Slant submanifolds of LP-contact manifolds

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Abstract. The purpose of the present paper is to study slant submanifolds in the setting of LP-contact manifolds. In particular, we have studied the slant submanifolds of an LP-Sasakian manifold via some curvature tensors.

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1 Introduction

The study of Lorentzian almost paracontact manifold was initiated by K. Matsumoto [9]. Later on several authors studied Lorentzian almost paracontact manifolds and their different classes including those of [5, 11].

B.Y. Chen [3] introduced the idea of slant-immersion in Complex manifolds, and was followed up by many geometers (cf., [1], [4], [8], [6], etc.). A. Lotta [7] extended the notion of slant immersion to the setting of almost contact metric manifolds and obtained results of fundamental importance. J.L. Cabrerizo et.al. [1] studied the geometry of slant submanifolds in the more specialized settings of K-contact and Sasakian manifolds. Our aim in the present note is to extend the study of slant submanifold to the setting of $LP-$contact manifolds.

In section 2, we recall some basic results of Lorentzian almost paracontact and Lorentzian para-Sasakian geometry, which are useful for further study. Section 3 deals with the study of slant submanifolds of LP-contact manifolds; in this section we obtain a characteristic equation of slant submanifolds in the setting of LP-contact manifolds. In the last section 4, slant submanifolds of LP-Sasakian manifolds are studied. Here we also obtain a characterization for slant submanifolds via some curvature tensors.

2 Preliminaries

Let $\bar{M}$ be an n dimensional differentiable manifold. An LP-contact structure $(\phi, \xi, \eta, \bar{g})$ on $\bar{M}$ consists of a tensor field ϕ of type$(1, 1)$, a vector field ξ, a 1-form η and a metric tensor field $\bar{g}$ on $\bar{M}$ such that

$$\phi^2 X = X + \eta(X)\xi,$$

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$$\eta(\xi) = -1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0 \tag{2.1}$$

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

$$\eta(X) = \bar{g}(X, \xi) \tag{2.3}$$

for any $X, Y \in T\bar{M}$. In this case

$$\bar{g}(\phi X, Y) = \bar{g}(X, \phi Y). \tag{2.4}$$

Moreover, On $\bar{M}$

$$(\bar{\nabla}_X \phi)Y = \bar{g}(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{2.5}$$

$$\bar{\nabla}_X \xi = \phi X, \tag{2.6}$$

for any vector fields $X, Y \in T\bar{M}$, then $\bar{M}$ is said to be LP-Sasakian manifold (cf., [10], [9]). Let $\bar{M}$ be an n-dimensional LP-Sasakian manifold with structure (ϕ, ξ, η, g), then we have ([5], [10])

$$\bar{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.7}$$

$$\bar{R}(\xi, X)Y = g(X, Y)\xi - \eta(Y)X \tag{2.8}$$

$$\bar{R}(\xi, X)\xi = X + \eta(X)\xi \tag{2.9}$$

$$S(X, \xi) = (n-1)\eta(X)$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y) \tag{2.10}$$

where $\bar{R}(X, Y)Z$ is the Riemannian curvature tensor and S is the Ricci tensor.

Let M be an m-dimensional immersed submanifold of the n-dimensional differentiable manifold $\bar{M}$. If $\bar{M}$ is a Lorentzian manifold endowed with the Lorentzian metric $\bar{g}$, then M also admits a Lorentzian metric induced from $\bar{M}$ which is denoted by the symbol g; also we denote the curvature tensor of the submanifold M by R. Then the Gauss and the Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.11}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.12}$$

for any $X, Y \in TM$ and $V \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_V is the Weingarten map associated with V via

$$g(h(X, Y), V) = g(A_V X, Y). \tag{2.13}$$

For any $x \in M$, $X \in T_xM$ and $V \in T^{\perp}{}_xM$, we write

$$\phi X = TX + NX, \tag{2.14}$$

$$\phi V = tV + nV, \tag{2.15}$$

where TX (respectively tV) denotes the tangential part of ϕX (respectively ϕV) and NX (respectively nV) denotes the normal part of ϕX (respectively ϕV).

3 Slant immersions in LP-contact manifolds

Hereafter, for a submanifold M of an LP-contact manifold, we assume that the structure vector field ξ is tangential to the submanifold M, whence the tangent bundle TM decomposes as

$$TM = D \oplus < \xi >$$

where $\langle \xi \rangle$ is the 1-dimensional distribution on M spanned by the structure vector field ξ; we assume as well that $g(X, X) \geq 0 \; \forall \; X \in TM \backslash \langle \xi \rangle$.

Let M be an immersed submanifold of $\bar{M}$. For any $x \in M$ and $X \in T_x M$, if the vectors X and ξ are linearly independent, then the angle $\theta(X) \in [0, \pi/2]$ between ϕX and $T_x M$ is well defined; if $\theta(X)$ does not depend on the choice of $x \in M$ and $X \in T_x M$, then M is *slant* in $\bar{M}$. The constant angle $\theta(X)$ is then called the slant angle of M in $\bar{M}$ and which in short we denote by $Sla(M)$.

For any $x \in M$ taking $X \in T_x M$ we put $\phi X = TX + NX, TX \in T_x M$ and $NX \in {T_x}^{\perp} M$, thus defining the endomorphism $T : T_x M \to T_x M$, whose square T^2 will be denoted by Q. Then the tensor fields on M of type $(1, 1)$ determined by these endomorphisms will be denoted by the same letters, respectively T and Q.

It is easy to prove that for every $x \in M$, we have $g(TX, Y) = g(X, TY)$, which implies that Q is symmetric. Moreover, in the following we prove that the eigenvalues of Q always belong to $[0, 1]$, for any vector $X \in T_x M - \langle \xi \rangle$

$$g(QX, X) = \|TX\|^2$$

but,

$$\|TX\| \leq \|\phi X\| = \|X\|;$$

therefore,

$$g(QX, X) = \lambda(X)\|X\|^2$$

where $0 \leq \lambda(X) \leq 1$ and λ depend on X. In other words, each eigenvalue of Q lies in $[0, 1]$.

Theorem 3.1. *Let $x \in M$ and let $X \in T_x M$ be an eigenvector of Q associated to the eigenvalue $\lambda(X)$. Suppose that X is linearly independent from ξ_x; then,*

$$\cos\theta(X) = \sqrt{\lambda(X)} \frac{|X|}{|\phi X|} \tag{3.1}$$

Proof. We have $|TX|^2 = g(TX, TX) = \lambda(X)|X|^2$; using the definition of $\theta(X)$, we get

$$\cos\theta(X) = \frac{g(\phi X, TX)}{|TX||\phi X|} = \sqrt{\lambda(X)} \frac{|X|}{|\phi X|},$$

which proves the theorem. □

Lemma 3.1. *Let M be a slant submanifold of an $LP-$contact manifold $\bar{M}$ and let $\theta = Sla(M) \neq \pi/2$. Then Q admits the real number $\cos^2\theta$ as the only non-vanishing eigenvalue, for any $x \in M$. Moreover, the related eigenspace H satisfies $H \subset D$, where $D = Span(\xi_x)^{\perp} \subset T_x \bar{M}$.*

Proof. Let $x \in M$. From equation (3.1), $Ker(Q) \neq T_xM$, otherwise $Sla(M) = \pi/2$ which contradicts the assumption. So let λ be an arbitrary non-vanishing eigenvalue of Q and let H be the corresponding eigenspace, Now we have $dim(D) = n - 1$, and $dim(H)$ is even. If n is even or odd, in both cases $dim(H \cap D) \geq 1$. Let $X \in H \cap D$ be a unit vector, then ϕX is also a unit vector, and from equation (3.1) follows

$$\cos\theta = \sqrt{\lambda(X)},$$

which proves the first assertion. Moreover, for any $X \in H$, formula (3.1) yields $|\phi X| = |X|$ which allows us to conclude that $g(X, \xi) = 0$, and hence $H \subset D$. □

We have noted that, invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant immersion which is neither invariant nor anti-invariant, is called a proper slant immersion. In the case of an invariant submanifold, $T = \phi$ and hence

$$T^2 = \phi^2 = I + \eta \otimes \xi,$$

while in case of anti-invariant submanifold, we have $T^2 = 0$. In fact, we have the following general result which characterize slant immersions.

Theorem 3.2. *Let M be a submanifold of an LP-Contact manifold $\bar{M}$ such that $\xi \in TM$. Then M is a slant submanifold if and only if there exist a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(I + \eta \otimes \xi). \tag{3.2}$$

Furthermore, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Proof. The necessary condition is obvious. For the sufficiency, suppose that there exist a constant λ such that $T^2 = \lambda(I + \eta \otimes \xi)$. Then for any $X \in TM\backslash\langle\xi\rangle$, we have

$$\cos\theta(X) = \frac{g(\phi X, TX)}{|TX||\phi X|} = \frac{g(X, T^2X)}{|TX||\phi X|} = \lambda\frac{g(X, \phi^2 X)}{|TX||\phi X|} = \lambda\frac{|\phi X|^2}{|TX||\phi X|} = \lambda\frac{|\phi X|}{|TX|}$$

and also $\cos\theta(X) = \frac{|TX|}{|\phi X|}$. Therefore $\lambda = \cos^2\theta$. Hence, $\theta(X)$ is constant so M is slant. □.

We have the following corollary, which can be easily proved

Corollary 3.1. *Let M be a slant submanifold of an LP-contact manifold $\bar{M}$ with slant angle θ. Then for any $X, Y \in TM$, we have*

$$g(TX, TY) = \cos^2\theta(g(X, Y) + \eta(X)\eta(Y)) \tag{3.3}$$

$$g(NX, NY) = \sin^2\theta(g(X, Y) + \eta(X)\eta(Y)). \tag{3.4}$$

4 Slant immersions in LP-Sasakian manifolds

In this section, we will study slant submanifolds of LP-Sasakian manifolds.

Theorem 4.1. *Let M be a slant submanifold of an LP-Sasakian manifold $\bar{M}$. Then Q is parallel if and only if M is anti-invariant.*

Proof. Let θ be the slant angle of M in $\bar{M}$; then for any X, Y in TM, by equation (3.2) we infer

$$T^2Y = QY = \cos^2\theta(Y + \eta(Y)\xi) \tag{4.1}$$

$$Q\nabla_X Y = \cos^2\theta(\nabla_X Y + \eta(\nabla_X Y)\xi) \tag{4.2}$$

By taking the covariant derivative of (4.1) with respect to $X \in TM$, we get

$$\nabla_X QY = \cos^2\theta(\nabla_X Y + \eta(\nabla_X Y)\xi + g(Y, \nabla_X\xi)\xi + \eta(Y)\nabla_X\xi) \tag{4.3}$$

and from (4.2) and (4.3),

$$(\bar{\nabla}_X Q)Y = \cos^2\theta(g(Y, \nabla_X\xi)\xi + \eta(Y)\nabla_X\xi). \tag{4.4}$$

Further, on using $\nabla_X\xi = TX$, we get

$$(\bar{\nabla}_X Q)Y = \cos^2\theta(g(Y, TX)\xi + \eta(Y)TX) \tag{4.5}$$

and the assertion follows by using the equation (4.5). □

We further investigate the existence of a slant submanifold via some curvature measures of the submanifold. To this end, we state first some formulae for the curvature tensor, in the next lemma.

Lemma 4.1. *Let M be an immersed submanifold of an LP-Sasakian manifold $\bar{M}$ such that ξ is tangent to M. Then for any $X, Y \in TM$ we have*

$$R(X, Y)\xi = (\nabla_X T)Y - (\nabla_Y T)X, \tag{4.6}$$

where ∇, R are respectively the Levi-Civita connection and the curvature tensor field associated to the metric induced by $\bar{M}$ on M. Moreover,

$$R(\xi, X)\xi = QX + (\nabla_\xi T)X \tag{4.7}$$

$$R(X, \xi, X, \xi) = g(QX, X). \tag{4.8}$$

Proof. We have $\bar{\nabla}_X\xi = \phi X$; using equation (2.11) we infer

$$\phi X = \nabla_X\xi + h(X, \xi). \tag{4.9}$$

From the definition of T, we get

$$TX = \nabla_X\xi$$

and hence $(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y = \nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi$. Similarly, we have $(\nabla_Y T)X = \nabla_Y TX - T\nabla_Y X = \nabla_Y\nabla_X\xi - \nabla_{\nabla_Y X}\xi$. Substituting these equations in the definition of $R(X,Y)\xi$, it is easy to get (4.6). Rewriting (4.6) for $X = \xi$ and $Y = X$ and using (4.9), we obtain

$$R(\xi, X)\xi = (\nabla_\xi T)X - (\nabla_X T)\xi = (\nabla_\xi T)X + QX.$$

Now taking the scalar product of the above equation with X, and using

$$g((\nabla_\xi T)X, X) = g(\nabla_\xi X, TX) - g(TX, \nabla_\xi X) = 0$$

we proves (4.8). □

Theorem 4.2. *Let M be an immersed submanifold of an LP-Sasakian manifold $\bar{M}$, such that the characteristic vector field ξ of $\bar{M}$ is tangent to M. If $\theta \in (0, \frac{\pi}{2})$, then the following statements are equivalent:*

(a) *M is slant, with slant angle θ;*

(b) *for any $x \in M$, the sectional curvature of any 2-plane of T_xM containing ξ_x equals $\cos^2\theta$.*

Proof. Let the statement (a) be true; then for any $X \perp \xi$, by equation (3.2) we get

$$QX = \cos^2\theta X,$$

which by virtue of (4.8) yields

$$R(X, \xi, X, \xi) = \cos^2\theta \tag{4.10}$$

and hence (b) is proved. Conversely, assuming that (b) is true, then for any $X \in TM$, we may write

$$X = X_\xi + {X_\xi}^\perp, \tag{4.11}$$

where $X_\xi = \eta(X)\xi$ and ${X_\xi}^\perp$ is the component of X perpendicular to the ξ; using (4.10) and (4.11) we infer

$$\frac{R({X_\xi}^\perp, \xi, {X_\xi}^\perp, \xi)}{|X_\xi|^2} = \cos^2\theta$$

$$R({X_\xi}^\perp, \xi, {X_\xi}^\perp, \xi) = \cos^2\theta |X_\xi|^2. \tag{4.12}$$

Let X be a unit vector such that $QX = 0$. Then from (4.6) and (4.12) we have

$$\cos^2\theta |{X_\xi}^\perp|^2 = 0. \tag{4.13}$$

If $\cos\theta \neq 0$, then from the above equation we get $X = X_\xi$. This proves that at each point $x \in M$,

$$Ker(Q) = \langle \xi_x \rangle. \tag{4.14}$$

More generally, Let A be the matrix of the endomorphism Q at $x \in M$, then for a unit vector field X on M, $QX = AX$, and as $Q(X_\xi) = 0, X = X_\xi$. Then by (4.8) and (4.12), we get

$$A = \cos^2\theta I \tag{4.15}$$

Choosing $\lambda = \cos^2\theta$, we conclude that for any $x \in M$, This fact together with (4.14) and (3.2) shows that M is slant in $\bar{M}$ with slant angle θ. Finally, suppose that $\cos\theta = 0$ and X is an arbitrary unit vector field such that $QX = \lambda X$ where $\lambda \in C^\infty(M)$. Then, from (4.8) and (4.12), we infer $g(QX, X) = 0$, that is $\lambda = 0$ and therefore $Q = 0$, which means M is anti-invariant. □

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