, Z, U) - \frac{1}{n-2}[g(Y, Z)\text{Ric}(X, U) - g(X, Z)\text{Ric}(Y, U)] \\ &\quad + \text{Ric}(Y, Z)w(X) - \text{Ric}(X, Z)w(Y) + \frac{r}{(n-1)(n-2)}[g(Y, Z)w(X) - g(X, Z)w(Y)] \end{aligned}$$

Now, $r = na + b$ and $\text{Ric}(X, U) = (a+b)w(X)$, so,

$$\begin{aligned} w(C(X, Y)Z) &= w(R(X, Y)Z) - \frac{2a+b}{n-2}[g(Y, Z)w(X) - g(X, Z)w(Y)] \\ &\quad + \frac{na+b}{(n-1)(n-2)}[g(Y, Z)w(X) - g(X, Z)w(Y)], \text{ i.e.,} \end{aligned}$$

$$w(C(X, Y)Z) = w(R(X, Y)Z) - \frac{a+b}{n-1}[g(Y, Z)w(X) - g(X, Z)w(Y)]$$

Therefore, we can state the following

Theorem 2.2. *In a quasi Einstein manifold the Riemannian curvature tensor and the conformal curvature tensor satisfies the condition $w(C(X, Y)Z) = w(R(X, Y)Z) - \frac{a+b}{n-1}[g(Y, Z)w(X) - g(X, Z)w(Y)]$.*

It is known to us that, in a quasi Einstein manifold the curvature and the Ricci-transformation commutes if and only if the relation $w(R(X, Y)Z) = 0$ holds [4]. Using this, we easily get the following theorem:

Theorem 2.3. *In a conformally flat quasi Einstein manifold the curvature and the Ricci-transformation commutes i.e. $R(X, Y) \circ Q = Q \circ R(X, Y)$ if and only if $a+b = 0$.*

3 Geometrical significance of the condition $\nabla_X U = 0$ in a quasi Einstein manifold

At first we are going to show that, a vector field is a parallel vector field iff it is an irrotational Killing vector field.

A vector field U is a Killing vector field if

$$g(\nabla_X U, Y) + g(\nabla_Y U, X) = 0$$

for all vector field X, Y .

Again, a vector field is irrotational if

$$g(\nabla_X U, Y) - g(\nabla_Y U, X) = 0$$

for all vector field X, Y . Therefore, the vector field U is a parallel vector field iff it is an irrotational Killing vector field. Thus, we have the following:

Theorem 3.1. *If the generator of a quasi Einstein manifold is parallel vector field then it is irrotational Killing vector field.*

Again, we know that, if the generator of a quasi Einstein manifold is a Killing vector field, then the manifold satisfies cyclic Ricci tensor [6], i.e.

$$(\nabla_X Ric)(Y, Z) + (\nabla_Y Ric)(Z, X) + (\nabla_Z Ric)(X, Y) = 0$$

Therefore, we have

Theorem 3.2. *If the generator of a quasi Einstein manifold is parallel vector field then the manifold satisfies cyclic Ricci tensor.*

Now, let U^\perp denote the $n-1$ -dimensional distribution in quasi Einstein manifold, orthogonal to U and if X and Y belongs to U^\perp then

$$g(X, U) = g(Y, U) = 0$$

By $(\nabla_Y g)(X, U) = 0$ we have

$$g(\nabla_Y X, U) + g(X, \nabla_Y U) = 0$$

and if $\nabla_X U = 0$, we get

$$g(\nabla_Y X, U) = 0$$

Similarly we get $g(\nabla_X Y, U) = 0$.

Hence, if $\nabla_X U = 0$, we find that

$$g(\nabla_X Y, U) = g(\nabla_Y X, U)$$

i.e. $g(\nabla_X Y - \nabla_Y X, U) = 0$

i.e. $g([X, Y], U) = 0$

So, $[X, Y] \in U^\perp$ when $X, Y \in U^\perp$.

Hence we can state the following theorem with the help of Frobenius theorem

Theorem 3.3. *If the generator of a quasi Einstein manifold is parallel vector field then the manifold is locally a product manifold of the one dimensional distribution U and $n - 1$ dimensional distribution U^\perp where U^\perp is involutive and integrable [3].*

4 Quasi Einstein spacetime

A spacetime is a time oriented 4-dimensional (M_4, g) manifold with Lorentz metric g and index 1 i.e. with signature $(+, +, +, -)$ and perfect fluid spacetime is a spacetime whose matter content is a perfect fluid. Further, a perfect fluid quasi Einstein spacetime is a perfect fluid spacetime which is also quasi Einstein i.e. whose Ricci tensor is of the form (1.1) and the generator vector field U is the unit speed time like vector field which is every where tangent to the flow lines of the perfect fluid and defined by

$$(4.1) \quad g(X, U) = w(X), g(U, U) = -1.$$

In a spacetime, Einstein's equation without cosmological constant, the scalar curvature r [1] and symmetric $(0, 2)$ type energy momentum tensor T are respectively given by

$$(4.2) \quad Ric(X, Y) - \frac{r}{2}g(X, Y) = kT(X, Y)$$

$$r = \sum_{i=1}^4 g(e_i, e_i) Ric(e_i, e_i)$$

and

$$T(X, Y) = (\sigma + p)w(X)w(Y) + pg(X, Y)$$

where k is the gravitational constant, $\{e_i\}$ is an orthonormal basis vector field with $e_4 = U$, p is the isotropic pressure of the fluid and σ is the energy density.

Ricci tensor of a spacetime controls the geometry of the spacetime and the energy momentum tensor signifies the physical aspects of the spacetime. The importance of the study of the quasi Einstein spacetime lies in the fact that this spacetime represents the present state of the universe when the effects of viscosity and the heat flux have become negligible and the matter content of the universe may be considered as a perfect fluid.

Now, in a perfect fluid spacetime the following results are obvious.

Theorem 4.1. *A perfect fluid spacetime without cosmological constant is a quasi Einstein manifold whose associated scalars a and b are given by $a = \frac{r}{2} + kp$, $b = k(\sigma + p)$ with $\sigma + p \neq 0$, where r is the scalar curvature of the manifold, k is the gravitational constant, p is the isotropic pressure of the fluid and σ is the energy density.*

We, now state the following results proved by Konar et.al. [9].

Theorem 4.2. *The energy momentum tensor T in a general relativistic spacetime without cosmological constant is recurrent if and only if the spacetime is Ricci recurrent.*

Theorem 4.3. *In a general relativistic spacetime if the energy momentum tensor T without cosmological constant is recurrent, then $\det T = 0$.*

Let us now assume that in a quasi Einstein spacetime the generator vector field U is a parallel vector field, i.e. $\nabla_X U = 0$, for all vector field X and ∇ is the Levi-Civita connection. Consequently we have

$$R(X, Y)Z = \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X, Y]} U = 0$$

and hence

$$(4.3) \quad Ric(X, U) = 0$$

for all vector fields X . Also, we have from (1.1) and (4.1)

$$(4.4) \quad Ric(X, U) = (a - b)w(X)$$

From (4.3) and (4.4) it is clear that $a = b$. Thus, we have the following:

Theorem 4.4. *If the generator U of a quasi Einstein spacetime with associated scalars a, b is parallel vector field then $a = b$.*

Next, let us assume that U is a parallel vector field i.e. $\nabla_Z U = 0$ for every vector field Z i.e. for all X and Z

$$(4.5) \quad g(\nabla_Z U, X) = (\nabla_Z w)(X) = 0.$$

Also, by theorem 4.4, $a = b$, so

$$(4.6) \quad Ric(X, Y) = a[g(X, Y) + w(X)w(Y)].$$

Therefore, differentiating the above relation and using (4.5), we find that

$$(\nabla_Z Ric)(X, Y) = \frac{da(Z)}{a} Ric(X, Y) = B(Z) Ric(X, Y), \text{ (where } B(Z) = \frac{da(Z)}{a}\text{)}$$

which shows that the manifold is Ricci-recurrent. Thus, we have

Theorem 4.5. *The quasi Einstein spacetime is Ricci-recurrent if the generator vector field U is parallel.*

Thus, by the help of theorem 4.2, 4.3 and 4.5, we can state that

Theorem 4.6. *If in a quasi Einstein spacetime the generator vector field is parallel, then $\det T = 0$.*

Also, in a quasi Einstein spacetime, we can deduce a simple result as follows:

Theorem 4.7. *In a quasi Einstein spacetime, the Ricci tensor has two distinct eigenvalues $a - b$ and a , of which the former is simple and the later is of multiplicity 3.*

Since the product of the eigenvalues expresses the value of the determinant, we have

Theorem 4.8. *If in a quasi Einstein spacetime the associated scalars are equal, then $\det(Ric) = 0$.*

again by the help of theorem 4.4, we have

Corollary 4.8.1. *If in a quasi Einstein spacetime the generator vector field is parallel then $\det(Ric) = 0$.*

Now, we are going to show that

Theorem 4.9. *The quasi Einstein spacetime with associated scalars a and b is Ricci symmetric if and only if the generator vector field U is parallel and $a = b = \text{constant}$.*

Proof. From relation (1.1), we get

$$(4.7) \quad (\nabla_Z Ric)(X, Y) = da(Z)g(X, Y) + db(Z)w(X)w(Y) \\ + b(\nabla_Z w)(X)w(Y) + bw(X)(\nabla_Z w)(Y) .$$

If U is a parallel vector field and a and b are constants, then $(\nabla_Z w)(X) = 0$ and $da(Z) = db(Z) = 0$, so we get

$$(\nabla_Z Ric)(X, Y) = 0$$

Thus, the manifold becomes Ricci-symmetric.

Conversely, let $(\nabla_Z Ric)(X, Y) = 0$. Thus, we have

$$da(Z)g(X, Y) + db(Z)w(X)w(Y) + b(\nabla_Z w)(X)w(Y) + bw(X)(\nabla_Z w)(Y) = 0$$

putting $X = U$ and applying $g(\nabla_Z U, U) = (\nabla_Z w)(U) = 0$, we get

$$(4.8) \quad d(a - b)(Z)w(Y) = b(\nabla_Z w)(Y)$$

Now, putting $Y = U$ and applying $(\nabla_Z w)(U) = 0$, we get

$$d(a - b)(Z) = 0$$

So, we are getting $a - b = \text{constant}$. Therefore by relation (4.8), we have $(\nabla_Z w)(Y) = 0$ (as $b \neq 0$) i.e. U is a parallel vector field. As U is a parallel vector field implies $a = b$, we find that

$$(\nabla_Z Ric)(X, Y) = 0 = da(Z)[g(X, Y) + w(X)w(Y)]$$

$$\text{i.e. } \frac{da(Z)}{a} Ric(X, Y) = 0$$

Since $Ric(X, Y) \neq 0$, we have $\frac{da(Z)}{a} = 0$, i.e. $a = \text{constant}$ or $a = \text{constant} = b$. \square

In [5], Chaki et.al proved that, if a spacetime obeying Einstein's equation without cosmological constant has parallel energy momentum tensor, then the spacetime is Ricci symmetric. We can easily show that the converse of the above theorem is also true for a quasi Einstein spacetime.

By theorem 4.9, we know that, a quasi Einstein spacetime is Ricci symmetric only when $a = b = \text{constant}$ and U is a parallel vector field.

So, in a Ricci symmetric quasi Einstein spacetime we have

$$(4.9) \quad (\nabla_Z Ric)(X, Y) = 0,$$

and

$$(4.10) \quad r = 3a = \text{constant}, \text{ i.e. } dr = 0$$

By equation (4.2), we find that

$$(\nabla_Z Ric)(X, Y) - \frac{1}{2}dr(Z)g(X, Y) = k(\nabla_Z T)(X, Y)$$

Applying equation (4.9) and (4.10), we have

$$(\nabla_Z T)(X, Y) = 0$$

since $k \neq 0$. Thus we have the following

Theorem 4.10. *The quasi Einstein spacetime without cosmological constant has parallel energy momentum tensor if and only if the space time is Ricci symmetric.*

Therefore, combining these results, we have

Theorem 4.11. *In a quasi Einstein spacetime the following statements are equivalent*

- i) The spacetime is Ricci symmetric.*
- ii) The energy momentum tensor is parallel.*
- iii) The associated scalars satisfies the condition $a = b = \text{constant}$ with U being a parallel vector field.*

Again, as a consequence of theorem 3.1 and theorem 3.3, we can state the following

Theorem 4.12. *In a quasi Einstein spacetime satisfying the condition $\nabla_X U = 0$, for an observer field U , the following are equivalent [10]*

- i) There is a rest space of U through each point $p \in M$.*
- ii) If the vector fields X and Y are orthogonal to U , then so is $[X, Y]$.*
- iii) U is irrotational, that is $\text{curl } U$ is zero on vector fields orthogonal to U .*

Since we have $\text{curl } U$ is zero, we can state that

Theorem 4.13. *An observer field U in a quasi Einstein spacetime satisfying the condition $\nabla_X U = 0$ is geodesic and irrotational [10].*

Finally, we like to make study of a Perfect Fluid quasi Einstein spacetime following the results of Thomas John I'Anson Bromwich [2] on the classification of quadratic forms by means of invariant factors and the result of Konar et. al [9].

The geometric nature of the spacetime is dominated by Einstein's tensor

$$G(X, Y) = Ric(X, Y) - \frac{1}{2}rg(X, Y).$$

So, to determine the geometric nature of the spacetime under the condition $\nabla_X U = 0$ i.e. $\det T = 0$ (by theorem 4.6), we should consider the Einstein equation

$$G = kT$$

and from $\det T = 0$, we get $\det G = 0$, which gives

$$\det(Ric - \frac{1}{2}rg) = 0$$

On using local coordinates this implies that

$$\det(R_{ij} - \rho g_{ij}) = 0, \rho = \frac{1}{2}r$$

Bromwich has shown, if one or more of the elementary divisors of the equation $\det(R_{ij} - \rho g_{ij}) = 0$ are multiple and real at a point p of the space, a real coordinate system can be chosen for which components of the tensors R_{ij} and g_{ij} are of different types along with their subcases.

Using the relevant result of Bromwich [2] and Konar et. al. [9], we have the following:

Theorem 4.14. *In a perfect fluid quasi Einstein spacetime for which the generator vector field is parallel, a real coordinate system can be chosen such that the components of the tensors R_{ij} and g_{ij} are of the type $g_{12} = 1$, $g_{33} = K_3 (< 0)$, $g_{44} = K_4 (< 0)$, $R_{11} = K_1$, $R_{12} = a$, $R_{33} = aK_3$, $R_{44} = 0$. where K 's are scalars, all other g_{ij} 's and R_{ij} 's are zero ($i, j = 1, 2, 3, 4$) and the elementary divisors are $(\rho - a)^2$, $(\rho - a)$ and ρ where a and 0 are the roots of the equation $\det(R_{ij} - \rho g_{ij}) = 0$ and the contravariant components of the corresponding flow vectors (principal directions) are $(0, 0, 0, (-K_4)^{-\frac{1}{2}})$, and any linear combination of $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$; the corresponding energy density and isotropic pressure are given by $\sigma = \frac{3}{2}ka$ and $p = -\frac{1}{2}ka$.*

References

- [1] J. K. Beem, Paul E. Ehrlich and Kevin L. Easley, *Global Lorentzian Geometry*, Marcel Dekker, Inc., New York 1996.
- [2] T.J.F.A. Bromwich, *Quadratic Forms and their Classification by Means of Invariant Factors*, Cambridge Tracts No.3, Cambridge University Press 1906.
- [3] F. Brickell and R.S. Clark, *Differentiable Manifolds*, Van Nostrand Reinhold Comp., London 1978, 194-197.
- [4] M.C. Chaki and R.K. Maity, *On Quasi-Einstein manifold*, Publ. Math. Debrecen 57 (2000), 297-306.
- [5] M.C. Chaki and S. Ray, *Space-times with covariant-constant energy-momentum tensor*, International Journal of Theoretical Physics 35, 5 (1996), 1027-1032.

- [6] U.C. De and B.K. De, *On quasi-Einstein manifolds*, Commun. Korean Math. Soc. 23, 3 (2008), 413-420.
- [7] P. Debnath and A. Konar, *Quasi-Einstein hypersurfaces of Euclidean space*, Bull. Calcutta Math. Soc. 100 (6) (2008), 711-722.
- [8] R. Deszcz, M. Glogowska, M. Hotlos, Z. Senturk, *On certain quasi-Einstein semi-symmetric hypersurfaces*, Annales Univ. Sci. Budapest, Eotvos sect. Math 41 (1998), 151-164.
- [9] A. Konar and B. Biswas, *Space-time with covariant-recurrent energy-momentum tensor*, Journal of the Calcutta Mathematical Society 1 (2004), 75-90.
- [10] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York 1983.
- [11] R.N. Sen, *On a characterization of conformally flat Riemannian spaces of class one*, The Journal of Australian Mathematical Society VI, 2 (1968), 172-178.

Authors' addresses:

Pratyay Debnath
Hooghly Engineering and Technology College,
Vivekananda Road, Pipulpati, Hooghly, 712103, India.
E-mail: pratyay2004@yahoo.co.in

Arabinda Konar
Department of Mathematics, University of Kalyani,
Nadia, 741235, India.
E-mail: arabindakonar@indiatimes.com