Weil-Petersson metric and infinite conformal iterated schemes

Alejandro Mesón and Fernando Vericat

Abstract. In this article we extend results of C. McMullen about the relationship among the Weil-Petersson metric on Teichm*ü*ller space, elements from dimension theory and Ruelle thermodynamic formalism. In particular we work in a more general setting in which the dynamics are implemented by infinite iterated schemes $\Phi = \left\{ f_i : \overline{H^2} \to \overline{H^2} : i \in N \right\}$. Expressions relating the Hausdorff dimension of the limit set of Φ and the variance of Gibbs states for potentials are obtained.

M.S.C. 2000: 37C45,37F30, 37F15.
Key words: Dimension theory; thermodynamic formalism; Teichmüller spaces.

1 Introduction

In a recent article C. McMullen[4] has presented a very interesting insight on how to reconstruct the Weil-Petersson metric on Teichm*ü*ller spaces from elements of the thermodynamical formalism and dimension theory: Let X_λ be a smooth family of genus $g-$Riemann surfaces belonging to the *Teichmüller space* τ_g. The tangent vector $\frac{dX_\lambda}{d\lambda} \mid_{\lambda=0}$ is uniquely represented by the Beltrami equation $\mu = \rho^{-2} F$, where F is a holomorfic quadratic differential and ρ is the hyperbolic metric, thus the Weil-Petersson metric can be expressed as

$$\left\| \frac{dX_\lambda}{d\lambda} \mid_{\lambda=0} \right\|_{WP}^2 = \|\mu\|_{WP}^2 = \int_{X_0} \rho^2 \left|\mu\right|^2 = \int_{X_0} \rho^2 \left|F\right|^2 .$$

Each member of the family X_λ is expressed as a quotient $X_\lambda = H^2/G_\lambda$, where G_λ is a family of Fuchsian groups smoothly varying with λ. Recall that Fuchsian groups are Kleinian groups whose limit set is contained in a circle. For a Kleinian hyperbolic group on H^2 the limit set must be contained in S^1. Also the limit set of an action group is the accumulation points of the action orbits. According to McMullen work, there exists an unique isotopy $h_\lambda : S^1 \to S^1$ which transports the action of G_0 to that of G_λ. By means of this isotopy it can be obtained a family of quasifuchsian groups Γ_λ acting on $\widehat{C}$[4]. Let now L^λ be the limit set of G_λ, which is a Jordan curve

Differential Geometry - Dynamical Systems, Vol.12, 2010, pp. 128-144.

with $L^0 = S^1$ and let m_λ be the pushforward by h_λ of the Lebesgue measure in S^1. McMullen showed that

$$\frac{d^2(\dim_H L^\lambda)}{d\lambda^2}\,|_{\lambda=0}= -\frac{1}{4}\frac{d^2(\dim_H\ m_\lambda)}{d\lambda^2}\,|_{\lambda=0}=$$

$$\frac{1}{3area\,(X_0)}\left\|\frac{dX_\lambda}{d\lambda}\,|_{\lambda=0}\right\|^2_{WP}, \tag{1.1}$$

where $\dim_H$ means Hausdorff dimension.

McMullen also considers a family of Blaschke products $f_\lambda : H^2 \to H^2$ and an isotopy h_λ conjugating f_λ with f_0 with $h_0 = id_{S^1}$. These conjugations allow to extend the maps to $F_\lambda : \widehat{C} \to \widehat{C}$. If $\mathcal{J}(F_\lambda)$ is the Julia set of F_λ , then relationships between the dimension of $\mathcal{J}(F_\lambda)$ and thermodynamical objects are obtained[4].

The aim of this article is to obtain also relationships between Teichm*ü*ller structures and thermodynamic objects, but now working within the more general context of infinite *conformal iterated schemes.* These will be given by a infinite family of injective contractions, the generators $f_i : X \to X$, with $X \subset \widehat{C}$.

We shall consider particular families of smooth iterated schemes

$$\Phi_\lambda = \left\{f_{i,\lambda} : H^2 \to H^2 : i \in N\right\}$$

in which the maps $f_{i,\lambda}$ are degree d proper holomorphic maps. The functions of the iterated scheme can be expressed as a Blaschke products

$$f_{i,\lambda}(z) = z\prod_{j=2}^{d}\left(\frac{z - c_{j,i,\lambda}}{1 - \overline{c_{j,i,\lambda}}z}\right), c_{j,i,\lambda} \in H^2,$$

so they can be seen as rational maps in the Riemann sphere. Themodynamic properties of hyperbolic rational maps, with statistical connections, were studied by Sridharan [8].

There are isotopies $h_{i,\lambda} : S^1 \to S^1$ carrying the action of the non-perturbed scheme $\{f_{i,0}\}$ to that of $\{f_{i,\lambda}\}$ and with $h_{i,0} = id$, for any i. From this can be obtained a family of iterated schemes of rational maps $F_{i,\lambda} : \widehat{C} \to \widehat{C}$ as generators, and with $f_{i,0} = F_{i,0}$, for any i, we still call this family Φ_λ. Consequently the induced maps $f_\lambda = f_{\lambda,\Phi_\lambda}$ and the induced maps F_λ from the scheme $\left\{F_{i,\lambda} : \widehat{C} \to \widehat{C}\right\}$ are defined.

Let $H_\lambda : H^2 \to \widehat{C}$ be the induced isotopies which conjugate the maps F_0 and F_λ and let V be the vector field $V = \dfrac{dH_\lambda}{d\lambda}\,|_{\lambda=0}$. The *asymptotic growth* of V, is given by

$$I_0(V) = \lim_{r\to 1}\frac{1}{2\pi\left|\log(1-r)\right|}\int_{C_r}\left|V(z)\right|^2\left|dz\right|, C_r = \{z : |z| = r\}$$

In the setting of iterated schemes we will relate the dimension of the limit set Λ^λ with asymptotic growth. To this, some intermediate results will be obtained for more general iterated schemes than those with Blashke products as generators.

For a family of iterated schemes is in general not defined a Weil-Petersson metric like is in the families of surfaces $X_\lambda \in \tau_g$. For schemes of Blaschke products

$\left\{f_{i,\lambda}: H^2 \to H^2 : i \in N\right\}$, with induced families $\{f_\lambda\}$, according the McMullen suggestion can be set

$$\left\| \frac{df_\lambda}{d\lambda} \mid_{\lambda=0} \right\|_{WP}^2 := \frac{d^2 \dim_H \Lambda^\lambda}{d\lambda^2} \mid_{\lambda=0} .$$

However it is unknown, to our knowledge , if this metric has similar properties as $\left\| \frac{dX_\lambda}{d\lambda} \mid_{\lambda=0} \right\|_{WP}^2$.

Thus, to obtain a relationship between dimension and Weil-Petersson metric, special iterated schemes should be considered: Let us take families $\{G_{\lambda,i} : i \in N\}$ of Fuchsian groups acting on H^2. If L_i^λ is the limit set of $G_{i,\lambda}$, then, by the Bowen-Series theory, there is a Markov map $g_{i,\lambda} : S^1 \to S^1$ modelling the $G_{i,\lambda}$ action which is conjugated to an analytic Markov map $f_{i,\lambda} : L_i^\lambda \to L_i^\lambda$. From the scheme $\left\{f_{i,\lambda} : L_i^\lambda \to L_i^\lambda\right\}$ the family of rational maps$\left\{F_{i,\lambda} : \widehat{C} \to \widehat{C}\right\}$ is obtained. As it was explained earlier we can define the respective induced maps f_λ and F_λ. . There is a family of groups corresponding to $\{f_\lambda\}$ with limit set L^λ and a family of quasifuchsian groups$\{\Gamma_\lambda\}$ corresponding the maps $\{F_\lambda\}$,according to the Bowen-Series theory. These families of iterated schemes together with the surfaces $X_\lambda = H^2/G_\lambda \in \tau_g$ associated to them is what we need.

A special case of conformal iterated schemes is derived from an expanding analytic Markov map $g : X \to X$. This dynamical system admits a Markov partition $\{X_i\}$and the maps of the scheme are constructed as the inverses branches of the restriction of g to each X_i. An interesting case is when $X = S^2$ and $g : S^2 \to S^2$ is a rational map whose Julia set admits a Markov partition. These are essentially the dynamics considered in the McMullen article, so a generalization of a part of the McMullen results to more general dynamical maps may be obtained. The relationships are similar to those of the McMullen work and the proofs go along the lines in that article up to the necessary modifications due to the consideration of infinite iterated schemes.

One of the interesting aspects of working with iterated schemes is their possible applications in the field of Statistical Mechanics. A lattice system in Statistical Mechanics is characterized by a set Ω , such that any site i has a spin $\omega_i \in \Omega$, the spins interacting among them. Also the system has a function $A : \Omega \times \Omega \to \{0,1\}$, which is called the *adjacency or transition matrix* since it gives the transition rules for the interactions between sites. The infinite, or bi-infinite sequences of elements in Ω are the "configurations ". Thus we have a symbolic space $\Sigma^\infty = \{\omega = (\omega_i)_{i \in Z^+} : \omega_i \in \Omega\}$, so that this way a Statistical Mechanics lattice system can be constructed from a iterated scheme, with the matrix A given by the rules of the system. Any configuration $\omega \in \Sigma^\infty$ can be interpreted as an "infinite length" path in a graph $\mathcal{G} = \mathcal{G}_A$ constructed by taking as vertices the elements of Ω and where there are $A(r,s)$ edges from a vertex r to a vertex s. In [5] we have assigned a weight $w_A(e_{r,s})$ to any pair of vertices r, s of $\mathcal{G}$, in that case finite, or more precisely to any edge $e_{r,s}$. In particular, for lattices models induced from iterated schemes can be considered a weigh defined from the maps of the scheme, for instance $w(i, i+1) = \log |f_i'|$. Markov processes correspond to cases when a probability $P_{r,s}$ is assigned to the edge $e_{r,s}$.

2 Preliminar definitions and facts

An *iterated scheme* is a set

Definition 2.1.

$$\Phi = \{f_i : X \to X\}_{i \in N}\,, \tag{2.1}$$

where the generators f_i are uniformly injective contractions, i.e.

$$d_{eucl}\left(f_i(x), f_i(y)\right) \leq \lambda d_{eucl}(x,y)$$

for some $\lambda \in (0,1)$ and for any $x, y \in X$.

From the iterated scheme $\Phi = \{f_i : X \to X : i \in N\}$ a Markov space Σ^∞ consisting of sequences $\omega{=}\omega_0\omega_1...$, $\omega_j \in N$ can be formed. A truncation of ω, to its $n-$first symbols is a finite string denoted by $[\omega]_n = \omega_0\omega_1...\,\omega_{n-1}$, and its length will be denoted by $|[\omega]_n| = n$.

The following conditions will be imposed for the maps:

$C1$) *Open set condition*
$f_i(IntX) \cap f_j(intX) = \varnothing$, $i \neq j$.
$C2$) *Bounded distortion*
There is a constant $K \geq 1$ such that
$\left|f'_{[\omega]_n}(x)\right| \leq K \left|f'_{[\omega]_n}(x)\right|$, for any sequence $\omega \in \Sigma^\infty$ and for any $x, y \in X$.
$C3$) *Conformality*
There is an open connected set $U \supset X$ such that holds
$0 < \inf\limits_{x \in U} |f'_i(x)| \leq \sup\limits_{x \in U} |f'_i(x)| < 1$.
$C4$) *Cone condition*
There exist numbers $r, s > 0$, such that for any $x \in \partial X$ the interior of X contains the cone set
$Con(x,u,r) := \{y \in X : 0 \leq \langle y - x, u\rangle \leq |y-x| \cos r < t\}$.

The *limit set* of the iterated scheme Φ is defined as

$$\Lambda := \bigcap_{n=1}^{\infty} \overline{\bigcup_{\omega \in \Sigma^\infty} f_{[\omega]_n}(X)}.$$

By the uniform contraction property of the generators $diam\left(f_{[\omega]_n}(X)\right) \to 0$, as $n \to \infty$, and so $\bigcap\limits_{n=1}^{\infty} \overline{f_{[\omega]_n}(X)} = \{x\}$, which allows to have a well defined map

$$\pi : \Sigma^\infty \to \Lambda$$

$$\pi\left(\omega{=}\omega_0\omega_1...\right) = x$$

which is called a *coding map.*

Since the maps are injective contractions the sets $A_i = f_i(X)$ form a partition of X, then can be well defined a function $f(x) = f_\Phi(x) := f_i^{-1}(x)$ when $x \in A_i$. We shall call the map $f = f_\Phi$ the *map induced by the iterated scheme* Φ. Thus are obtained "potentials"

$$\phi : \Sigma^\infty \to R$$

$$\phi(\omega) = \log\left|f'(\pi(\sigma(\omega)))\right|, \tag{2.2}$$

where $\sigma : \Sigma^\infty \to \Sigma^\infty$ is the usual Bernoulli shift.

As we have mentioned in the introduction the aim is to relate dimensions with Teichm$\ddot{u}$ller structures, one main ingredient to make the link is the Mauldin and Urba$\acute{n}$ski thermodynamic formalism[3], whose main aspects we revise next. Let us consider the symbolic space

$\Sigma^\infty = \left\{\omega = (\omega_n)_{n\in N} : \omega_n \in \mathcal{A}\right\}$, where $\mathcal{A}$ is the alphabet ,$\mathcal{A}$ is at most countable set and it can be taken as the the natural numbers. The dynamical structure is given by the shift $\sigma : \Sigma^\infty \to \Sigma^\infty$. In the symbolic space is put the metric $d_t(\omega, \eta) = \exp(-t(N-1))$, where N is the length of the maximal initial common of symbols of sequences ω and η.

If $\omega \in \Sigma^\infty$, then the *cylinder of length* n which contains ω is the set

$$C^n(\omega) = \{\eta : [\omega]_n = [\eta]_n\}$$

and by C^n is denoted the set of all cylinders of length n.

For a potential $\phi : \Sigma^\infty \to R$, let us consider a "partition function"

$$Z_n(\phi) = \sum_{[\omega]_n} \exp\left(\sup_{\eta\in C^n(\omega)} \{(S_n(\phi))(\eta)\}\right)$$

where $S_n(\phi)(x) := \sum_{i=0}^{n-1} \phi\left(\sigma^i(x)\right)$ (*statistical sum). The topological pressure* of ϕ is the number

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log\left[Z_n(\phi)\right], \tag{2.3}$$

A potential ϕ is called *acceptable* if it is uniformly continuous with respect to the metric d_t and

$osc(\phi) := \sup_i \{\sup(\phi|_{C_i}) - \inf(\phi|_{C_i})\} < \infty$, where $C_i = \{\omega : \omega_0 = i\}$.

Let F a finite subset of the alphabet $\mathcal{A}$, the topological pressure restricted to F is

$$P_F(\phi) = \lim_{n\to\infty} \frac{1}{n} \log\left[\sum_{[\omega]_n} \exp\left(\sup_{\eta\in C_n(\omega\cap F)} \{(S_n(\phi))(\eta)\}\right)\right], \tag{2.4}$$

where $C^n(\omega \cap F) = \left\{\eta \in F^{\mathcal{A}} : [\omega]_n = [\eta]_n\right\}$.

Theorem 2.2. *(Approximation theorem[3]) If* $\phi : \Sigma^{\infty} \to R$ *is acceptable then*

$$P(\phi) = \{P_F(\phi) : F \subset \mathcal{A} \text{ finite}\}$$

Theorem 2.3. *(Variational principle[3]) If the potential* $\phi : \Sigma^{\infty} \to R$ *is acceptable then is valid*

$$P(\phi) = \sup\left\{h_\mu(\sigma) + \int \phi d\mu\right\},$$

where the supremum is taken over all the $\sigma-$*invariant measures with* $\int \psi d\mu > -\infty$ *and* $h_\mu(\sigma)$ *is the usual measure theoretic entropy.*

Let us call $\mathcal{S}$ to the class constituted by the potentials $\phi : \Sigma^{\infty} \to R$ satisfying the next conditions:

1) $V(\phi) := \sum\limits_{n=1}^{\infty} [\sup\limits_{\omega,\eta:[\omega]_n=[\eta]_n} \{|\phi(\omega) - \phi(\eta)|\}] < \infty.$

2) $\sum\limits_{n=1}^{\infty} \exp(\sup(\phi|_{C_n})) < \infty$, where $C_n = \{\omega : \omega_0 = n\}$.

The Ruelle operator[6] associated to the iterated scheme Φ is

$$\mathcal{L} = \mathcal{L}_{\phi,\Phi} : C(\Sigma^{\infty}) \to C(\Sigma^{\infty}) \tag{2.5}$$

$$\mathcal{L}_{\phi,\Phi}(\psi)(\omega) = \sum_{n=1}^{\infty} \exp[\phi(\omega)] \psi(f_n(\pi(\sigma(\omega))))$$

Let $\mathcal{L}^*_{\phi,\Phi}$ be the adjoint of $\mathcal{L}$, if the operator is restricted to the class $\mathcal{S}$ then there is an eigenmeasure μ_ϕ,with eigenvalue $\lambda = \exp(P(\phi))$[3], i.e.

$\mathcal{L}^*_{\phi,\Phi}(\mu_\phi) = \lambda\mu_\phi$.

To establish the analytical properties of the pressure is followed also[3]:

Let us consider the space

$\mathcal{H}_0 = \{\phi : \mathcal{A}^{\infty} \to C \text{ bounded and continuous}\}$, here we are taking $\mathcal{A} = N$. and let $M_t(\phi) := \inf\{L : |\phi(\omega) - \phi(\eta)| \le L d_t(\omega,\eta),\ \omega_1 = \eta_1\} < \infty$

Now let

$\mathcal{H}_t := \{\phi \in \mathcal{H}_0 : M_t(\phi) < \infty\}$ which endowed with the norm

$\|\phi\|_t := \|\phi\|_0 + M_t(\phi)$ becomes a Banach space.

If $\phi \in \mathcal{H}_t \cap \mathcal{S}$, for some $t > 0$, and the incidence matrix is irreducible then there exist an eigenmeasure of the conjugate operator $\mathcal{L}^*_\phi$ which is a Gibbs state for ϕ.

Now we consider a renormalized operator $\mathcal{L}_0 = \exp(P(\phi))\mathcal{L}_{\phi,\Phi}$, the number $\exp(P(\phi))$ is precisely the spectral radius of $\mathcal{L}_{\phi,\Phi}$[3]. Also is valid the following estimation [3] for $\mathcal{L}_0 : \mathcal{H}_t \to \mathcal{H}_t$

$$\left\|\mathcal{L}_0^n(\psi) - \int \phi d\mu_\phi\right\|_t \le K\gamma^n \|\psi\|_t,$$

for some constants K, γ and μ_ϕ the Gibbs state associated to the potential $\phi \in \mathcal{H}_t \cap \mathcal{S}_t$ From this estimation deduces that the spectrum of $\mathcal{L}_0$ consists in an isolated simple eigenvalue $\lambda = \exp(P(\phi))$ and a set contained in the disc $\{|z| < \gamma\lambda\}$.

Let the operator $\mathcal{L}_{q,\psi} \in L(\mathcal{H}_t \cap \mathcal{S})$ given by

$$\mathcal{L}_{q,\phi,\zeta}(\psi) = \sum_{n=1}^{\infty} \exp\left[\phi(\omega) + q\zeta(\omega)\right] \psi\left(f_n(\pi(\sigma(\omega)))\right),$$

then the map $q \mapsto \mathcal{L}_{q,\phi,\zeta}$ is differentiable[3]

$q > \theta(\phi, \zeta) :=$

$\max\{\inf\{q : \sup\{q\zeta\} < \infty\}, \ \inf\{q : \sup\{\phi\} + \sup\{q\zeta\}\} < \infty\}$.

By the perturbation theory of analytic dependence of an isolated eigenvalue, holds that the map

$(q, s) \mapsto P(q\phi + s\zeta, \sigma)$ is real analytic in both the two variables for $q > \theta(\phi) := \inf\{q : P(q\phi) < \infty\}$, $s > \theta(\phi, \zeta)$

The derivative with respect to q of the pressure function is given by

$$\frac{dP(q\phi + s\zeta)}{dq} \mid_{q=q_0} = \int \phi d\mu_{q_0\phi + s\zeta},$$

for $q > \theta(\phi)$ and where $\mu_{q_0\phi+s\zeta}$ is the Gibbs state for the potential $q_0\phi + s\zeta \in \mathcal{H}_t \cap \mathcal{S}$. Also the maps ϕ, ζ must verify that $\int (|\phi| + \tau\zeta) d\mu_{q_0\phi - s\zeta} < \infty$.

Let the "free energy" $T(q) = T_\upsilon(q) := P(q\phi) = \lim_{n\to\infty} \frac{1}{n} \log[Z_n(\phi)]$, now if the potentials are in the class $\mathcal{H}_t \cap \mathcal{S}$ then the free energy is real analytic and

$T^{\acute{}}(q) = \int \phi d\mu_q$, where μ_q is the Gibbs state associated to the potential $q\phi$.

3 Results connecting thermodynamics and Weil-Petersson metrics for a class of iterated schemes

Let $\Phi_\lambda = \{f_{i,\lambda} : X \to X : i \in N\}$ with $\lambda \in (-\delta, \delta)$, be a smooth family of iterated schemes seen as a perturbation of a fixed scheme Φ_0. Let Λ^λ be the limit set of Φ_λ and $\pi_\lambda : \Sigma^\infty \to \Lambda^\lambda$ the coding corresponding to each member of the family. Consequently, from the schemes are defined, as we saw, a family of maps$\{f_\lambda\}$ and then family of potentials $\phi_\lambda(\omega) = \log|f'_\lambda(\pi_\lambda(\sigma(\omega)))|$ also seen as a perturbation of a fixed potential ϕ_0. We call to $\{f_\lambda\}$ the family of iterated schemes induced from the family $\{f_{i,\lambda} : X \to X : i \in N\}$.

The family of free energies $\tau(\lambda, q) = P(q\phi_\lambda)$, $\lambda \in (-\delta, \delta)$, is analytical in the variable q, for $q > \theta(\phi_\lambda)$, with fixed $\lambda \in (-\delta, \delta)$ due to the conditions imposed of the maps of the iterated scheme. Also from the formalism displayed in the previous section any potential ϕ_λ has a Gibbs state μ_{ϕ_λ}

The *Variance* of the potential ψ with respect to the measure μ is defined as

$$Var(\psi, \mu) := \int \psi^2 d\mu - \left(\int \psi d\mu\right)^2$$

If $\lambda \to \phi_\lambda$ is C^2 then the second derivative of the free energy with respect to λ can be calculated as

$$\frac{d^2\tau\left(\lambda,q\right)}{d\lambda^2}\mid_{\substack{\lambda=0\\q=1}}= Var\left(\frac{d\phi_{_\lambda}}{d\lambda}\mid_{\lambda=0},\mu_{\phi_0}\right)+\mu_{\phi_0}\left(\frac{d^2\phi_{_\lambda}}{d\lambda^2}\mid_{\lambda=0}\right)$$

Let $D(\lambda) := \dim_H \Lambda^\lambda$, by the Bowen equation (in its version for infinite iterated schemes and so for countable alphabet symbolics) we have that $P\left(D(\lambda)\phi_\lambda\right) = 0, i.e.$ $\tau\left(\lambda, D(\lambda\right) = 0$, so that differentiating with respect to λ leads to

$$\frac{dD\left(\lambda\right)}{d\lambda}\mid_{\lambda=0}=\frac{-D\left(0\right)\mu\left(\dfrac{\partial\phi_{_\lambda}}{\partial\lambda}\mid_{\lambda=0}\right)}{\mu_{\phi_0}\left(\phi_0\right)} \tag{3.1}$$

$$\begin{aligned}\frac{d^2D\left(\lambda\right)}{d\lambda^2}\mid_{\lambda=0}=&\left\{-Var\left(\frac{dD\left(\lambda\right)}{d\lambda}\mid_{\lambda=0}+D\left(0\right)\frac{d\phi_{_\lambda}}{d\lambda}\mid_{\lambda=0},\mu_{\phi_0}\right)-\right.\\ &\left.-2\frac{dD\left(\lambda\right)}{d\lambda}\mid_{\lambda=0}\times\mu_{\phi_0}\left(\frac{d\phi_{_\lambda}}{d\lambda}\mid_{\lambda=0}\right)D\left(0\right)\mu_{\phi_0}\left(\frac{d^2\phi_{_\lambda}}{d\lambda^2}\mid_{\lambda=0}\right)\right\}\times\frac{1}{\mu_{\phi_0}\left(\phi_0\right)}.\end{aligned} \tag{3.2}$$

Here μ_{ϕ_0} denotes the Gibbs state for the potential $D\left(0\right)\,\phi_0$

Remark 3.1. For the special Markov maps considered in [4] the above formula reduces to

$$\frac{d^2D\left(\lambda\right)}{d\lambda^2}\mid_{\lambda=0}=-\frac{Var\left(\dfrac{d\phi_{_\lambda}}{d\lambda}\mid_{\lambda=0},\mu_{\phi_0}\right)}{\mu_{\phi_0}\left(\phi_0\right)}.$$

Let μ_λ be the pushforward of μ_{ϕ_0} by π_λ, and denote $\mu_\lambda = (\pi_\lambda)_*(\mu_{\phi_0})$, in the case of [4] the coding is taking over$\mathcal{J}(F_{_\lambda})$ with $\mathcal{J}(F_0) = S^1$ and μ_λ is equivalent to the Lebesgue measure *Leb* on S^1 and result $\mu_\lambda = (h_\lambda)_*(Leb)$.

Proposition 3.2. *For a family of iterated schemes*

$$\Phi_\lambda = \{f_{i,\lambda} : X \to X : i \in N\}_{\lambda\in(-\delta,\delta)}$$

with limit set Λ^λ *and associated family of potentials* $\phi_\lambda\left(\omega\right) = \log\left|f'_\lambda(\pi_\lambda\left(\sigma(\omega)\right))\right|$*, the following relationship holds*

$$\begin{aligned}\frac{d^2\dim_H\mu_\lambda}{d\lambda^2}\mid_{\lambda=0}=&\left[P(\phi_0)+\int\phi_0 d\mu_{\phi_0}\right]\times\frac{1}{D(0)}\times\frac{1}{\left[\int\phi_0 d\mu_{\phi_0}\right]^2}\times\\ &\left[\mu_{\phi_0}\left(\phi_0\right)\frac{d^2D\left(\lambda\right)}{d\lambda^2}\mid_{\lambda=0}+Var\left(\frac{dD\left(\lambda\right)}{d\lambda}\mid_{\lambda=0}+D\left(0\right)\frac{d\phi_{_\lambda}}{d\lambda}\mid_{\lambda=0},\mu_{\phi_0}\right)+\right.\\ &\left.2\frac{dD\left(\lambda\right)}{d\lambda}\mid_{\lambda=0}\times\mu_{\phi_0}\left(\frac{d\phi_{_\lambda}}{d\lambda}\mid_{\lambda=0}\right)\right],\end{aligned} \tag{3.3}$$

where $D(\lambda) := \dim_H \Lambda^\lambda$, $\dim_H \mu = \max\left\{\dim_H Z : \mu(Z) = 1\right\}$ *and* μ_{ϕ_0} *is the Gibbs state for the potential* $D\left(0\right)\,\phi_0$.

Proof of Proposition 3.2. The Lyapunov exponent for the measure μ_λ is given by $\chi(\Phi^\lambda, \mu_\lambda = (\pi_\lambda)_*(\mu_{\phi_0})) = -\int \phi_\lambda d\mu_{\phi_0}$ and by the known relationship between Lyapunov exponents and entropy (for this special case can see [7] or [2]) gets

$$\dim_H \mu_\lambda = \dim_H (\pi_\lambda)_*(\mu_{\phi_0}) = \frac{h(\sigma)}{-\int \phi_\lambda d\mu_{\phi_0}} = \frac{P(\phi_0) + \int \phi_0 d\mu_{\phi_0}}{-\int \phi_\lambda d\mu_{\phi_0}}, \tag{3.4}$$

so that

$$\frac{d^2 \dim_H \mu_\lambda}{d\lambda^2} \mid_{\lambda=0} = \left[P(\phi_0) + \int \phi_0 d\mu_{\phi_0}\right] \times \frac{-\int \frac{d^2\phi_\lambda}{d\lambda^2} \mid_{\lambda=0} d\mu_{\phi_0}}{\left[\int \phi_0 d\mu_{\phi_0}\right]^2} \tag{3.5}$$

and finally, from eq. 3.2, is obtained

$$\begin{aligned} \frac{d^2 \dim_H \mu_\lambda}{d\lambda^2} \mid_{\lambda=0} &= \left[P(\phi_0) + \int \phi_0 d\mu_{\phi_0}\right] \times \frac{1}{D(0)} \times \frac{1}{\left[\int \phi_0 d\mu_{\phi_0}\right]^2} \times \\ &\left[\mu_{\phi_0}(\phi_0) \frac{d^2 D(\lambda)}{d\lambda^2} \mid_{\lambda=0} + Var\left(\frac{dD(\lambda)}{d\lambda} \mid_{\lambda=0} + D(0) \frac{d\phi_\lambda}{d\lambda} \mid_{\lambda=0}, \mu_{\phi_0}\right) + \right. \\ &\left. 2 \frac{dD(\lambda)}{d\lambda} \mid_{\lambda=0} \times \mu_{\phi_0}\left(\frac{d\phi_\lambda}{d\lambda} \mid_{\lambda=0}\right)\right]. \end{aligned} \tag{3.6}$$

□

In the case of $D(0) = 1$ is $P(\phi_0) = 0$. If besides is $\frac{dD(\lambda)}{d\lambda} \mid_{\lambda=0} = 0$ then we have

$$\frac{d^2 D(\lambda)}{d\lambda^2} \mid_{\lambda=0} = -\frac{Var\left(\frac{d\phi_\lambda}{d\lambda} \mid_{\lambda=0}, \mu_{\phi_0}\right)}{\mu_{\phi_0}(\phi_0)} \tag{3.7}$$

as is displayed in the McMullen article.

Recall that for familes of Fuchsian groups $\{G_{i,\lambda} : i \in N\}$ acting on the hyperbolic disc by the Bowen-Series theory can be constructed Markov maps $g_{i,\lambda} : S^1 \to S^1$ conjugated to analytic Markov maps $f_{i,\lambda} : L_i^\lambda \to L_i^\lambda$, where L_i^λ is the limit set of the group $G_{i,\lambda}$. So that we have special conformal iterated schemes $\left\{f_{i,\lambda} : L_i^\lambda \to L_i^\lambda\right\}$, the geneartors maps have the conformal properties required for the iterated schemes Then is defined the induced family of schemes $\{f_\lambda\}$ in the already explained way. As it was previously pointed out from the family of Fuchsian groups $\{G_{i,\lambda}\}$, acting on H^2, can be obtained a family of quasifuschsian groups $\{\Gamma_{i,\lambda}\}$ acting on $\widehat{C}$. Recall also that there are corresponding family of rational maps $\left\{F_\lambda : \widehat{C} \to \widehat{C}\right\}$. These maps originate families of quasifuchsian groups Γ_λ whose action is modelled by them. Let L^λ be the limit set of Γ_λ and let $\pi_\lambda : \Sigma^\infty \to L^\lambda$ be the coding from the Bowen-Series theory, like above is defined the family of potentials ϕ_λ and the measures $\mu_\lambda = (\pi_\lambda)_*(\mu_{\phi_0})$. Now if $D(\lambda) := \dim_H L^\lambda$ and μ_λ is the pushforward of μ_{ϕ_0} by π_λ, then, for these special iterated schemes, a proposition similar to proposition 3.2,can be established:

Proposition 3.3. *Let* $\left\{F_{i,\lambda} : \widehat{C} \to \widehat{C} : i \in N\right\}$ *be a family of iterated schemes originated from a family of quasifuchsian groups* $\{\Gamma_{i,\lambda}\}$, *let* L^{λ} *be the limit set of the family of groups. If* $D(\lambda) := \dim_H L^{\lambda}$ *then holds*

$$\frac{d^2 \dim_H \mu_\lambda}{d\lambda^2} |_{\lambda=0} = \left[P(\phi_0) + \int \phi_0 d\mu_{\phi_0}\right] \times \frac{1}{D(0)} \times \frac{1}{\left[\int \phi_0 d\mu_{\phi_0}\right]^2} \times$$

$$\left[\mu_{\phi_0}(\phi_0) \frac{d^2 D(\lambda)}{d\lambda^2} |_{\lambda=0} + Var\left(\frac{dD(\lambda)}{d\lambda} |_{\lambda=0} + D(0) \frac{d\phi_\lambda}{d\lambda} |_{\lambda=0}, \mu_{\phi_0}\right) + \right.$$

$$\left. 2\frac{dD(\lambda)}{d\lambda} |_{\lambda=0} \times \mu_{\phi_0}\left(\frac{d\phi_\lambda}{d\lambda} |_{\lambda=0}\right)\right] \tag{3.8}$$

Propositions 3.2 and 3.3 generalize o iterated schemes and symbolics with countable alphabet results of the section 2 in [4].

Now we are going to relate dimensions with Weil-Petersson metrics. In the first stage we consider smooth families of iterated schemes

$$\Phi_\lambda = \left\{f_{i,\lambda} : H^2 \to H^2 : i \in N\right\}$$

with the $f_{i,\lambda}$ being degree d-holomorphic maps and where those conjugated by an automorphism of H^2 will be identified. As we mentioned in the introduction each representative in these classes can be expressed as a Blaschke product

$$f_{i,\lambda}(z) = z \prod_{j=2}^{d} \left(\frac{z - c_{j,i,\lambda}}{1 - \overline{c}_{j,i,\lambda} z}\right), c_{j,i,\lambda} \in H^2. \tag{3.9}$$

In this step are related dimensions with asymptotic growth of the vector field V. To obtain a link with the metric, as we commented in the introduction, are considered schemes defined from quasi-Fuchsian groups.

Let ζ be a fixed map in the class $\mathcal{H}_t \cap \mathcal{S}$ and a fixed equilibrium measure on Σ^∞, the *stopping time* $N(n,\omega) = N(n,\omega,\zeta)$ is defined as the small integer such that

$S_{N(n,\omega)}(\zeta) > n \int \zeta d\mu$, the map ζ is called the *roof function.* It holds that when n increases, $N(n,\omega)/n$ goes asymptotically to 1[4].

Let $\phi \in \mathcal{H}_t \cap \mathcal{S}$ with associated Gibbs state μ_ϕ, the statistical sum ϕ relative to ζ is the map

$$\omega \longmapsto S_{N(n,\omega,\zeta)}(\phi)(\omega)) = \sum_{i=0}^{N(n,\omega,\zeta)-1} \phi\left(\sigma^i(\omega)\right),$$

and the variance of a potential ψ and the measure μ_ϕ relative to ζ is

$$Var_\zeta(\psi, \mu_\phi) = \lim_{n\to\infty} \frac{1}{n} \left\|S_{N(n,\omega,\zeta)}(\phi)\right\|_2^2.$$

In [4] is established that

$$Var_\zeta(\psi, \mu_\phi) = Var(\psi, \mu_\phi) \tag{3.10}$$

for any roof function ζ and

$$Var\,(\psi,\mu_{\phi})=\frac{\lim\limits_{n\to\infty}\left\langle \psi,\frac{S_{N(n,\omega,\zeta)}(\phi)^{2}}{n}\right\rangle_{\mu_{\phi}}}{\mu_{\phi}(\psi)},\tag{3.11}$$

where $\langle\psi,\xi\rangle_{\mu_{\phi}}:=\int\psi\xi d\mu_{\phi}$.

Let $\Phi=\{f_i:X\to X:i\in N\}$, be an iterated scheme, let $H=\{h_i:X\to Y\}$ and $\Psi=\{g_i:X\to Y\}$, it says that H and Ψ satisfy the *coboundary equation,* with respect to the iterated scheme Φ, if

$$h_i(x)=g_{,i}(x)-g_i(f_i(x)),x\in X,\ i\in N.\tag{3.12}$$

In this case it will say that H and Ψ are $\Phi-$ *coboundaries.*

Recall from the introduction the following

Definition 3.4. The *asymptotic growth of* $g:H^2\to C$ is
$I_0(g)=\lim\limits_{r\to 1}\frac{1}{2\pi\left|\log(1-r)\right|}\int_{C_r}\left|g(z)\right|^2\left|dz\right|$, where $C_r=\{|z|=r\}$.

Let us particularly consider $f_i:H^2\to H^2$, and let A, B be small annular neighborhoods of S^1, with $\overline{A}\subset B$, we shall taken stopping times with respect to the map $\zeta=\phi\left(\omega\right)=\log\left|f_{\lambda}'(\pi_{\lambda}\left(\sigma(\omega)\right))\right|$, the potential with respect to the induced map f. Let $R_n=1-\exp(-n\overline{\phi})$, with $\overline{\phi}=\int\phi d\mu_{\phi}$ and let $Z(n,\omega):=R_n(\pi(\omega))\in H^2$. Now let $E(n,\omega)$ be the small integer such that $f_{[\omega]_E}(Z)\notin A$.

Lemma 3.5. *For any sequence* ω *holds* $E(n,\omega)=N(n,\omega)+O(1)$.

Proof of Lemma 3.5. We have that $f_{[\omega]_E}(Z)\in H^2.-A$, so that $1-\left|f_{[\omega]_E}(Z)\right|$ behaves asymptotically as 1, now by the bounded distortion property for iterated schemes we have

$$1-\left|f_{[\omega]_E}(Z)\right|\left|\left(f_{[\omega]_E}\right)'(z)\right|=\exp(-n\overline{\phi})\left|\left(f_{[\omega]_E}\right)'(z)\right|(z=\pi(\omega)).$$

Taking *log*

$$\log\left|\left(f_{[\omega]_E}\right)'(z)\right|=S_{E(n,\omega)}(\phi)(\omega))=n\overline{\phi}+O(1),$$

and so

$$E(n,\omega)=N(n,\omega)+O(1)$$

□

In the next proposition the relation between variance and growth is emphasized.

Proposition 3.6. *Let the iterated system* $\Phi=\left\{f_i:H^2\to H^2:i\in N\right\}$*, and let* $\Psi=\left\{g_i:H^2\to H^2\right\}$ *and* $H=\left\{h_i:H^2\to H^2\right\}$, $\Phi-$*coboundaries, where the* h_i *are injective contractions and such that* $h\circ\pi\in\mathcal{H}_t\cap\mathcal{S}$*, where* h *is the induced from the scheme* $\{h_i\}$, *then*

$$\frac{Var\,(h\circ\pi,\mu_{\phi})}{-\int\phi d\mu_{\phi}}=I_0(g).$$

Proof of Proposition 3.6. We continue with the potential ϕ associated to Φ as roof function, now

$$S_{N(n,\omega)}(h\circ\pi)(\omega)) = \sum_{k=0}^{N-1}(h\circ\pi)(\sigma^{k-1}(\omega)) =$$

$$= \sum_{k=0}^{E(n\omega i)-1}(h\circ\pi)(\sigma^{k-1}(\omega)) + O(1) = \sum_{i=0}^{E-1} h\circ f_{[\omega]_i}(z) + O(1),$$

with $z=\pi(\omega)$. By the contractibility of the maps of the iterated scheme, there is a $\theta\in(0,1)$ such that for any $i=0,1,...,E$ and for any sequence ω holds

$$\left|f_{[\omega]_i}(z) - f_{[\omega]_i}(Z = Z(n,\omega))\right| = O(\theta^{E-i}).$$

Since $h\circ\pi\in\mathcal{H}_t\cap\mathcal{S}$ then we have

$$\sum_{i=0}^{E-1} h\circ f_{[\omega]_i}(z) = \sum_{i=0}^{E-1} h\circ f_{[\omega]_i}(Z) + O(\theta^{tE-i}) =$$

$$O(1) + \sum_{i=0}^{E-1} h\circ f_{[\omega]_i}(Z) - g(f_{[\omega]_i}(Z)) = O(1) + g(Z) - g(f_{[\omega]_i}(Z)),$$

with $i=0,1,,,E$. Therefore is established that

$$S_{N(n,\omega)}(h\circ\pi)(\omega)) = g(Z(n,\omega)) + O(1) \tag{3.13}$$

Before completing the proof, let us recall the following facts from the Ruelle thermodynamic formalism: let $\mathcal{L}_\phi$ be the Ruelle operator whose spectral radius $\lambda(\mathcal{L}_\phi)$ equals $exp(P(\phi))$. By a generalization of the Perron-Frobenius theorem there is positive eigenfunction ζ such that $\mathcal{L}_\phi(\exp(\zeta)) = \lambda(\mathcal{L}_\phi)\exp(\zeta)$.Thus, when $P(\phi)=0, i.e.$ $\mathcal{L}_\phi(\exp(\zeta)) = \exp(\zeta)$ there exists an unique measure ν such that $\int\mathcal{L}_\phi(\varphi)d\nu = \int\varphi d\nu$ and $\int\exp(-\zeta)d\nu = 1$. The Gibbs state for ϕ can be also obtained by $\mu_\phi = \exp(\zeta)\nu$ and $m=\pi_*(\nu)$ is the Lebesgue measure on S^1. In the case of [4] it has $\dim_H\Lambda = 1$ and so by the Bowen equation, the potentials considered there have pressure zero. Our potential ϕ does not necessarily satisfy this, nevertheless we may consider, if necessary, a “renormalized” potential $\widetilde{\phi} = \phi - P(\phi)$, which has zero topological pressure. Also, because of the translation invariance property of the Gibbs states, we have $\mu_\phi = \mu_{\widetilde{\phi}}$. Besides $N(n,\omega,\phi) = N(n,\omega,\widetilde{\phi})$, and so the main formulae are not altered by a possible renormalization changes. The variance of $h\circ\pi$ can be computed by the eq. (3.11) as

$$Var\,(h\circ\pi,\mu_\phi) = \lim_{n\to\infty}\left\langle\exp(\zeta), \frac{S_{N(n,\omega)}(h\circ\pi)^2}{n}\right\rangle_{\mu_\phi}$$

and, by eq. (3.13), $S_{N(n,\omega)}(h\circ\pi)^2$ can be replaced by $g(Z(n,\omega))^2$. So that

$$Var\,(h\circ\pi,\mu_\phi) = \lim_{n\to\infty}\frac{1}{n}\int g(Z(n,\omega))^2 dm.$$

If we do the change of variables $z\to R := R_n z = (1-\exp(-n\overline{\phi}))z$, then

$$\frac{Var\,(h\circ\pi,\mu_\phi)}{-\int\phi d\mu_\phi} = \frac{Var\,(h\circ\pi,\mu_\phi)}{r/n} = \lim_{r\to\infty}\frac{1}{2\pi r}\int_{C_R}|g(z)|^2\,|dz| = I_0(g).$$

□

The result for schemes defined from groups is obtained in the already described way of constructing the Markov maps from Fuchsian groups.

Recall that for a family iterated schemes $\Phi_\lambda = \left\{ F_{i,\lambda} : \widehat{C} \to \widehat{C} \right\}$ defined from the family $\left\{ f_{i,\lambda} : H^2 \to H^2 : i \in N \right\}$ there are isotopies $H_{i,\lambda} : H^2 \to \widehat{C}$ conjugating the maps $F_{i,0}$ and $F_{i,\lambda}$. Further are defined isotopies H_λ conjugating the actions, namely $F_{i,\lambda} \circ H_{i,\lambda} = H_{i,\lambda} \circ F_{i,0}$ and, consequently, $F_\lambda \circ H_\lambda = H_\lambda \circ F_0$ Let V_i be the vector field $V_i = \frac{dH_{i,\lambda}}{d\lambda} \mid_{\lambda=0}$ and let

$$h_i(z) := \frac{d(\log F'_{i,\lambda}(z))}{d\lambda} \mid_{\lambda=0} .$$

Proposition 3.7. *Let* $\Phi_\lambda = \left\{ f_{i,\lambda} : H^2 \to H^2 : i \in N \right\}$ *be a family of iterated schemes and let* $\phi_{i,\lambda}(\omega) = \log \left| f'_{i,\lambda}(\pi_\lambda(\sigma(\omega))) \right|$ *be the family of potentials associated to* Φ_λ, *then holds:*

$$\frac{Var\left(\frac{d\phi_{i,\lambda}}{d\lambda} \mid_{\lambda=0}, \mu_{\phi_0} \right)}{-\int \phi_0 d\mu_{\phi_0}} = I_0(V_i(z)).$$

Proof of Proposition 3.7. We have

$$F'_{i,\lambda}(H_{i,\lambda}(z)) H'_{i,\lambda}(z) = H'_{i,\lambda}(F_{i,0}(z)) F'_{i,0}(z)$$

or

$$F'_{i,\lambda}(H_{i,\lambda}(z)) = \left(H'_{i,\lambda}(z) \right)^{-1} H'_{i,\lambda}(F_{i,0}(z)) F'_{i,0}(z),$$

so that

$$log\left(F'_{i,\lambda}(H_{i,\lambda}(z)) \right) = -\log\left(H'_{i,\lambda}(z) \right) + \log(H'_{i,\lambda}(F_{i,0}(z))) + \log(F'_{i,0}(z))$$

Thus differentiating with respect to λ

$$\frac{d(\log(F'_{i,\lambda}(H_{i,\lambda}(z)))}{d\lambda} \mid_{\lambda=0} = \frac{1}{H'_{i,0}(z)} \frac{d(H'_{i,\lambda}(z))}{d\lambda} \mid_{\lambda=0} + \frac{\frac{d(H'_{i,\lambda}(F_{i,0}(z)))}{d\lambda} \mid_{\lambda=0}}{H'_{i,0}(F_{i,0}(z))},$$

therefore

$$h_i(z) = \frac{d(H'_{i,\lambda}(z))}{d\lambda} \mid_{\lambda=0} + \frac{d(H'_{i,\lambda}(F_{i,0}(z)))}{d\lambda} \mid_{\lambda=0} = V'_i(z) + V'_i(F_{i,0}(z)).$$

Let $\phi_{i,\lambda}(\omega) = \log \left| f'_{i,\lambda}(\pi_\lambda(\sigma(\omega))) \right|$, the we have

$$\frac{d\phi_{i,\lambda}}{d\lambda} \mid_{\lambda=0} = \mathrm{Re}(h_i(\pi(\sigma(\omega)))$$

and since h_i and V_i' are coboundaries, we arrive, by Proposition 3.7, to

$$\frac{Var\left(\frac{d\phi_{i,\lambda}}{d\lambda}\mid_{\lambda=0},\mu_{\phi_0}\right)}{-\int\phi_0 d\mu_{\phi_0}}=I_0(V_i'(z)).$$

Passing to the induced schemes $\Phi_\lambda=\left\{F_\lambda:\widehat{C}\rightarrow\widehat{C}\right\}$, to the vector field $V=\frac{dH_\lambda}{d\lambda}\mid_{\lambda=0}$ and to the maps $h(z):=\frac{d(\log F_\lambda'(z))}{d\lambda}\mid_{\lambda=0}$ we have

$$\frac{Var\left(\frac{d\phi_{\lambda}}{d\lambda}\mid_{\lambda=0},\mu_{\phi_0}\right)}{-\int\phi_0 d\mu_{\phi_0}}=I_0(V'(z)).$$

□

The variance $Var\left(\frac{d\phi_{\lambda}}{d\lambda}\mid_{\lambda=0},\mu_{\phi_0}\right)$ is related with $\frac{d^2D(\lambda)}{d\lambda^2}\mid_{\lambda=0}$, thus can be obtained a relationship between dimension and growth.

Now, we are going to the result linking dimension and metric in Teichmüller structures. Let $\Omega^k(H^2)$ be the space of holomorphic $k-$forms on the hyperbolic disc, an element of Ω^k is $\omega(z)dz^k$, we consider the differential operator

$\mathcal{D}:\Omega^k(H^2)\rightarrow\Omega^{k+1}(H^2)$ with $\mathcal{D}\omega=D\omega(z)dz^{k+1}$. For $W\in\Omega^k(H^2)$ and $U\in\Omega^l(H^2)$, let

$\langle W,U\rangle_{k,l}=\lim_{r=\rightarrow\infty}\frac{1}{2\pi r}\int_{C_r}W\overline{U}dz,$.

For any holomorphic map $f_i:H^2\rightarrow H^2$, $f_i(z)=\sum_{n=0}^{\infty}a_{n,i}z^n$, let

$I_{k+l}(f_i)=\left\langle D^kf_i,D^lf_i\right\rangle_{k,l}$, it can be checked that

$I_m(f_i)=\lim_{r=\rightarrow\infty}\exp(-rm)\sum_{n=0}^{\infty}n^m\left|a_{n,,i}\right|^2r^{2m}$, for $m>0$, while for $m=0$

$I_0(f_i)=\lim_{r=\rightarrow\infty}\frac{1}{r}\sum_{n=0}^{\infty}\left|a_{n,i}\right|^2r^{2n}$, giving the growth earlier defined.

Lemma 3.8. *[4] For any natural i, if $I_m(f_i)$ exists then $I_{m-1}(f_i)$, $I_{m-2}(f_i),...,I_0(f_i)$ exist and holds $I_m(f_i)=\frac{m!}{2^m}I_0(f_i)$.*

Proof of Lemma 3.8. Let us assume that $I_m(f_i)$ exists for $m>0$, therefore

$$\sum_{n=0}^{\infty}n^m\left|a_{n,i}\right|^2r^{2m}$$

asymptotically behaves as

$$I_m(f_i)\exp(rm)$$

with $r \to \infty$ or as

$$I_m(f_i) \times \frac{1}{(1-r)^m}$$

with $r \to 1$. Thus

$$\sum_{n=0}^{\infty} 2n^m \left|a_{n,i}\right|^2 \int_0^r t^{2m} dt$$

asymptotically behaves as

$$2I_m(f_i) \times \int_0^r \frac{1}{(1-t)^m} dt,$$

or

$$\sum_{n=0}^{\infty} n^{m-1} \left|a_{n,i}\right|^2 r^{2m}$$

asymptotically behaves as

$$2I_m(f_i) \times \frac{1}{(1-r)^{m-1}} = 2I_m(f_i) \exp(r(m-1))$$

for $m > 0$ and as $|\log(r-1)|$ for $m = 0$. So that if $I_m(f_i)$ exists for $m > 0$ then $I_{m-1}(f_i)$ does exist and holds $I_m(f_i) = \dfrac{m-1}{2} I_m(f_i)$ for $m > 1$and $I_1(f_i) = \dfrac{1}{2} I_0(f_i)$, hence by induction the result is proved □

Now we state:

Theorem 3.9. *Let* $\left\{f_{i,\lambda} : H^2 \to H^2\right\}$ *a smooth family iterated schemes where the* $f_{i,\lambda}$ *are proper* $d-$*holomorphic maps, represented as Blashke products which leads to a family of rational maps* $\left\{F_{i,\lambda} : \widehat{C} \to \widehat{C}\right\}$. *Let* $\{f_\lambda\}$ *and* $\Phi_\lambda = \left\{F_\lambda : \widehat{C} \to \widehat{C}\right\}$ *the respective induced families, and* $\phi_\lambda(\omega) = \log\left|f'_\lambda(\pi_\lambda(\sigma(\omega)))\right|$ *the associated potentials. If* Λ^λ *be the limit set of* Φ_λ *and* $D(\lambda) := \dim_H \Lambda^\lambda$, $\dfrac{dD(\lambda)}{d\lambda} \mid_{\lambda=0} = 0$, *then the the Hausdorff dimension of* Λ^λ *is related with the growth of* V *as*

$$\frac{d^2 D(\lambda)}{d\lambda^2} \mid_{\lambda=0} = \frac{d^2 \dim_H \mu_\lambda}{d\lambda^2} \mid_{\lambda=0} - \frac{1}{2} D^2(0) I_0(g)$$

where μ_λ *is the pushforward of* μ_{ϕ_0} *by* π_λ.

Proof of Theorem 3.9. The proof follows by the above results (propositions 3.2, 3.3, and 3.7 and Lemmas 3.5 and 3.8). □

Theorem 3.10. *Let* $\left\{F_{i,\lambda} : \widehat{C} \to \widehat{C} : i \in N\right\}$ *be a family of iterated schemes originated from a family of quasifuchsian groups* $\{\Gamma_{i,\lambda}\}$ *acting on* $\widehat{C}$ *and let* L^λ *be the limit set of the family of groups. If* $D(\lambda) := \dim_H L^\lambda$, *with* $\dfrac{dD(\lambda)}{d\lambda} \mid_{\lambda=0} = 0$, *then the dimension of the measures* μ_λ, *the dimension of the limit set and Weil-Petersson* *metric for the corresponding smooth path* X_λ *in the Teichmüller space* τ_g *are related as*

$$\frac{d^2 \dim_H \mu_\lambda}{d\lambda^2}\,|_{\lambda=0} = \frac{d^2 \dim_H L^\lambda}{d\lambda^2}\,|_{\lambda=0} + \frac{4}{3A(X_0)}\left\| \frac{dX_\lambda}{d\lambda}\,|_{\lambda=0} \right\|^2_{WP},$$

where $A(X_0) = 4\pi(g-1)$.

Proof of Theorem 3.10. By the above theorem the proof will be established proving

$$\frac{1}{A(X_0)}\left\| \frac{dX_\lambda}{d\lambda}\,|_{\lambda=0} \right\|^2_{WP} = \frac{3}{2} I_0(V(z)).$$

Now, let us consider the differentiable quadratic form $W_i = -2D^3V_i(z)(z)dz^2$, and surfaces $X_{i,\lambda} = H^2/\Gamma_{i,\lambda}$, thus we have

$$\frac{1}{A(X_{i,0})}\left\| \frac{dX_{i,\lambda}}{d\lambda}\,|_{\lambda=0} \right\|^2_{WP} = \frac{1}{A(X_{i,0})}\int_{X_{i,0}} \rho^{-4}\left|W_i\right|^2 \rho^2 \left|dz\right|^2$$

and

$$\left\langle \widetilde{W_i}, \widetilde{W_i} \right\rangle_{2,2} = \lim_{r=\to\infty} \frac{1}{2\pi r}\int_{C_r} \rho^{-4}\left|\widetilde{W_i}\right|^2 \left|z\right|^4 \left|dz\right|^2,$$

where $\widetilde{W_i} = \pi^*(W_i)$, being π the natural projection $\pi : H^2 \to X_{i,0} = H^2/\Gamma_{i,0} \in \tau(S_g)$. The average of $\rho^{-4}\left|\widetilde{W_i}\right|^2$ in C_r converges to the average of $\rho^{-4}\left|W_i\right|$ in $X_{i,0}$ provided the projection $\pi\left(C_r\right)$ be equidistributed in $X_{i,0}$, which occurs by the mixing property of the geodesic flow in $X_{i,0}$[1]. Thus

$$\lim_{r=\to\infty} \frac{1}{2\pi r}\int_{C_r} \rho^{-4}\left|\widetilde{W_i}\right|^2 \left|z\right|^4 \left|dz\right|^2 = \frac{1}{A(X_{i,0})}\int_{X_{i,0}} \rho_i^{-4}\left|W_i\right|^2 \rho_i^2 \left|dz\right|^2,$$

so

$$\frac{1}{A(X_{i,0})}\left\| \frac{dX_{i,\lambda}}{d\lambda}\,|_{\lambda=0} \right\|^2_{WP} = \left\langle \widetilde{W_i}, \widetilde{W_i} \right\rangle_{2,2} = 4\left\langle \mathcal{D}^2(DV_i(z)), \mathcal{D}^2(DV_i(z)) \right\rangle_{2,2}$$

$$= 4I_4(DV_i(z))\frac{3}{2}I_0(V_i(z)).$$

□

Acknowledgement. Support of this work by Consejo Nacional de Investigaciones Científicas y Técnicas (PIP 112-200801-01192), Universidad Nacional de La Plata (Grant 11/I108) and Agencia Nacional de Promoción Científica y Tecnológica of Argentina (PICT 2007-00908) is greatly appreciated.

References

[1] A. Eskin and C. McMullen, *Mixing, counting and equidistribution in Lie Groups*, Duke Math. Journal 71, (1993) 181-209.

[2] A. Manning, *The dimension of the maximal measure for polynomial map*, Ann. of Math. 119, (1984), 425-430.

[3] R. D. Mauldin and M. Urbanski, *Gibbs states on the symbolic space over an infinite alphabet*, Israel J. of Math. 125 (2001), 93-130.

[4] C. McMullen, *Thermodynamcis, dimension and the Weil-Petersson metric*, Invent. Math. 173, (2009), 365-425

[5] A.M. Mesón and F.Vericat, *A Statistical Mechanics approach for a rigidity problem*, J. Stat. Phys. 126, (2007), 391-417.

[6] D. Ruelle, Thermodynamic Formalism, Enciclopedia of Mathematics, Addison-Wesley, 1978.

[7] D. Ruelle, *Repellers for real analytic maps*, Ergodic Theory Dyn. Systems, 2 (1982), 99-107.

[8] S. Sridharan, *Statistical properties of hyperbolic Julia sets*, Diff. Geom. Dyn. Syst., 11 (2009), 175-184.

Authors' addresses:

Alejandro Mesón and Fernando Vericat
Instituto de Fisica de Líquidos y Sistemas Biológicos (IFLYSIB),
CONICET La Plata-UNLP and Grupo de Aplicaciones Matemáticas
y Estadísticas de la Facultad de Ingeniería (GAMEFI), UNLP, La Plata, Argentina.
E-mail: meson@iflysib.unlp.edu.ar, vericat@iflysib.unlp.edu.ar