# On weakly pseudo-projectively symmetric manifolds

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**Abstract.** The object of present paper is to study weakly pseudo-projectively symmetric manifolds and pseudo-projectively flat weakly Ricci-symmetric manifolds.

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**Key words**: Weakly symmetric manifold; weakly pseudo-projectively symmetric manifold; weakly Ricci-symmetric manifold; quasi-Einstein manifold.

## 1 Introduction

The notion of weakly symmetric Riemannian manifolds was introduced by L. Tamassy and T. Q. Binh [10] and also studied in [1].

A non-flat Riemannian manifold $(M^n, g)$ $(n > 2)$ is called weakly symmetric manifold if its curvature tensor $K$ of type $(0,4)$ satisfies the condition

$$\begin{aligned}(\bigtriangledown_X K)(Y,Z,U,V) =& A(X)K(Y,Z,U,V) + B(Y)K(X,Z,U,V)\\ &+ C(Z)K(Y,X,U,V) + D(U)K(Y,Z,X,V)\\ &+ E(V)K(Y,Z,U,X),\end{aligned} \tag{1.1}$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where $A, B, C, D$ and $E$ are 1-forms (not simultaneously zero) and $\bigtriangledown$ the operator of covariant differentiation with respect to the Riemannian metric $g$. The 1-forms are called the associated 1-forms of the manifold and an $n$ dimensional manifold of this kind is denoted by $(WS)_n$.

Tamassy and Binh [11] further studied weakly symmetric Sasakian manifolds and proved that such a manifold does not always exist. In [3] the authors established the existence of $(WS)_n$ by an example and proved that in $(WS)_n$, the associated 1-forms $B = C$ and $D = E$. So (1.1) reduces to the following form

$$\begin{aligned}(\bigtriangledown_X K)(Y,Z,U,V) =& A(X)K(Y,Z,U,V) + B(Y)K(X,Z,U,V)\\ &+ B(Z)K(Y,X,U,V) + D(U)K(Y,Z,X,V)\\ &+ D(V)K(Y,Z,U,X).\end{aligned} \tag{1.2}$$

Some authors like De and Bandyopadhyay [4], Shaikh and Baishya [9] extended this notion for conformal curvature tensor, quasi-conformal curvature tensor respectively. Recently Malik and Samavaki [7] have also studied weakly symmetric Riemannian manifolds.

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In 2002 Prasad [8] defined and studied a tensor field $\bar{P}$ on a Riemannian manifold of dimension $n$ $(n > 2)$ which includes the projective curvature tensor $P$. This tensor field $\bar{P}$ is known as pseudo-projective curvature tensor and given by

$$\begin{aligned}\bar{P}(X,Y,Z) =& aK(X,Y,Z) + b[R(Y,Z)X - R(X,Z)Y] \\ & - \frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1.3}$$

where $a$ and $b$ are constants such that $a, b \neq 0$, $K$ is the curvature tensor, $R$ is the Ricci tensor and $r$ is the scalar curvature.

A non-pseudo projectively flat Riemannian manifold $(M^n, g)$ $(n > 2)$ is said to be weakly pseudo-projectively symmetric manifold if the pseudo-projective curvature tensor $\bar{P}$ of type $(0,4)$ satisfies the condition

$$\begin{aligned}(\bigtriangledown_X \bar{P})(Y,Z,U,V) =& A(X)\bar{P}(Y,Z,U,V) + B(Y)\bar{P}(X,Z,U,V) \\ & + C(Z)\bar{P}(Y,X,U,V) + D(U)\bar{P}(Y,Z,X,V) \\ & + E(V)\bar{P}(Y,Z,U,X),\end{aligned} \tag{1.4}$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where $A, B, C, D$ and $E$ are defined as before. Such an n-dimensional manifold is denoted by $(WPPS)_n$.

Section 2 is concerned with preliminaries. It is shown that like $(WS)_n$, in a $(WPPS)_n$ we have always $B = C$ and $D = E$ and hence (1.4) reduces to the form

$$\begin{aligned}(\bigtriangledown_X \bar{P})(Y,Z,U,V) =& A(X)\bar{P}(Y,Z,U,V) + B(Y)\bar{P}(X,Z,U,V) \\ & + B(Z)\bar{P}(Y,X,U,V) + D(U)\bar{P}(Y,Z,X,V) \\ & + D(V)\bar{P}(Y,Z,U,X),\end{aligned} \tag{1.5}$$

where $A, B, D$ are non-zero 1-forms.

In section 3 we have investigated the nature of scalar curvature of a $(WPPS)_n$. It is proved that if in a $(WPPS)_n$ the Ricci tensor is of Codazzi type or constant scalar curvature then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor $R$ corresponding to the eigenvector $Q$ defined by $g(X, Q) = \lambda(X)$, for all $X$. Further we have studied some other properties.

Section 4 is devoted in the study of pseudo-projectively flat $(WRS)_n$. At first we have proved that in a pseudo-projectively flat $(WRS)_n$ the vector field $\rho$ defined by $g(X, \rho) = H(X)$ is a proper concircular vector field. Finally we have shown that a pseudo-projectively flat $(WRS)_n$ $(n > 2)$ is a quasi-Einstein manifold.

## 2 Preliminaries

In this section we derive some formulae which will be required to the study of a $(WPPS)_n$. Let $\{e_i, i = 1, 2, ...., n\}$ be an orthonormal basis of the tangent at any point of the manifold. Then from (1.3), we have the following

$$\sum_{i=1}^{n} \bar{P}(e_i, Y, Z, e_i) = [a + (n-1)b]P(Y,Z), \tag{2.1}$$

where

$$P(Y,Z) = R(Y,Z) - \frac{r}{n} g(Y,Z),$$

(2.2) $$\sum_{i=1}^{n} \bar{P}(X,Y,e_i,e_i) = 0,$$

(2.3) $$\sum_{i=1}^{n} P(e_i,e_i) = 0.$$

It can be easily seen that the pseudo-projective curvature tensor $\bar{P}$ is skew symmetric with respect to first two indices but neither symmetric nor skew-symmetric with respect to last two indices. Also neither symmetric nor skew-symmetric in first and last two indices.

**Proposition 2.1.** *In a Riemannian manifold* $(M^n, g)$ $(n > 2)$ *the pseudo-projective curvature tensor satisfies the following relations*

(2.4) $$\begin{aligned}(I)\bar{P}(X,Y,Z,U) + \bar{P}(Y,Z,X,U) + \bar{P}(Z,X,Y,U) = 0,\\ (II)\bar{P}(X,Y,U,Z) + \bar{P}(Y,Z,U,X) + \bar{P}(Z,X,U,Y) = 0.\end{aligned}$$

**Proposition 2.2.** *The defining condition of* $(WPPS)_n$ *can always be expressed in the form (1.5).*

*Proof.* Interchanging $Y$ and $Z$ in (1.4), we get

(2.5) $$\begin{aligned}(\nabla_X \bar{P})(Z,Y,U,V) =& A(X)\bar{P}(Z,Y,U,V) + B(Z)\bar{P}(X,Y,U,V)\\ &+ C(Y)\bar{P}(Z,X,U,V) + D(U)\bar{P}(Z,Y,X,V)\\ &+ E(V)\bar{P}(Z,Y,U,X).\end{aligned}$$

Adding (1.4) and (2.5) then using skew-symmetric property of $\bar{P}$ , we obtain

(2.6) $$\mu(Y)\bar{P}(X,Z,U,V) + \mu(Z)\bar{P}(X,Y,U,V) = 0,$$

where $\mu(X) = B(X) - C(X), \forall X$.

Now we choose a particular vector field $\rho$ such that $\mu(\rho) \neq 0$. Substituting $Y = Z = \rho$ in (2.6) we get $\bar{P}(X,\rho,U,V) = 0$. Again putting $Z = \rho$ in (2.6) we get $\bar{P}(X,Y,U,V) = 0$ for all vector fields $X, Y, U$ and $V$ which contradicts our assumption that the manifold in not pseudo projectively flat. Hence we must have $\mu(X) = 0$ for all $X$, and $B = C$. Similarly, by interchanging $U$ and $V$ in (1.4) and proceeding as above, it can be easily seen that $D = E$. Thus all the associated 1-forms $A, B, C, D$ and $E$ coincide, since $B = C$ and $D = E$. Therefore (1.4) can be written as (1.5). □

## 3 The nature of scalar curvature

Let $L$ be the symmetric endomorphism of the tangent space at any point of the manifold corresponding to the Ricci tensor $R$ i.e.

$$g(LX,Y) = R(X,Y).$$

**Theorem 3.1.** *If in a Riemannian manifold $(M^n, g)$ $(n > 2)$, the Ricci tensor if of Codazzi type then relation (3.4) holds. Converse also holds if the scalar curvature is constant.*

*Proof.* From (1.3) it follows by virtue of Bianchi identity that

$$\begin{aligned}
&(\bigtriangledown_X \bar{P})(Y,Z,U,V) + (\bigtriangledown_Y \bar{P})(Z,X,U,V) + (\bigtriangledown_Z \bar{P})(X,Y,U,V)\\
&=b[g(Y,V)\{(\bigtriangledown_X R)(Z,U) - (\bigtriangledown_Z R)(X,U)\}\\
&\quad + g(Z,V)\{(\bigtriangledown_Y R)(X,U) - (\bigtriangledown_X R)(Y,U)\}\\
&\quad + g(X,V)\{(\bigtriangledown_Z R)(Y,U) - (\bigtriangledown_Y R)(Z,U)\}]\\
&\quad - \frac{1}{n}\left[\frac{a}{n-1} + b\right][dr(X)\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\}\\
&\quad + dr(Y)\{g(X,U)g(Z,V) - g(Z,U)g(X,V)\}\\
&\quad + dr(Z)\{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}].
\end{aligned} \tag{3.1}$$

If the Ricci tensor is of Codazzi type [5] i.e. if

$$(\bigtriangledown_X R)(Z,U) = (\bigtriangledown_Z R)(X,U), \tag{3.2}$$

which implies

$$dr(X) = 0, \forall X. \tag{3.3}$$

By virtue of (3.2) and (3.3), (3.1) becomes

$$(\bigtriangledown_X \bar{P})(Y,Z,U,V) + (\bigtriangledown_Y \bar{P})(Z,X,U,V) + (\bigtriangledown_Z \bar{P})(X,Y,U,V) = 0. \tag{3.4}$$

Conversely suppose that in a Riemannian manifold (3.4) holds, then (3.1) becomes

$$\begin{aligned}
&b[g(Y,V)\{(\bigtriangledown_X R)(Z,U) - (\bigtriangledown_Z R)(X,U)\}\\
&\quad + g(Z,V)\{(\bigtriangledown_Y R)(X,U) - (\bigtriangledown_X R)(Y,U)\}\\
&\quad + g(X,V)\{(\bigtriangledown_Z R)(Y,U) - (\bigtriangledown_Y R)(Z,U)\}]\\
&=\frac{1}{n}\left[\frac{a}{n-1} + b\right][dr(X)\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\}\\
&\quad + dr(Y)\{g(X,U)g(Z,V) - g(Z,U)g(X,V)\}\\
&\quad + dr(Z)\{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}].
\end{aligned} \tag{3.5}$$

Putting $Y = V = e_i$ in (3.5) and then taking summation over $i, 1 \leq i \leq n$, we obtain

$$\begin{aligned}
&b[(\bigtriangledown_X R)(Z,U) - (\bigtriangledown_Z R)(X,U)]\\
&=\frac{1}{n}\left[\frac{a}{n-1} + b\right]\{dr(X)g(Z,U) - dr(Z)g(X,U)\}.
\end{aligned} \tag{3.6}$$

Since $r$ is constant, then (3.6) shows that Ricci tensor is of Codazzi type. □

**Theorem 3.2.** *If in a $(WPPS)_n$ the Ricci tensor is of Codazzi type or the scalar curvature is constant then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor R corresponding to the eigenvector Q defined by $g(X,Q) = \lambda(X)$, for all X provided that $a + \frac{(n-2)b}{2} \neq 0$.*

*Proof.* First we Suppose that the Ricci tensor is of Codazzi type so by virtue of (3.4) and (1.5), we have

$$\lambda(X)\bar{P}(Y,Z,U,V)+\lambda(Y)\bar{P}(Z,X,U,V)+\lambda(Z)\bar{P}(X,Y,U,V)=0, \tag{3.7}$$

where

$$\lambda(X)=A(X)-2B(X),\forall X.$$

Putting $Y=V=e_i$ in (3.7) and then taking summation over $i, 1\leq i\leq n$, we obtain by virtue of (2.1) that

$$\{a+(n-1)b\}[\lambda(X)P(Z,U)-\lambda(Z)P(X,U)]+\lambda(\bar{P}(Z,X,U))=0. \tag{3.8}$$

Again putting $X=U=e_i$ in (3.8) and then taking summation over $i, 1\leq 1\leq n$ and then using (2.3) we obtain

$$\lambda(LZ)=\frac{r}{n}\lambda(Z), \tag{3.9}$$

provided that $a+\frac{(n-2)b}{2}\neq 0$, i.e.

$$R(Z,Q)=\frac{r}{n}g(Z,Q).$$

Next suppose that scalar curvature is constant. By virtue of (3.1) and (1.5), we obtain

$$\begin{aligned}
&\lambda(X)\bar{P}(Y,Z,U,V)+\lambda(Y)\bar{P}(Z,X,U,V)+\lambda(Z)\bar{P}(X,Y,U,V)\\
&\quad=b[g(Y,V)\{(\bigtriangledown_X R)(Z,U)-(\bigtriangledown_Z R)(X,U)\}\\
&\qquad+g(Z,V)\{(\bigtriangledown_Y R)(X,U)-(\bigtriangledown_X R)(Y,U)\}\\
&\qquad+g(X,V)\{(\bigtriangledown_Z R)(Y,U)-(\bigtriangledown_Y R)(Z,U)\}]\\
&\qquad-\frac{1}{n}\left[\frac{a}{n-1}+b\right][dr(X)\{g(Z,U)g(Y,V)-g(Y,U)g(Z,V)\}\\
&\qquad+dr(Y)\{g(X,U)g(Z,V)-g(Z,U)g(X,V)\}\\
&\qquad+dr(Z)\{g(Y,U)g(X,V)-g(X,U)g(Y,V)\}].
\end{aligned} \tag{3.10}$$

Setting $Y=V=e_i$ in (3.10) and then taking summation over $i, 1\leq i\leq n$, we get

$$\begin{aligned}
&\{a+(n-1)b\}[\lambda(X)P(Z,U)-\lambda(Z)P(X,U)]+\lambda(\bar{P}(Z,X,U))\\
&\quad=(n-2)\left\{\ b\,[(\bigtriangledown_X R)(Z,U)-\bigtriangledown_Z R)(X,U)]\right.\\
&\qquad\left.-\frac{1}{n}\left(\frac{a}{n-1}+b\right)[dr(X)g(Z,U)-dr(Z)g(X,U)]\right\}.
\end{aligned} \tag{3.11}$$

Again contracting over $X$ and $U$, we obtain

$$\frac{(n-2)}{2n}dr(Z)=\lambda(LZ)-\frac{r}{n}\lambda(Z), \tag{3.12}$$

provided $a+\frac{(n-2)b}{2}\neq 0$. Since the manifold is of constant scalar curvature then (3.12) reduces to (3.9) provided that $a+\frac{(n-2)b}{2}\neq 0$. □

**Theorem 3.3.** *If the scalar curvature of $(WPPS)_n$ vanishes then relation (3.17) holds provided that $a + \frac{(n-2)b}{2} \neq 0$. Converse also holds if $T(X) \neq 0$ for all $X$.*

*Proof.* Putting $Y = V = e_i$ in (1.5) and then taking summation over $i, 1 \leq i \leq n$, we get by virtue of (2.1) that

$$\begin{aligned}\{a+(n-1)b\}(\nabla_X P)(Z,U) = [a+(n-1)b]\{A(X)P(Z,U)+B(Z)P(X,U) \\ + D(U)P(Z,X)\} + B(\bar{P}(X,Z,U)) + D(\bar{P}(X,U,Z)).\end{aligned} \tag{3.13}$$

Let $\rho_1, \rho_2, \rho_3$ be the vector fields associated to the 1-forms $A, B$ and $D$ respectively. Therefore we have $A(X) = g(X,\rho_1)$, $B(X) = g(X,\rho_2)$, $D(X) = g(X,\rho_3)$. Substituting $Z = U = e_i$ in (3.13) and then taking summation over $i, 1 \leq i \leq n$, we get

$$P(X,\rho_2) + P(X,\rho_3) = 0, \tag{3.14}$$

provided $a + \frac{(n-2)b}{2} \neq 0$.
Which implies that

$$R(X,\rho_2) + R(X,\rho_3) = \frac{r}{n}[g(X,\rho_2) + g(X,\rho_3)]. \tag{3.15}$$

In view of (3.15), we have

$$R(X,\widetilde{\rho}) = \frac{r}{n} g(X,\widetilde{\rho}), \tag{3.16}$$

where $g(X,\widetilde{\rho}) = T(X) = B(X) + D(X)$, $\widetilde{\rho} = \rho_2 + \rho_3$. Since the scalar curvature $r$ of $(WPPS)_n$ is zero then from (3.16) $R(X,\widetilde{\rho}) = 0$ and so by virtue of (1.3), we have

$$\bar{P}(X,Y,\widetilde{\rho},U) = aR(X,Y,\widetilde{\rho},U). \tag{3.17}$$

Also if (3.17) holds in $(WPPS)_n$, then by virtue of (3.16) it follows from (1.3) that $r = 0$ for $T(X) \neq 0$ for all $X$. □

## 4 Pseudo projectively flat weakly Ricci-symmetric manifolds

The notion of weakly Ricci symmetric manifolds was introduced by Tamassy and Binh ([11]).

A non-flat Riemannian manifold $(M^n, g)$ $(n > 2)$ is called weakly Ricci symmetric if its Ricci tensor of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X R)(Y,Z) = A(X)R(Y,Z) + B(Y)R(X,Z) + D(Z)R(Y,X), \tag{4.1}$$

where $A, B, D$ and $\nabla$ are as before. Such an n-dimensional manifold is denoted by $(WRS)_n$. In [6] Jana and Shaikh have studied quasi-conformally flat weakly Ricci symmetric manifolds.

**Proposition 4.1.** *In a $(WRS)_n$ with $\sigma(X) \neq 0$ the scalar curvature can not be zero and the Ricci tensor will be of the form $R(X,Y) = rH(X)H(Y)$ where the vector field associated with the 1-form $H$ is a unit vector field.*

*Proof.* From (4.1) it follows that

$$(\triangledown_X R)(Y,Z) - (\triangledown_X R)(Z,Y) = [B(Y) - D(Y)]R(X,Z) + [D(Z) - B(Z)]R(X,Y). \tag{4.2}$$

Since $R$ is symmetric, (4.2) can be written as

$$[B(Y) - D(Y)]R(X,Z) = [B(Z) - D(Z)]R(X,Y). \tag{4.3}$$

Let $\sigma(X) = B(X) - D(X)$ for any vector field $X$. Then (4.3) becomes

$$\sigma(Y)R(X,Z) = \sigma(Z)R(X,Y). \tag{4.4}$$

Let $\{e_i, i = 1,2,....,n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = Z = e_i$ in (4.4) and then taking summation over $i, 1 \leq i \leq n$, we get

$$r.\sigma(Y) = \sigma(RY), \tag{4.5}$$

where $\sigma(X) = g(X,\delta)$ for any vector field $X$ and $r$ is the scalar curvature. From (4.4) we have

$$\sigma(\delta)R(X,Z) = \sigma(Z)R(X,\delta) = \sigma(Z)\sigma(RX). \tag{4.6}$$

Using (4.5) in (4.6), we get

$$R(X,Z) = \frac{r}{\sigma(\delta)}\sigma(Z)\sigma(X) = rH(X)H(Z), \tag{4.7}$$

where $H(X) = \frac{\sigma(X)}{\sqrt{\sigma(\delta)}}$ and $g(X,\rho) = H(X)$, $\rho$ is a unit vector field. Now from (4.7) it follows that if $r = 0$, then $R(X,Z) = 0$ which is inadmissible by the definition of $(WRS)_n$. So $r \neq 0$. □

**Proposition 4.2.** *In a $(WRS)_n$ with $\sigma(X) \neq 0$, $r$ is an eigenvalue of the Ricci tensor corresponding to the eigenvector $\delta$.*

*Proof.* From (4.5) it follows that $r.g(Y,\delta) = R(Y,\delta)$, which shows that $r$ is an eigenvalue of the Ricci tensor corresponding to the eigenvector $\delta$. □

**Theorem 4.3.** *In a pseudo projectively flat $(WRS)_n, (n > 2)$ with $\sigma(X) \neq 0$, $a + (n-1)b \neq 0$, $a + b \neq 0$, the vector field $\rho$ defined by $g(X,\rho) = H(X)$ is a proper concircular vector field.*

*Proof.* Differentiating (1.3) covariantly we have

$$\begin{aligned}(\triangledown_W \bar{P})(X,Y,Z) =& a(\triangledown_W K)(X,Y,Z) + b[(\triangledown_W R)(Y,Z)X - (\triangledown_W R)(X,Z)Y] \\ &- \frac{1}{n}\left[\frac{a}{(n-1)} + b\right] dr(W)[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{4.8}$$

Contracting above we get

$$\begin{aligned}(div\bar{P})(X,Y,Z) =& a(divK)(X,Y,Z) + b[(\triangledown_X R)(Y,Z) - (\triangledown_Y R)(X,Z)] \\ &- \frac{1}{n}\left[\frac{a}{(n-1)} + b\right][dr(X)g(Y,Z) - dr(Y)g(X,Z)].\end{aligned} \tag{4.9}$$

We know that in a Riemannian manifold

$$(divK)(X,Y,Z) = (\bigtriangledown_X R)(Y,Z) - (\bigtriangledown_Y R)(X,Z). \tag{4.10}$$

Using (4.10) in (4.9), we have

$$\begin{aligned}(div\bar{P})(X,Y,Z) =&(a+b)[(\bigtriangledown_X R)(Y,Z) - (\bigtriangledown_Y R)(X,Z)] \\ &- \frac{1}{n}\left[\frac{a}{(n-1)} + b\right][dr(X)g(Y,Z) - dr(Y)g(X,Z)],\end{aligned} \tag{4.11}$$

since $div\bar{P} = 0$, so (4.11) becomes

$$\begin{aligned}(a+b)[(\bigtriangledown_X R)(Y,Z) - (\bigtriangledown_Y R)(X,Z)] =& \\ \frac{1}{n}\left[\frac{a}{(n-1)} + b\right]&[dr(X)g(Y,Z) - dr(Y)g(X,Z)].\end{aligned} \tag{4.12}$$

Now (4.7) implies

$$\begin{aligned}(\bigtriangledown_Y R)(X,Z) = dr(Y)H(X)H(Z) + &r[(\bigtriangledown_Y H)(X)H(Z) \\ &+ (\bigtriangledown_Y H)(Z)H(X)].\end{aligned} \tag{4.13}$$

In view (4.13), (4.12) becomes

$$\begin{aligned}&(a+b)[dr(X)H(Y)H(Z) - dr(Y)H(X)H(Z) \\ &\quad + r\{(\bigtriangledown_X H)(Y)H(Z) + (\bigtriangledown_X H)(Z)H(Y) \\ &\quad - (\bigtriangledown_Y H)(X)H(Z) - (\bigtriangledown_Y H)(Z)H(X)\}] \\ &\quad = \frac{1}{n}\left[\frac{a}{(n-1)} + b\right][dr(X)g(Y,Z) - dr(Y)g(X,Z)].\end{aligned} \tag{4.14}$$

Putting $Y = Z = e_i$ in the above expression and then taking summation over $i, 1 \leq i \leq n$, we get

$$\begin{aligned}(a+b)[dr(\rho)H(X) + &r\{(\bigtriangledown_\rho H)(X) + H(X)\sum_{i=1}^{n}(\bigtriangledown_{e_i} H)(e_i)\}] \\ &= \left\{\frac{a(n-1)+b}{n}\right\} dr(X).\end{aligned} \tag{4.15}$$

Now substituting $Y = Z = \rho$ in (4.14) gives

$$r(a+b)(\bigtriangledown_\rho H)(X) = \left[\frac{a(n^2-n-1) + b(n-1)^2}{n(n-1)}\right]\{dr(X) - dr(\rho)H(X)\}. \tag{4.16}$$

By virtue of (4.16), (4.15) assumes the form

$$\begin{aligned}&\left\{\frac{a+(n-1)b}{n(n-1)}\right\}\{(n-2)dr(X) + dr(\rho)H(X)\} \\ &\qquad + r(a+b)H(X)\sum_{i=1}^{n}(\bigtriangledown_{e_i} H)(e_i) = 0.\end{aligned} \tag{4.17}$$

Now putting $X = \rho$ in (4.17), we have

$$\left\{\frac{a + (n-1)b}{n}\right\} dr(\rho) = -r(a+b)\sum_{i=1}^{n}(\bigtriangledown_{e_i} H)(e_i). \tag{4.18}$$

From (4.17) and (4.18) it follows that

$$dr(X) = dr(\rho)H(X), \tag{4.19}$$

if $a + (n-1)b \neq 0$.
Again setting $Z = \rho$ in (4.14) and then using (4.19) we get

$$r(a+b)\{(\bigtriangledown_X H)(Y) - (\bigtriangledown_Y H)(X)\} = 0,$$

which shows that

$$(\bigtriangledown_X H)(Y) - (\bigtriangledown_Y H)(X) = 0, \tag{4.20}$$

if $a + b \neq 0$ (since $r \neq 0$).
By virtue of (4.19), it follows from (4.16) that

$$(\bigtriangledown_\rho H)(X) = 0, \tag{4.21}$$

provided that $a + b \neq 0$.
Again substituting $Y = \rho$ in (4.14) and then using (4.19) and (4.21) we get

$$(\bigtriangledown_X H)(Z) = \left\{\frac{a + (n-1)b}{n(n-1)}\right\}\left\{\frac{dr(\rho)}{r(a+b)}\right\}[H(X)H(Z) - g(X,Z)]. \tag{4.22}$$

Now we consider the scalar function

$$f = \left\{\frac{a + (n-1)b}{n(n-1)}\right\}\left\{\frac{dr(\rho)}{r(a+b)}\right\},$$

then we have

$$\begin{aligned}(\bigtriangledown_X f) = &-\left\{\frac{a + (n-1)b}{n(n-1)}\right\}\left\{\frac{dr(\rho)}{r^2(a+b)}\right\} dr(X) \\ &+\left\{\frac{a + (n-1)b}{n(n-1)}\right\}\left\{\frac{1}{r(a+b)}\right\} d^2r(\rho, X).\end{aligned} \tag{4.23}$$

By virtue of (4.19) we get

$$d^2r(X,Y) = d^2r(\rho,Y)H(X) + dr(\rho)(\bigtriangledown_Y H)(X). \tag{4.24}$$

Now in a Riemannian manifold the second covariant differentiation of any function $h \in C^\infty(M)$ is defined by

$$d^2h(X,Y) = X(Yh) - (\bigtriangledown_X Y)h,$$

for all $X, Y \in \chi(M)$, which shows that

$$d^2h(X,Y) = d^2h(Y,X),$$

for all $X, Y \in \chi(M)$.
And hence by virtue of (2.24) and (2.20), we have

$$d^2r(\rho, Y)H(X) = d^2r(\rho, X)H(Y). \tag{4.25}$$

Putting $Y = \rho$ in above, we get

$$d^2r(\rho, X) = d^2r(\rho, \rho)H(X) = \phi H(X), \tag{4.26}$$

where $\phi = d^2r(\rho, \rho)$ is a scalar function.
Now in consequence of (4.26) and (4.19), (4.23) assumes the form

$$\bigtriangledown_X f = \nu H(X), \tag{4.27}$$

where

$$\nu = \left\{\frac{a + (n-1)b}{n(n-1)}\right\}\left\{\frac{1}{r^2(a+b)}\right\}[r\phi - \{dr(\rho)\}^2].$$

Now we consider a 1-form $\alpha$ given by

$$\alpha(X) = \left\{\frac{a + (n-1)b}{n(n-1)}\right\}\left\{\frac{dr(\rho)}{r(a+b)}\right\}H(X) = fH(X). \tag{4.28}$$

In view of (4.28), (4.27) and (4.20) we have

$$d\alpha(X, Y) = 0, \tag{4.29}$$

i.e. the 1-form $\alpha$ is closed. So (4.22) can be written as follows

$$(\bigtriangledown_X H)(Y) = \alpha(X)H(Y) - fg(X, Y), \tag{4.30}$$

where $\alpha$ is closed. But this means that the vector field $\rho$ corresponding to the 1-form $H$ defined by $g(X, \rho) = H(X)$ is a proper concircular vector field [12]. □

**Theorem 4.4.** *A pseudo projectively flat $(WRS)_n$ $(n > 2)$ with $\sigma(X) \neq 0$, $a + (n-1)b \neq 0$, $a + b = 0$ is of constant scalar curvature and Ricci tensor is symmetric along the direction of the unit vector field $\rho$ defined by $g(X, \rho) = H(X)$ for all $X$.*

*Proof.* Suppose $a + b = 0$ and $a + (n-1)b \neq 0$, then (4.12) becomes

$$dr(X)g(Y, Z) - dr(Y)g(X, Z) = 0, \tag{4.31}$$

which gives

$$dr(X) = 0, \tag{4.32}$$

for all $X$. Now setting $Y = Z = e_i$ in (4.13) and then taking summation over $i, 1 \leq i \leq n$, we get by virtue of (4.32) that

$$(\bigtriangledown_\rho H)(X) + H(X)\sum_{i=1}^{n}(\bigtriangledown_{e_i} H)(e_i) = 0. \tag{4.33}$$

Putting $X = \rho$ in above we get

$$\sum_{i=1}^{n} (\bigtriangledown_{e_i} H)(e_i) = 0.$$

Using above relation in (4.33) we obtain

$$(\bigtriangledown_{\rho} H)(X) = 0. \tag{4.34}$$

Again setting $Y = \rho$ in (4.13) and then using (4.34) and (4.32) we get

$$(\bigtriangledown_{\rho} R)(X, Z) = 0,$$

for all $X, Z$. The proof is complete. □

**Theorem 4.5.** *In a pseudo projectively flat $(WRS)_n$ $(n > 2)$ with $\sigma(X) \neq 0$, $a+b \neq 0$ and $a + (n-1)b = 0$ the vector field $\rho$ defined by $g(X, \rho) = H(X)$ is a unit parallel vector field.*

*Proof.* Suppose $a + b \neq 0$ and $a + (n-1)b = 0$ then (4.12) becomes

$$(\bigtriangledown_X R)(Y, Z) = (\bigtriangledown_Y R)(X, Z), \tag{4.35}$$

for all $X, Y$ and $Z$, which gives

$$dr(X) = 0,$$

for all $X$.
Hence (4.14) assumes the form

$$\begin{aligned} &\{(\bigtriangledown_X H)(Y)H(Z) + (\bigtriangledown_X H)(Z)H(Y) \\ &- (\bigtriangledown_Y H)(X)H(Z) - (\bigtriangledown_Y H)(Z)H(X)\} = 0. \end{aligned} \tag{4.36}$$

Putting $Y = Z = \rho$ in above, we get

$$(\bigtriangledown_{\rho} H)(X) = 0, \tag{4.37}$$

for all $X$. Again putting $Y = \rho$ in (4.36) and using (4.37) we obtain $(\bigtriangledown_X H)(Z) = 0$, for all $X, Z$; this implies $g(Z, \bigtriangledown_X \rho) = 0$, for all $X, Z$. Since $g$ is non-degenerate, it follows that $\bigtriangledown_X \rho = 0$, for all $X$, which shows the result. □

**Remark 4.6.** In a pseudo projectively flat $(WRS)_n$ $(n > 2)$ the case $a + (n-1)b = 0$ and $a + b = 0$ can not occur simultaneously. Suppose if possible they occur simultaneously then we have $a = -b$ and $a = (1-n)b$ which gives $n = 2$, which is contradiction the fact that $n > 2$.

**Theorem 4.7.** *A pseudo projectively flat $(WRS)_n$ $(n > 2)$ is a quasi-Einstein manifold.*

*Proof.* Since the manifold is pseudo projectively flat $(WRS)_n$ then from (1.3)

$$\begin{aligned} 'K(X, Y, Z, W) = &-\frac{b}{a}[R(Y, Z)g(X, W) - R(X, Z)g(Y, W)] \\ &+ \frac{r}{an}\left[\frac{a}{n-1} + b\right][g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \end{aligned} \tag{4.38}$$

Now using (4.7) in above equation we get

$$'K(X,Y,Z,W) = p\,[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + q\,[g(X,W)H(Y)H(Z) - g(Y,W)H(X)H(Z)], \tag{4.39}$$

where $p = \frac{r}{n}\left[\frac{1}{n-1} + \frac{b}{a}\right]$, $q = \frac{-br}{a}$. Now substituting $X = W = e_i$ and then taking summation over $i, 1 \leq i \leq n$, we get

$$R(Y,Z) = p'g(Y,Z) + q'H(Y)H(Z), \tag{4.40}$$

where $p' = (n-1)p$, $q' = (n-1)q$. Hence the manifold is quasi-Einstein [2]. □

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