

Symmetry analysis of quadratic homogeneous differential systems

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Abstract. In this paper, the symmetry group of a differential system of n quadratic homogeneous ordinary differential equations of n variables is studied. For this purpose, we consider the action of both point and contact transformations to signify the corresponding Lie algebras. We also find the independent differential invariants of these actions.

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1 Introduction

A quadratic homogeneous differential system (QHDS) is a generalization of second degree polynomial system of differential equations that is rooted in natural science, and also in mathematics by virtue of Hilbert's sixteen problem of his famous list of problems (see, e.g., [1] and the references therein). They are studied in some different viewpoints of their applications. For example, in physical sciences, in the Van der Pol oscillator as an important example of qualitative theory of ordinary differential equations (ODEs), in mathematical ecology and in particular in Volterra-Lotka equations, and many other applications in astrophysics and fluid mechanics [1].

There are many aspects to consider while studying a QHDS. For instance, in [4], the geometric classification of trajectories of a given two dimensional QHDS was studied for determining the invariant lines through the origin of the QHDS, and also its location in the plane of parameters for classification of the geometry of its trajectories. In [10], the necessary conditions for the existence of polynomial first integrals and a polynomial symmetry field of an n -dimensional QHDS were given.

In the present work, the symmetry group of a QHDS of a finite dimension is studied. We specify the symmetry groups of both point and contact transformation groups (similar works to this paper can be found in [7, 6]).

Assume that we have a differential system \mathcal{E} of n quadratic homogeneous first

order ODEs of n variables, in which each of them is a function of t :

$$(1.1) \quad \mathcal{E} : \dot{x}_i = F_i(x_1, \dots, x_n), \quad F_i = \sum_{j,k} a_{jk}^i x_j x_k,$$

where the coefficients $a_{j,k}^i$ ($1 \leq i, j, k \leq n$) are arbitrary real constants. The solution space of the differential system (1.1) is an immersed submanifold of the first order jet space $J^1(\mathbb{R}, \mathbb{R}^n)$ of dimension $2n + 1$ and local coordinates $(t, \mathbf{x}, \mathbf{p}) := (t, x_1, \dots, x_n, p_1, \dots, p_n)$, where $p_j = \dot{x}_j$.

According to [8], the action of the contact transformations group is introduced on the jet space $J^1(\mathbb{R}, \mathbb{R}^n)$, and each element of this group is introduced as the following transformation

$$(1.2) \quad T = \chi(t, \mathbf{x}, \mathbf{p}), \quad X_i = \psi_i(t, \mathbf{x}, \mathbf{p}), \quad P_j = \pi_j(t, \mathbf{x}, \mathbf{p}),$$

where $1 \leq i, j \leq n$. An important special case of the contact transformation group is the point transformation group, where the action reduces to the below change of coordinates

$$T = \chi(t, \mathbf{x}), \quad X_i = \psi_i(\mathbf{x}, \mathbf{p}),$$

when the variables come from jet space $J^0(\mathbb{R}, \mathbb{R}^n)$.

Remark 1.1 *In general, the point and the contact symmetry groups of a geometric object are not necessarily equal. A jet space $J^k(\mathbb{R}^m, \mathbb{R}^n)$ of order k can be imbedded in other jet spaces $J^l(\mathbb{R}^m, \mathbb{R}^n)$ of order $l > k$ as a submanifold. Also a contact transformation of $J^l(\mathbb{R}^m, \mathbb{R}^n)$ can be restricted to $J^k(\mathbb{R}^m, \mathbb{R}^n)$ and then acts as a point transformation.*

Since the number of independent variables in our problem is equal to one, then there are first order contact transformations which do not come from point transformations (see e.g. [8, Theorem 6.46]).

In the next section, we will see that the point symmetry group of differential system \mathcal{E} is of dimension 1, while its contact symmetry group has infinite dimension.

2 The point symmetry of a QHDS

As indicated above, an important question of a quadratic homogeneous system is about its symmetry group. There are various methods for signifying symmetry group of infinitesimal generators via Lie's method of infinitesimals [2, 3, 5, 7, 8]. In the present work, we follow the method of [8] to find the point and contact symmetries.

In the jet space $J^1(\mathbb{R}, \mathbb{R}^n)$, the system of equations (1.1) can be written as

$$(2.1) \quad p_i = \sum_{j,k} a_{jk}^i x_j x_k, \quad 1 \leq i, j, k \leq n.$$

For investigating the symmetry group of the system \mathcal{E} , we consider the general form of point infinitesimal generators as

$$(2.2) \quad v = T \frac{\partial}{\partial t} + \sum_{i=1}^n K_i \frac{\partial}{\partial x_i},$$

in which, T and K_i s are smooth functions of variables t, x_1, \dots, x_n . The first order prolongation of v is as the following expression

$$(2.3) \quad v^{(1)} = v + \sum_{i=1}^n K_i^t \frac{\partial}{\partial p_i},$$

where for $1 \leq i \leq n$ we assumed

$$(2.4) \quad \begin{aligned} K_i^t &= \mathcal{D}_t(Q^i) + p_{i,t} T \\ &= K_{i,t} + \sum_j p_j [K_{i,x_j} - p_i T_{x_j}] - p_i T_t, \end{aligned}$$

where \mathcal{D}_t is the total derivative with respect to the variable t and $Q^i = K_i - p_i T$ is the characteristic of infinitesimal generator v (see [8, Theorem 4.16]).

According to [8], $v^{(1)}$ as a prolongation of v is a symmetry of system \mathcal{E} if and only if it satisfies the relation $v^{(1)}[\mathcal{E}] = 0$. By effecting $v^{(1)}$ on the equations of the system we conclude the following system of equations for $i = 1, \dots, n$:

$$(2.5) \quad \begin{aligned} &K_1 \left(\sum_k (a_{1k}^i + a_{k1}^i) x_k \right) + \dots + K_n \left(\sum_k (a_{nk}^i + a_{kn}^i) x_k \right) \\ &- K_{i,t} - p_1 K_{i,x_1} - p_2 K_{i,x_2} - \dots - p_n K_{i,x_n} \\ &+ p_i (T_t + p_1 T_{x_1} + \dots + p_n T_{x_n}) = 0. \end{aligned}$$

In the last equation, since we can prescribe t, x_i and p_j arbitrarily and the functions K_i and N_j only depend on t, x_i , the equations (2.5) will be satisfied if and only if the individual coefficients of the powers of p_j vanish, i.e., we have the following system of equations

$$(2.6) \quad K_1 \left(\sum_k (a_{1k}^i + a_{k1}^i) x_k \right) + \dots + K_n \left(\sum_k (a_{nk}^i + a_{kn}^i) x_k \right) - K_{i,t} = 0,$$

$$(2.7) \quad K_{i,x_1} = 0, \quad K_{i,x_2} = 0, \quad \dots, \quad K_{i,x_{i-1}} = 0, \quad K_{i,x_{i+1}} = 0, \quad \dots, \quad K_{i,x_n} = 0,$$

$$(2.8) \quad T_t - K_{i,x_i} = 0, \quad T_{x_1} = 0, \quad \dots, \quad T_{x_n} = 0.$$

for every $1 \leq i \leq n$.

From (2.7)–(2.8) we understand that T depends only on t and for each i , K_i is a function of t and x_i . For a fixed $1 \leq i \leq n$, by differentiating with respect to x_i of (2.6) and then by replacing $T_t = K_{i,x_i}$ in this equation, we deduce the following expression

$$T_t \left(\sum_k (a_{ik}^i + a_{ki}^i) x_k \right) = T_{tt}.$$

By solving the last equation in terms of t , we conclude the following solution

$$T(t) = c_1 \exp \left(t \sum_k (a_{ik}^i + a_{ki}^i) x_k \right) + c_2,$$

for constants c_1 and c_2 . Since T is only a function of t , c_1 must be zero and therefore

$$T(t, x_1, \dots, x_n) = c,$$

for an arbitrary constant c . Then for each i , we see that $K_{i,x_i} = 0$ and hence it only depends to t . But by solving (2.6) with respect to $K_i(t)$, we obtain the below solutions (where i varies between 1 and n)

$$(2.9) \quad K_i(t) = \exp\left(t \sum_k (a_{ik}^i + a_{ki}^i)x_k\right) \cdot \left(\int \exp\left(-t \sum_k (a_{ik}^i + a_{ki}^i)x_k\right) F dt + c_3\right),$$

in which we assumed that $F = \sum_{j \neq i} K_j(t) \left(\sum_k (a_{jk}^i + a_{kj}^i)x_k\right)$ and c_3 are arbitrary constants. From (2.9) and knowing the fact that all of K_i s depend only on t , we infer

$$K_i(t, x_1, \dots, x_n) = 0 \quad \text{for } 1 \leq i \leq n,$$

where c is an arbitrary constant. The symmetry group of point infinitesimal generators is corresponding to the point Lie algebra $g = \langle v := c \frac{\partial}{\partial t} \rangle$ is, in exact, the translation of t by a constant coefficient of the parameter s of one-parameter subgroup

$$(t, x_1, \dots, x_n) \mapsto (cs + t, x_1, \dots, x_n).$$

Theorem 2.1 *The set of all infinitesimal generators of the point symmetry group of system (1.1) is a one-dimensional Lie algebra with the basis $\langle \frac{\partial}{\partial t} \rangle$. Therefore, the symmetry group of the system (1.1) is a one-dimensional Lie group of time translations.*

Theorem 2.2 *The independent first integrals of the characteristic system of the infinitesimal generator $\frac{\partial}{\partial t}$ are the invariants*

$$(2.10) \quad \begin{aligned} I_1(t, x_1, x_2, \dots, x_n) &= x_1 = d_1, \\ &\vdots \\ I_n(t, x_1, x_2, \dots, x_n) &= x_n = d_n. \end{aligned}$$

for arbitrary constants d_i .

Proof. Relating to [8, Theorem 2.74], the invariants $u = I(t, x_1, x_2, \dots, x_n)$ of the one-parameter groups corresponding to infinitesimal generators in the form (2.2) satisfy a linear, homogeneous partial differential equation of the first order:

$$v[I] = 0.$$

The solutions of the last equation are found by the method of characteristics (See [3] and [8] for details). So we can replace the last equation by the following characteristic system of ordinary differential equations

$$(2.11) \quad \frac{dt}{T} = \frac{dx_1}{K_1} = \frac{dx_2}{K_2} = \dots = \frac{dx_n}{K_n}.$$

By solving (2.11) of differential generator $v = c \frac{\partial}{\partial t}$, we (locally) find the following general solutions

$$I_1(t, x_1, x_2, \dots, x_n) = x_1 = d_1, \quad \dots \quad I_n(t, x_1, x_2, \dots, x_n) = x_n = d_n,$$

for constants d_i where $i = 1, \dots, n$. □

3 The contact symmetry of a QHDS

On the other side, we deal with the infinitesimal generators of contact transformations. Strictly speaking, the contact transformation group acts on the jet space J^1 of the system (1.1). Hence, we may suppose the general form of an infinitesimal generator as

$$(3.1) \quad v = T \frac{\partial}{\partial t} + \sum_{i=1}^n K_i \frac{\partial}{\partial x_i} + \sum_{j=1}^n P_j \frac{\partial}{\partial p_j},$$

for arbitrary smooth functions T, K_i, P_j in terms of variables t, x_i, p_j .

Since this expression depends on variables of the 1-jet space and is a infinitesimal generator of the jet space, thus the prolongation process is not necessary. Therefore, according to [8], v is a symmetry of \mathcal{E} if and only if it satisfies the relation $v[\mathcal{E}] = 0$. After applying v on the system we obtain

$$K_1 \sum_k (a_{1k}^i + a_{k1}^i) x_k + \cdots + K_n \sum_k (a_{nk}^i + a_{kn}^i) x_k - P_i = 0,$$

where i varies between 1 and n .

Thus, the general form of each symmetry of the system has the form

$$(3.2) \quad v = T \frac{\partial}{\partial t} + \sum_{i=1}^n \left\{ K_i \frac{\partial}{\partial x_i} + \left(\sum_{j,k} K_j (a_{jk}^i + a_{kj}^i) x_k \right) \frac{\partial}{\partial p_i} \right\},$$

in which T, K_1, K_2, \dots, K_n are arbitrary functions of t, x_1, x_2, \dots, x_n . One may divide v to the following vector fields

$$(3.3) \quad \begin{aligned} v_1 &= T \frac{\partial}{\partial t}, \\ v_2 &= K_1 \left(\frac{\partial}{\partial x_1} + \sum_{j,k} (a_{1k}^j + a_{k1}^j) x_k \frac{\partial}{\partial p_j} \right), \\ &\vdots \\ v_{n+1} &= K_n \left(\frac{\partial}{\partial x_n} + \sum_{j,k} (a_{nk}^j + a_{kn}^j) x_k \frac{\partial}{\partial p_j} \right). \end{aligned}$$

The Lie bracket of each two of v_1, \dots, v_{n+1} is defined to be the commutator $[v_i, v_j] := v_i v_j - v_j v_i$ ($1 \leq i, j \leq n+1$). It is not hard to see that the commutator of each two of v_1, \dots, v_{n+1} is a linear combination of themselves and hence v_1, \dots, v_{n+1} generate a Lie algebra of infinitesimals. Roughly speaking, where $1 \leq \alpha, \beta \leq n$, assuming

$$\begin{aligned} v_{\alpha+1} &= K_\alpha \left(\frac{\partial}{\partial x_\alpha} + \sum_{j,k} (a_{\alpha k}^j + a_{k\alpha}^j) x_k \frac{\partial}{\partial p_j} \right), \\ v_{\beta+1} &= K_\beta \left(\frac{\partial}{\partial x_\beta} + \sum_{l,m} (a_{\beta m}^l + a_{m\beta}^l) x_m \frac{\partial}{\partial p_l} \right), \end{aligned}$$

Table 1: The commutators table of Lie algebra of equation (3.1)

$[,]$	v_1	v_2	\cdots	v_n	v_{n+1}
v_1	0	$v_2 - v_1$	\cdots	$v_n - v_1$	$v_{n+1} - v_1$
v_2	$v_1 - v_2$	0	\cdots	$v_n - v_2$	$v_{n+1} - v_2$
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
v_n	$v_n - v_1$	$v_2 - v_n$	\cdots	0	$v_{n+1} - v_n$
v_{n+1}	$v_1 - v_{n+1}$	$v_2 - v_{n+1}$	\cdots	$v_n - v_{n+1}$	0

one finds that

$$\begin{aligned}
[v_{\alpha+1}, v_{\beta+1}] &= \{K_\alpha K_{\beta, x_\alpha} + K_\alpha \sum_{j,k} K_{\beta, p_j} (a_{\alpha k}^j + a_{k\alpha}^j) x_k\} \frac{\partial}{\partial x_\beta} \\
&\quad - \{K_\beta K_{\alpha, x_\beta} + K_\beta \sum_{j,k} K_{\alpha, p_j} (a_{\beta k}^j + a_{k\beta}^j) x_k\} \frac{\partial}{\partial x_\alpha} \\
&\quad + \sum_j \left\{ K_\alpha K_{\beta, x_\alpha} \sum_k (a_{\beta k}^j + a_{k\beta}^j) x_k \right. \\
&\quad \left. - K_\beta K_{\alpha, x_\beta} \sum_k (a_{\alpha k}^j + a_{k\alpha}^j) x_k \right. \\
&\quad \left. + \left(K_\alpha \sum_{l,m} K_{\beta, p_l} (a_{\alpha m}^j + a_{m\alpha}^j) x_m \right) \left(\sum_k (a_{\beta k}^j + a_{k\beta}^j) x_k \right) \right. \\
&\quad \left. - K_\beta \left(\sum_{l,m} K_{\alpha, p_l} (a_{m\beta}^j + a_{\beta m}^j) x_m \right) \left(\sum_k (a_{\alpha k}^j + a_{k\alpha}^j) x_k \right) \right\} \frac{\partial}{\partial p_j}.
\end{aligned}$$

In addition, one can compute the table of Lie symmetry algebra of commutators arising from infinitesimal operators (3.3) which is given in Table 1.

Theorem 3.1 *The symmetry group of the system \mathcal{E} of quadratic homogeneous differential system (1.1) under the action of contact transformations is an infinite dimensional Lie algebra generated by the infinitesimal operators (3.3).*

Theorem 3.2 *The functions of the form*

$$\begin{aligned}
(3.4) \quad I_1(t, \mathbf{x}, \mathbf{p}) &= \int (K_1 dt - T dk_1) = d_1, \dots, I_n(t, \mathbf{x}, \mathbf{p}) = \int (K_n dt - T dk_n) = d_n, \\
I_{n+1}(t, \mathbf{x}, \mathbf{p}) &= \sum_{j,k=1}^n (a_{jk}^1 + a_{kj}^1) x_k d_j \\
&\quad + \int T \left(\sum_{j,k=1}^n (a_{jk}^1 + a_{kj}^1) x_k dx_j - dp_1 \right) = d_{n+1} \dots \\
I_{2n}(t, \mathbf{x}, \mathbf{p}) &= \sum_{j,k=1}^n (a_{jk}^n + a_{kj}^n) x_k d_j \\
&\quad + \int T \left(\sum_{j,k=1}^n (a_{jk}^n + a_{kj}^n) x_k dx_j - dp_n \right) = d_{2n},
\end{aligned}$$

for constants d_i ($1 \leq i \leq 2n$) are independent first integrals (invariants) of the characteristic system of infinitesimal generator (3.2).

Proof. According to [8, Theorem 2.74] and considering the infinitesimal operators in the form (3.2), the invariants $I(t, \mathbf{x}, \mathbf{p})$ of the corresponding one-parameter groups are determined from the characteristic system of ordinary differential equations ($i, j, k = 1, \dots, n$):

$$(3.5) \quad \frac{dt}{T} = \frac{dx_i}{K_i} = \frac{dp_j}{K_1 \sum_k (a_{1k}^j + a_{k1}^j)x_k + \dots + K_n \sum_k (a_{nk}^j + a_{kn}^j)x_k}$$

which tends to $2n$ invariant functions. Solving these equations, the result of the theorem will be derived. \square

According to [3, page 76], one can deduce the following important result for every QHDS.

Corollary 3.3 *The invariants introduced in (3.4) provide the following general solution for the quadratic homogeneous differential system (1.1)*

$$(3.6) \quad S(t, \mathbf{x}, \mathbf{p}) := \varphi(I_1(t, \mathbf{x}, \mathbf{p}), I_2(t, \mathbf{x}, \mathbf{p}), \dots, I_{2n}(t, \mathbf{x}, \mathbf{p})),$$

for an arbitrary function φ which satisfies the relation $v[\varphi] = 0$ and therefore,

$$(3.7) \quad T \frac{\partial S}{\partial t} + \sum_{i=1}^n \left(K_i \frac{\partial S}{\partial k_i} + \left(\sum_{j,k} K_j (a_{jk}^i + a_{kj}^i)x_k \right) \frac{\partial S}{\partial p_i} \right) = 0.$$

The solution of the last equation is implicitly defined by $S(t, \mathbf{x}, \mathbf{p}) = 0$. If $\frac{\partial S}{\partial t} \neq 0$, then the solution can be explicitly written as $t = \gamma(\mathbf{x}, \mathbf{p})$.

Example 3.4 Considering $T = 1$ and $K_i = 0$, the problem turns back to a point symmetry, in which as we saw in the previous section, the symmetry group is provided by translating times and fixing the other variables.

Example 3.5 If we suppose $T = 1$ and $K_i = 1$ for some i and $K_j = 0$ for $j \neq i$, then we have the following expression of infinitesimal generators

$$(3.8) \quad v = \frac{\partial}{\partial t} + \frac{\partial}{\partial x_i} + \left(\sum_{j,k} (a_{jk}^i + a_{kj}^i)x_k \right) \frac{\partial}{\partial p_j}.$$

Thus its one-parameter group is

$$(3.9) \quad (t, \mathbf{x}, \mathbf{p}) \mapsto (t + s, x_1, \dots, x_i + s, \dots, x_n, p_m + a_{ii}^m s^2 + s \sum_k (a_{ik}^m + a_{ki}^m)x_k)$$

where s is the parameter of the flow and m changes through $1, \dots, n$. If the point $(t, \mathbf{x}, \mathbf{p})$ is fixed, then the flow of v is space parabolic, by restricting the jet space coordinates to t, x_i and p_j -axis for a $1 \leq j \leq n$ and a fixed value i .

Example 3.6 Let $T = t$, $K_i = 1$ for an index i and $K_j = 0$ for other indices. The infinitesimal operator v and its flow are respectively given by changing the coefficient of $\frac{\partial}{\partial t}$ in (3.8) to t and the component $t + s$ of the right hand side of (3.9) to $t e^s$. By fixing $(t, \mathbf{x}, \mathbf{p})$ and considering the action of the symmetry group, the projection of the flow of v to the coordinates (t, x_i, p_j) , where $1 \leq j \leq n$, is like the space curve $(s, e^s, s^2 + s)$, including two branches, where the first branch comes from the space exponential map and the second one comes from a space parabolic.

Example 3.7 For the case in which $T = ct$ for constant c , if we suppose that $K_i = 1$ ($i = 1, \dots, n$), the contact infinitesimal operator v is

$$v = ct \frac{\partial}{\partial t} + \sum_i \left\{ \frac{\partial}{\partial x_i} + \left(\sum_{j,k} (a_{jk}^i + a_{kj}^i) x_k \right) \frac{\partial}{\partial p_i} \right\}.$$

The one-parameter group of v is found to be

$$(t, \mathbf{x}, \mathbf{p}) \mapsto \left(t e^{cs}, \mathbf{x} + s, p_m + s^2 \sum_l a_{ll}^m + s \sum_{k,l} (a_{lk}^m + a_{kl}^m) x_k \right),$$

where m varies from 1 to n and we assumed that $1 \leq k, l \leq n$. When the symmetry group acts on a fixed point, the flow of the action when is projected to the three arbitrary axis (t, x_i, p_j) is similar to the 3-dimensional flow explained in Example 3.5. In the particular situation of this example, the characteristic equations tend to the following independent differential invariants ($1 \leq \alpha \leq n$)

$$I_\alpha(t, \mathbf{x}, \mathbf{p}) = t(1 - c k_\alpha),$$

$$I_{n+\alpha}(t, \mathbf{x}, \mathbf{p}) = \sum_{j,k} [(a_{jk}^\alpha + a_{kj}^\alpha)(t - ct(x_j - k_j))x_k] - ct p_\alpha.$$

Example 3.8 Let we consider $T = 0$ and $K_i = x_i$. Then we find the following infinitesimal operator

$$v = \sum_i \left\{ x_i \frac{\partial}{\partial x_i} + \left(\sum_{j,k} (a_{jk}^i + a_{kj}^i) x_j x_k \right) \frac{\partial}{\partial p_i} \right\},$$

and also its one-parameter group

$$(t, \mathbf{x}, \mathbf{p}) \mapsto \left(t, e^s \mathbf{x}, p_i + (e^{2s} - 1) \sum_{j,k} (a_{jk}^i + a_{kj}^i) x_j x_k \right),$$

where i varies among values $1, \dots, n$. Its flow also signifies a parabolic if one reduces the jet coordinate $(t, \mathbf{x}, \mathbf{p})$ to (t, x_l, p_k) for some specified values l and k . Thus the concluding invariants (for $1 \leq \alpha, \beta \leq n$) are

$$I_\alpha = x_\alpha t, \quad I_{n+\alpha} = \sum_{j,k} (a_{jk}^\alpha + a_{kj}^\alpha) x_j x_k t.$$

Finally, we present an example concerning the condition in which each K_i is a function of p_i .

Example 3.9 We suppose that $T = 0$ and $K_i = p_i$ ($i = 1, \dots, n$). So the infinitesimal generator is of the form

$$v = \sum_i \left\{ p_i \frac{\partial}{\partial x_i} + \left(\sum_{j,k} p_j (a_{jk}^i + a_{kj}^i) x_k \right) \frac{\partial}{\partial p_i} \right\},$$

which has the following trajectory in terms of the parameter s

$$(t, \mathbf{x}, \mathbf{p}) \mapsto (t, \mathbf{x} + \mathbf{p}s, \tilde{\mathbf{p}})$$

where $\tilde{\mathbf{p}}$ consists of n components with the i^{th} component

$$\tilde{p}_i = p_i + \left(\exp \left(\left(\sum_k (a_{ik}^i + a_{ki}^i) x_k \right) s \right) - 1 \right) \left(\frac{\sum_{j,k} p_j (a_{jk}^i + a_{kj}^i) x_k}{\sum_k (a_{ik}^i + a_{ki}^i) x_k} \right).$$

One can derive the following independent differential invariants

$$I_\alpha = p_\alpha t, \quad I_{n+\alpha} = \sum_{j,k} (a_{jk}^\alpha + a_{kj}^\alpha) x_k p_j t.$$

for $\alpha = 1, \dots, n$. Clearly, projecting the jet coordinate to the coordinate (x_m, p_l) , for specified amounts m and l , this is the graph of the exponential map.

According to [8] or [9], if G acts smoothly and transitively on a manifold M , then M is isomorphic to G/H as a homogeneous space, which is obtained by quotienting G by a closed Lie subgroup, that is, an isotropy subgroup provided by an element of G . Thus, one can use the exponential map $\exp : TM \rightarrow M$, defined in a small neighborhood of the identity element of M as a Lie group. From the Baker–Campbell–Hausdorff formula we know that for the vector fields X, Y in the Lie algebra relating to a Lie group, if $[X, Y] = 0$ then the flow of $X + Y$ is $\xi_t = \phi_t \circ \psi_t = \psi_t \circ \phi_t$ where ϕ_t and ψ_t are flows of resp. X and Y .

Remark 3.10 *Using a method similar to the above examples, by different selections of coefficients of (3.2) as linear combinations of 1-jet space variables, their corresponding invariants are of the forms (3.4). Furthermore, all proper projections of the trajectories of their flows to 2 or 3-dimensional space of variables (when their commutators are zero), are similar to a parabolic, an exponential map or a shape of two branches similar to them.*

This result is inferred, since the symmetric group prepared in the above examples acts transitively, if we add the condition that the Lie bracket of every two infinitesimal operators be zero, then the flow of every linear combination of any two of infinitesimal generators is a composition of flows made by these operators.

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