Linear Weingarten helicoidal surfaces in Minkowski 3-space

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Abstract. In this paper we construct linear Weingarten helicoidal surfaces under the cubic screw motion, i.e., the case with mean curvature H and Gauss curvature K satisfying $lH + K = c$, in 3-dimensional Minkowski space E_1^3, where $c =$ constant, $l =$ constant and $l \neq 0$. The solutions are presented explicitly by integrals.

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1 Introduction

A helicoidal surface is a natural generalization of rotation surface, As for helicoidal surface in R^3, the case with constant mean or Gauss curvature has been studied by Do Carmo and Dajczer [4], and that with prescribed mean or Gauss curvature has been constructed by Baikoussis and Koufogioros [1]. Also in [1], it was proved that, locally, there exist helicoidal surfaces with any given one-variable smooth function as mean curvature or Gauss curvature.

Denote by E_1^3 the *Minkowski 3-space* with an inner product of signature $(1, 2)$ given by

$$g(x, y) = -x_1y_1 + x_2y_2 + x_3y_3,$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$. A *helicoidal surface* in E_1^3 is defined as the orbit of a plane curve under a *Lorentzian screw motion*.

A Lorentzian screw motion can be defined as a Lorentzian rotation around an axis together with a translation in the direction of the axis. Helicoidal surfaces under a *so-called Lorentzian screw motion* with prescribed mean or Gauss curvatures have been studied by Beneki et al [2], Ji and Hou [5].

However, Dillen and Kühnel [3] pointed out that *a Lorentzian rotation around an null axis, together with a translation in the direction of the axis, is again a Lorentzian rotation around a null axis* (see Remark 2.1 in [6]), and there exist other non-trivial 1-parameter families of translations that, together with a Lorentzian rotation around

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a null axis, constitute a 1-parameter group of Lorentzian motions, the so-called *cubic screw motion* ([3]), which is expressed as:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1+\frac{s^2}{2} & -\frac{s^2}{2} & s \\ \frac{s^2}{2} & 1-\frac{s^2}{2} & s \\ s & -s & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + h \begin{pmatrix} \frac{s^3}{3}+s \\ \frac{s^3}{3}-s \\ s^2 \end{pmatrix}.$$

Obviously, a cubic screw motion is different from a non-cubic case. A non-cubic screw motion has the property that, if we take a point of the axis, then the orbit of that point is simply the axis (or the point itself if the screw motion is a rotation). A cubic screw motion does not have that property. In fact, the orbit of the origin under a cubic screw motion is a cubic null curve.

Ji and Hou [6] constructed helicoidal surfaces under the cubic screw motion with prescribed mean or Gauss curvatures. Also in [2], [5] and [6], It was proved that, Locally, there exist helicoidal surfaces of any type with prescribed smooth function as mean or Gauss curvature. In this paper, we study the local-existence and the construction of linear Weingarten helicoidal surfaces under a cubic screw motion, ie., the case with mean curvature and Gauss curvature satisfying

$$(*) \qquad lH + K = c,$$

where l and c are two constants and $l \neq 0$.

2 Helicoidal surfaces under a cubic screw motion in E_1^3

We denote by E_1^3 the 3-dimensional Minkowski space with the Lorentz metric

$$g(x, y) = -x_1y_1 + x_2y_2 + x_3y_3,$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$. An Lorentzian transformation of E_1^3 is a linear map that preserves the bilinear form g.

Now, we consider a pseudo-orthonormal basis of E_1^3, i.e. a basis $\{e_1, e_2, e_3\}$ such that

$$g(e_1, e_1) = g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_3) = 0, \quad g(e_1, e_3) = g(e_2, e_2) = 1.$$

Then the Lorentz metric can be expressed as

$$g(x, x) = 2x_1x_3 + {x_2}^2, \quad x = \Sigma x_k e_k.$$

Let

$$(e_1, e_2, e_3) = (\eta_1, \eta_2, \eta_3)X,$$

where $\{\eta_1, \eta_2, \eta_3\}$ is an orthonormal basis such that

$$g(\eta_i, \eta_j) = \varepsilon\delta_{ij}, \quad \varepsilon = \begin{cases} -1, & \text{if} \quad i = 1, \\ 1, & \text{if} \quad i = 2, 3 \end{cases}$$

and

$$X = \begin{pmatrix} -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{pmatrix}.$$

Then the cubic screw motion around the axis e_3 can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto X^{-1} \begin{pmatrix} 1+\frac{s^2}{2} & -\frac{s^2}{2} & s \\ \frac{s^2}{2} & 1-\frac{s^2}{2} & s \\ s & -s & 1 \end{pmatrix} X \begin{pmatrix} x \\ y \\ z \end{pmatrix} + hX^{-1} \begin{pmatrix} \frac{s^3}{3}+s \\ \frac{s^3}{3}-s \\ s^2 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto A(v) \begin{pmatrix} x \\ y \\ z \end{pmatrix} + h\beta(v), \tag{2.1}$$

where

$$A(v) = \begin{pmatrix} 1 & 0 & 0 \\ v & 1 & 0 \\ -\frac{v^2}{2} & -v & 1 \end{pmatrix}, \quad \beta(v) = \begin{pmatrix} v \\ \frac{v^2}{2} \\ -\frac{v^3}{6} \end{pmatrix}, \quad v = -\sqrt{2}s.$$

Definition 2.1. Let $\gamma : I = (a, b) \subset \mathbb{R} \to P$ be a curve in a plane P in E_1^3 and denote by L a straight line that does not intersect the curve γ. A helicoidal surface under a cubic screw motion in E_1^3 is defined as a non-degenerate surface that is generated by a cubic screw motion around L.

We distinguish the following two cases.

Case 1. Let $\gamma_1(u) = (u, 0, f(u)), u > 0$ be a curve in the Oe_1e_3 plane. Suppose that S_1 be the helicoidal surface generated by $\gamma_1(u)$ under a cubic screw motion with pitch h, the position vector r of which has the form

$$r(u,v) = \left(u + hv, uv + \frac{hv^2}{2}, f(u) - \frac{uv^2}{2} - \frac{hv^3}{6}\right). \tag{2.2}$$

We call S_1 *a helicoidal surface of type I.*

Case 2. Let $\gamma_2(u) = (0, u, f(u)), u > 0$ be a curve in the Oe_2e_3 plane. Suppose that S_2 be the helicoidal surface generated by $\gamma_2(u)$ under a cubic screw motion with pitch h, the position vector r of which has the form

$$r(u,v) = \left(hv, u + \frac{hv^2}{2}, f(u) - uv - \frac{hv^3}{6}\right). \tag{2.3}$$

We call S_2 *a helicoidal surface of type II.*

We say that a helicoidal surface in E_1^3 is of type I^+ or type I^- (resp. type II^+ or type II^-) if the discriminant $D = EG - F^2$ is positive or negative, where E, F, G are the coefficients of its first fundamental form.

Remark 2.1. Especially when $h = 0$, helicoidal surfaces given by (2.2) is called rotation surfaces with null axis. It can be easily seen from (2.3) that helicoidal surface of type V is degenerate when $h = 0$. Rotation surfaces will not be discussed in this paper. We also claim that, unless otherwise stated, all the helicoidal surfaces discussed in this paper are smooth non-degenerate surfaces in E_1^3.

3 Helicoidal surface of type I with $lH + K = c$

Let S_1 be the helicoidal surface given by (2.2). The first fundamental form I, the second fundamental form II of S_1 are

$$I = 2f'du^2 + 2hf'dudv + u^2dv^2$$

and

$$II = |D|^{-\frac{1}{2}}[uf''du^2 + 2hf'dudv + (h^2f' - u^2)dv^2],$$

where

$$D = EG - F^2 = 2u^2f' - h^2f'^2$$

and the *prime* denotes derivative with respect to u.

By a direct computation, we can see that the mean curvature H and the Gauss curvature K of S_1 are given by

$$H = \frac{u^3f'' - 2u^2f'}{2D|D|^{\frac{1}{2}}} \tag{3.1}$$

and

$$K = \frac{-u^3f'' + h^2uf'f'' - h^2f'^2}{D|D|}. \tag{3.2}$$

The solutions of equation $(*)$ for helicoidal surface of type I

Let

$$A = (h^2f' - u^2)|D|^{-\frac{1}{2}}. \tag{3.3}$$

For $u \in (0, +\infty)$, from (3.1) and (3.2) we have

$$H = \frac{1}{2u}(A)' \quad \text{and} \quad K = \frac{1}{2u}\left(\frac{A^2 + \varepsilon h^2}{u^2}\right)', \tag{3.4}$$

where

$$\varepsilon = \begin{cases} 1, & S_1 \text{ is spacelike} \\ -1, & S_1 \text{ is timelike.} \end{cases}$$

Combining $(*)$ with (3.4), we get

$$\frac{1}{2u}\left(lA + \frac{u^2 + \varepsilon h^2}{u^2}\right)' = c. \tag{3.5}$$

The general solution of the differential equation (3.5) is

$$A = \frac{-lu^2 \pm \left[(l^2 + 4c)u^4 + 4c_1u^2 - 4\varepsilon h^2\right]^{\frac{1}{2}}}{2}, \tag{3.6}$$

where c_1 is a integration constant. Combining (3.3) and (3.6), we get

$$f = \int \frac{u^2}{h^2}\left[1 \pm |A|(A^2 + \varepsilon h^2)^{-\frac{1}{2}}\right]du + c_2, \tag{3.7}$$

where A is given by (3.6).

Moreover, let $I \subset (0, +\infty)$ be an open interval and h, l, c be three given real numbers ($hl \neq 0$). Then for any $u_0 \in I$, there exist an open interval $I' \subset I$ containing $u_0 \in I$ and an open interval B of $\mathbb{R}$ containing

$$c_1{}' = \left[(l^2 + 4c)u^2 + \varepsilon h^2/u^2 + (|l|u^2 + 2h + 2)^2\right](u_0),$$

such that the function $F_1(u, c_1) > 0$ and $F_2(u, c_1) > 0$ for any $(u, c_1) \in I' \times B$, where

$$F_1(u, c_1) = A^2 + \varepsilon h^2 \quad \text{and} \quad F_2(u, c_1) = (l^2 + 4c)u^4 + 4c_1u^2 - 4\varepsilon h^2.$$

In fact, an easy computation leads to $F_1(u_0, c_1{}') > 0$ and $F_2(u_0, c_1{}') > 0$, by the continuity of F_1 and F_2, They are positive in a subset of $\mathbb{R}^2$ of the form $I' \times B$.

Hence for any $(u, c_1) \in I' \times B, h \in \mathbb{R}, c_2 \in \mathbb{R}$ and two given real numbers l, c ($l \neq 0$), we can define a two-parametric family of curves

$$\gamma(u, c_1, c_2) = \left(u, 0, \int \frac{u^2}{h^2}\left[1 \pm |A|(A^2 + \varepsilon h^2)^{-\frac{1}{2}}\right] du + c_2\right). \tag{3.8}$$

Applying a screw motion of pitch h on these curves we get a two-parametric family of helicoidal surfaces of type I with mean curvature H and Gauss curvature K satisfying $(*)$ and pitch h. So we have proved the following:

Theorem 3.1. *Let $[O; e_1, e_2, e_3]$ be the pseudo-orthogonal frame with e_3 as a null vector. Suppose that h is a non-zero constant and $H(u)$ and $K(u)$ are two smooth real valued functions satisfying $lH + K = c$, where $c = constant, l = constant$ and $l \neq 0$. Then for any $u_0 \in I$ there exists an open interval $I' \subset I$ containing u_0 so that we can construct a two-parametric family of helicoidal surfaces of type I^+ (resp. I^-) in E_1^3 generated by*

$$\gamma(u, c_1, c_2) = \left(u, 0, \int \frac{u^2}{h^2}\left[1 \pm |A|(A^2 + \varepsilon h^2)^{-\frac{1}{2}}\right] du + c_2\right),$$

where $A = \left\{-lu^2 \pm [(l^2 + 4c)u^4 + 4c_1u^2 - 4\varepsilon h^2]^{1/2}\right\}/2$, under a cubic screw motion with $H(u)$, $K(u)$ and h as the mean curvature, the Gauss curvature and the pitch, respectively.

4 Helicoidal surface of type *II* with $lH + K = c$

Let S_2 be the helicoidal surface given by (2.3). The first fundamental form I, the second fundamental form II of S_2 are

$$I = du^2 + 2hf'dudv - 2hudv^2$$

and

$$II = |D|^{-\frac{1}{2}}(-hf''du^2 + 2hdudv + h^2f'dv^2),$$

where

$$D = EG - F^2 = -2hu - h^2f'^2$$

and the prime denotes derivative with respect to u.

By a direct computation, we can see that the mean curvature H and the Gauss curvature K of S_2 are given by

$$H = \frac{h^2(2uf'' - f')}{2D|D|^{\frac{1}{2}}} \tag{4.1}$$

and

$$K = \frac{-h^2(hf'f'' + 1)}{D|D|}. \tag{4.2}$$

The solutions of equation $(*)$ for helicoidal surface of type II

$$A = f'|D|^{-\frac{1}{2}}. \tag{4.3}$$

For $u \in (0, +\infty)$, from (4.1) and (4.2) we have

$$H = -\frac{h}{2}A' \quad \text{and} \quad K = \frac{h}{2}(\frac{h^2A^2 + \varepsilon}{2hu})', \tag{4.4}$$

where

$$\varepsilon = \begin{cases} 1, & S_2 \text{ is spacelike} \\ -1, & S_2 \text{ is timelike.} \end{cases}$$

Combining $(*)$ with (4.4), we get

$$\frac{h}{2}\left(\frac{h^2A^2 + \varepsilon}{2hu} - lA\right)' = c. \tag{4.5}$$

The general solution of the differential equation (4.5) is

$$A = \frac{-lu^2 \pm \left[(l^2 + 4ch^2)u^2 + 2c_1u^2 - \varepsilon h^2\right]^{\frac{1}{2}}}{h^2}, \tag{4.6}$$

where c_1 is a integration constant. Combining (4.3) and (4.6), we get

$$f = \pm \int \left[\frac{-2huA^2}{h^2A^2 + \varepsilon u}\right]^{1/2} du + c_2, \tag{4.7}$$

where A is given by (4.6).

Moreover, let $I \subset (0, +\infty)$ be an open interval and h, l, c be three given real numbers $(hl \neq 0)$. Then for any $u_0 \in I$, we can always select ${c_1}'$ such that $F_1(u, {c_1}') > 0$ and $F_2(u, {c_1}') > 0$, where

$$F_1(u, c_1) = \frac{-2huA^2}{h^2A^2 + \varepsilon u} \quad \text{and} \quad F_2(u, c_1) = (l^2 + 4ch^2)u^2 + 2c_1u^2 - \varepsilon h^2.$$

Therefore, by the continuity of F_1 and F_2, They are positive in a subset of R^2 of the form $I' \times B$, where $I' \subset I$ is an open interval containing u_0 and an $B \subset \mathbb{R}$ is an open interval containing ${c_1}'$.

Hence for any $(u, c_1) \in I' \times B, h \in \mathbb{R}, c_2 \in \mathbb{R}$ and two given real numbers l, c $(l \neq 0)$, we can define a two-parametric family of curves

$$\gamma(u, c_1, c_2) = \left(u, 0, \pm \int \left[\frac{-2huA^2}{h^2A^2 + \varepsilon u}\right]^{1/2} du + c_2\right). \tag{4.8}$$

Applying a screw motion of pitch h on these curves we get a two-parametric family of helicoidal surfaces of type *II* with mean curvature H and Gauss curvature K satisfying $(*)$ and pitch h. So we have proved the following:

Theorem 4.1. *Let $[O; e_1, e_2, e_3]$ be the pseudo-orthogonal frame with e_3 as a null vector. Suppose that h is a given non-zero constant and $H(u)$ and $K(u)$ are two smooth real valued functions satisfying $lH + K = c$, where $c = constant, l = constant$ and $l \neq 0$. Then for any $u_0 \in I$ there exists an open interval $I' \subset I$ containing u_0 so that we can construct a two-parametric family of helicoidal surfaces of type II^+ (resp. II^-) in E_1^3 generated by*

$$\gamma(u, c_1, c_2) = \left(u, 0, \pm \int \left[\frac{-2huA^2}{h^2A^2 + \varepsilon u}\right]^{1/2} du + c_2\right),$$

where $A = \left\{-lu^2 \pm \left[(l^2 + 4ch^2)u^2 + 2c_1u^2 - \varepsilon h^2\right]^{\frac{1}{2}}\right\} / h^2$, under a cubic screw motion with $H(u)$, $K(u)$ and h as the mean curvature, the Gauss curvature and the pitch, respectively.

References

[1] C. Baikoussis, T. Koufogioros,T., *Helicoidal surface with prescribed mean or Gauss curvature*, J. Geom. 63 (1998), 25-29.

[2] C.C. Beneki, G. Kaimakamis, B.J. Papantoniou, *Helicoidal surface in three dimensional Minkowski space*, J. Math. Anal. Appl. 275 (2002), 586-614.

[3] F. Dillen, W. Kühnel, *Ruled Weingarten surfaces in Minkowski 3-space, Manuscripta Math.* 98, 3 (1999) 307-320.

[4] M. Do Carmo, M. Dajczer, *Helicoidal surface with constant mean curvature*, Töhoku. Math. J. 34 (1982), 425-435.

[5] F. Ji, Z.H. Hou, *A kind of helicoidal surfaces in 3 - dimensional Minkowski space*, J. Math. Anal. Appl. 304 (2005), 632-643.

[6] F. Ji, Z.H. Hou, *Helicoidal surfaces under a cubic screw motion in Minkowski 3-space*, J. Math. Anal. Appl. 318 (2006) 634-647.

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