Second order parallel tensor on $N(k)$-contact metric manifolds

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Abstract. The object of the present paper is to study the symmetric and skew-symmetric properties of a second order parallel tensor in a $N(k)$-contact metric manifold.

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1 Introduction

In 1926 H. Levy [10] proved that a second order symmetric parallel non-singular tensor on a space of constant curvature is a constant multiple of the metric tensor. In recent papers R. Sharma ([13], [14], [15]) generalized Levy's result and also studied a second order parallel tensor on Kaehler space of constant holomorphic sectional curvature as well as on contact manifolds. In 1996 U. C. De [8] studied a second order parallel tensor on a P-Sasakian manifold. Recently L. Das [7] studied a second order parallel tensor on a α-Sasakian manifold.

In this paper it is shown that if a $N(k)$-contact metric manifold admits a second order symmetric parallel tensor then either the manifold is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n \geq 1$ and flat for $n = 1$ or the second order symmetric parallel tensor is a constant multiple of the associated metric tensor. Further, it is shown that on a $N(k)$-contact metric manifold with $k \neq 0$ there is no nonzero second order skew-symmetric parallel tensor.

2 Contact Metric Manifolds

A $(2n + 1)$-dimensional manifold M is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying

$$(2.1) \qquad (a)\ \phi^2 = -I + \eta \otimes \xi,\ \ (b)\ \eta(\xi) = 1,\ \ (c)\ \phi\xi = 0,\ \ (d)\ \eta \circ \phi = 0.$$

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An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M \times \mathbb{R}$. Let g be a compatible Riemannian metric with almost contact structure (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (2.1) it can be easily seen that

$$(a) g(X, \phi Y) = -g(\phi X, Y), (b) g(X, \xi) = \eta(X),$$

for all vector fields X, Y. An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y),$$

for all vector fields X, Y. The 1-form η is then a contact form and ξ is its characteristic vector field. We define a $(1, 1)$ tensor field h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes the Lie-differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. We have $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. Also,

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.2}$$

holds in a contact metric manifold. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in TM,$$

where ∇ is Levi-Civita connection of the Riemannian metric g. A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which ξ is a killing vector is said to be a K-contact manifold. A Sasakian manifold is K-contact but not conversely. However a 3-dimensional K-contact manifold is Sasakian [9]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$ ([2]). On the other hand, on a Sasakian manifold the following holds:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

As a generalization of both $R(X, Y)\xi = 0$ and the Sasakian case; D. Blair, T. Koufogiorgos and B. J. Papantoniou [5] considered the (k, μ)-nullity condition on a contact metric manifold and gave several reasons for studying it. The (k, μ)-nullity distribution $N(k, \mu)$ ([5], [11]) of a contact metric manifold M is defined by

$$N(k, \mu) : p \longrightarrow N_p(k, \mu)$$
$$N_p(k, \mu) = \{W \in T_pM : R(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y)\},$$

for all $X, Y \in TM$, where $(k, \mu) \in \mathbb{R}^2$. A contact metric manifold M^{2n+1} with $\xi \in N(k, \mu)$ is called a (k, μ)-manifold. In particular on a (k, μ)-manifold, we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY].$$

On a (k, μ)-manifold $k \leq 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and μ is indeterminant) and if $k < 1$, the (k, μ)-nullity condition determines the curvature of M^{2n+1} completely [5]. In fact, for a (k, μ)-manifold, the condition of being a Sasakian manifold, a K-contact manifold, $k = 1$ and $h = 0$ are all equivalent.

The k-nullity distribution $N(k)$ of a Riemannian manifold M [16] is defined by

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold an $N(k)$-contact metric manifold [4]. If $k = 1$, then $N(k)$-contact metric manifold is Sasakian and if $k = 0$, then $N(k)$-contact metric manifold is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$. If $k < 1$, the scalar curvature is $r = 2n(2n - 2 + k)$. If $\mu = 0$, then a (k, μ)-contact metric manifold reduces to a $N(k)$-contact metric manifold.

In [1], $N(k)$-contact metric manifolds were studied in some detail. For more details we refer to [6], [3]. In an $N(k)$-contact metric manifold the following relations hold:

$$h^2 = (k - 1)\phi^2, \quad k \leq 1, \tag{2.3}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX),$$

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X], \tag{2.4}$$

$$S(X, \xi) = 2nk\eta(X),$$

$$\begin{aligned} S(X, Y) &= 2(n - 1)g(X, Y) + 2(n - 1)g(hX, Y) \\ &+ [2(1 - n) + 2nk]\eta(X)\eta(Y), \quad n \geq 1, \end{aligned}$$

$$r = 2n(2n - 2 + k),$$

$$S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 4(n - 1)g(hX, Y),$$

$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y),$$

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{2.5}$$

$$\eta(R(X, Y)Z) = k[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)].$$

3 Second order parallel tensors

Definition 3.1 A tensor α of second order is said to be a *parallel tensor* if $\nabla\alpha = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

Let α be a $(0,2)$-symmetric tensor field on an $N(k)$-contact metric manifold M, such that $\nabla\alpha = 0$. Then it follows that

$$\alpha(R(W,X)Y,Z) + \alpha(Y,R(W,X)Z) = 0, \tag{3.1}$$

for arbitrary vector fields W, X, Y, Z on M.

The substitution of $W = Y = Z = \xi$ in (3.1) which gives us $\alpha(\xi, R(\xi,X)\xi) = 0$, since α is symmetric, since the manifold is an $N(k)$-contact metric manifold. Using (2.4) in the above equation, we get

$$k\{g(X,\xi)\alpha(\xi,\xi) - \alpha(X,\xi)\} = 0. \tag{3.2}$$

From (3.2) it follows that either $k = 0$, or

$$g(X,\xi)\alpha(\xi,\xi) - \alpha(X,\xi) = 0. \tag{3.3}$$

Now, if $k = 0$, then $M^{2n+1}(\phi,\xi,\eta,g)$ is a contact metric manifold with $R(X,Y)\xi = 0$ for all vector fields X, Y. It is known [2] that if $R(X,Y)\xi = 0$, then M^{2n+1} is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive curvature 4.

Moreover, by differentiating (3.3) covariantly along Y, we get

$$\begin{aligned} g(\nabla_Y X,\xi)\alpha(\xi,\xi) &+ g(X,\nabla_Y\xi)\alpha(\xi,\xi) + 2g(X,\xi)\alpha(\nabla_Y\xi,\xi) \\ &- \alpha(\nabla_Y X,\xi) - \alpha(X,\nabla_Y\xi) = 0. \end{aligned} \tag{3.4}$$

Putting $X = \nabla_Y X$ in (3.3) yields

$$g(\nabla_Y X,\xi)\alpha(\xi,\xi) - \alpha(\nabla_Y X,\xi) = 0. \tag{3.5}$$

From (3.4) and (3.5), it follows that

$$g(X,\nabla_Y\xi)\alpha(\xi,\xi) + 2g(X,\xi)\alpha(\nabla_Y\xi,\xi) - \alpha(X,\nabla_Y\xi) = 0. \tag{3.6}$$

Using (2.2), we have from (3.6)

$$\begin{aligned} g(X,\phi Y)\alpha(\xi,\xi) &+ g(X,\phi hY)\alpha(\xi,\xi) + 2g(X,\xi)\alpha(\phi Y,\xi) \\ &+ 2g(X,\xi)\alpha(\phi hY,\xi) - \alpha(X,\phi Y) \\ &- \alpha(X,\phi hY) = 0. \end{aligned} \tag{3.7}$$

Replacing X by ϕY in (3.3) and using (2.1), we get

$$\alpha(\phi Y,\xi) = 0. \tag{3.8}$$

From (3.7) and (3.8) it follows that

$$\begin{aligned} g(X,\phi Y)\alpha(\xi,\xi) &+ g(X,\phi hY)\alpha(\xi,\xi) \\ &- \alpha(X,\phi Y) - \alpha(X,\phi hY) = 0. \end{aligned} \tag{3.9}$$

Replacing Y by ϕY in (3.9) and using $\phi h = -h\phi$, we obtain

$$\{g(X,\phi^2 Y) - g(X,h\phi^2 Y)\}\alpha(\xi,\xi) = \alpha(X,\phi^2 Y) - \alpha(X,h\phi^2 Y). \tag{3.10}$$

Also, using (2.1) we have from (3.10)

$$\begin{aligned}\{-g(X,Y) &+ \eta(Y)g(X,\xi) + g(X,hY)\}\alpha(\xi,\xi) \\ &= -\alpha(X,Y) + \eta(Y)\alpha(X,\xi) + \alpha(X,hY).\end{aligned} \tag{3.11}$$

Putting $Y = hY$ in (3.11), we get

$$\begin{aligned}\{-g(X,hY) &+ \eta(hY)g(X,\xi) + g(X,h^2 Y)\}\alpha(\xi,\xi) \\ &= -\alpha(X,hY) + \eta(hY)\alpha(X,\xi) + \alpha(X,h^2 Y).\end{aligned} \tag{3.12}$$

Applying the conditions (2.1), (2.3) and (3.3) in (3.12) gives

$$g(X,hY)\alpha(\xi,\xi) - \alpha(X,hY) = (k-1)\{\alpha(X,Y) - g(X,Y)\alpha(\xi,\xi)\}. \tag{3.13}$$

Now from (3.11) using (3.13), we obtain

$$\alpha(X,Y) = \alpha(\xi,\xi)g(X,Y). \tag{3.14}$$

Differentiating (3.14) covariantly along any vector field on M, it can be easily seen that $\alpha(\xi,\xi)$ is constant. Hence we can state the following theorem:

Theorem 3.1. *If an $N(k)$-contact metric manifold admits a second order symmetric parallel tensor, then either the manifold is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n \geq 1$ and flat for $n = 1$, or the second order symmetric parallel tensor is a constant multiple of the associated metric tensor.*

As an immediate corollary of Theorem 3.1, we have the following result:

Corollary 3.1. *Any Ricci symmetric ($\nabla S = 0$) $N(k)$-contact metric manifold with $k \neq 0$ is an Einstein manifold.*

The above Corollary has also been proved by D. Perrone [12].

Next, let M be an $N(k)$-contact metric manifold admitting a second order skew-symmetric parallel tensor. Putting $Y = W = \xi$ in (3.1) and using (2.4) and (2.5), we obtain

$$k\{\alpha(X,Z) + \eta(X)\alpha(Z,\xi) - \eta(Z)\alpha(X,\xi)\} = 0.$$

Then either $k = 0$, or

$$\alpha(X,Z) + \eta(X)\alpha(Z,\xi) - \eta(Z)\alpha(X,\xi) = 0. \tag{3.15}$$

Here we assume that $k \neq 0$.

Now, let A be a $(1,1)$ tensor field which is metrically equivalent to α, that is, $\alpha(X,Y) = g(AX,Y)$. Then from (3.15), we have

$$g(AX,Z) = \eta(X)g(A\xi,Z) - \eta(Z)g(A\xi,X),$$

and thus

$$AX = \eta(X)A\xi - g(A\xi, X)\xi. \tag{3.16}$$

Since α is parallel, then A is parallel. Hence, using (2.2), it follows that

$$\nabla_X(A\xi) = A(\nabla_X\xi) = -A(\phi X) + A(h\phi X).$$

Thus

$$\begin{aligned} \nabla_{\phi X}(A\xi) &= -A(\phi^2 X) + A(h\phi^2 X) \\ &= A(X) - \eta(X)A(\xi) - A(hX). \end{aligned} \tag{3.17}$$

Using (3.16) and (3.17), we obtain

$$\nabla_{\phi X}(A\xi) = -A(hX) - g(A\xi, X)\xi. \tag{3.18}$$

Also from (3.16) we get

$$g(A\xi, \xi) = 0. \tag{3.19}$$

From (3.18) and (3.19) we have

$$g(\nabla_{\phi X}(A\xi), A\xi) = -g(A(hX), A\xi)$$

or,

$$g(\nabla_{\phi X}(A\xi), A\xi) = g(hX, A^2\xi).$$

Thus,

$$g(\nabla_{\phi X}\xi, A^2\xi) = -g(hX, A^2\xi). \tag{3.20}$$

Now from (2.2) we get

$$\begin{aligned} \nabla_{\phi X}\xi &= -\phi^2 X + h\phi^2 X \\ &= X - hX - \eta(X)\xi. \end{aligned}$$

Using this in (3.20) yields

$$A^2\xi = g(\xi, A^2\xi)\xi = -\|A\xi\|^2\xi. \tag{3.21}$$

Differentiating (3.21) covariantly along X, it follows that

$$\nabla_X(A^2\xi) = A^2(\nabla_X\xi) = A^2(-\phi X - \phi hX) = -\|A\xi\|^2(\nabla_X\xi).$$

Hence

$$-A^2(\phi X) - A^2(\phi hX) = \|A\xi\|^2\phi X + \|A\xi\|^2\phi hX. \tag{3.22}$$

Replacing X by ϕX and using (3.21), we obtain from (3.22)

$$A^2(X) - A^2(hX) = -\|A\xi\|^2 X + \|A\xi\|^2 hX. \tag{3.23}$$

Putting X by hX in (3.23) and using (2.3), (3.21), we obtain

$$A^2(hX) + (k-1)A^2X = -\|A\xi\|^2 hX - (k-1)\|A\xi\|^2 X. \tag{3.24}$$

Using (3.23) from (3.24), we get

$$A^2X = -\|A\xi\|^2 X, \quad \textit{since} \quad k \neq 0.$$

Now, if $\|A\xi\| \neq 0$, then $J = \frac{1}{\|A\xi\|}A$ is an almost complex structure on M. In fact, (J, g) is a Kaehler structure on M. The fundamental second order skew-symmetric parallel tensor is $g(JX, Y) = \lambda g(AX, Y) = \lambda\alpha(X, Y)$, with $\lambda = \frac{1}{\|A\xi\|} = constant$. But (3.16) means $\alpha(X, Y) = \eta(X)\alpha(\xi, Y) - \eta(Y)\alpha(\xi, X)$ and thus α is degenerate, which is a contradiction. Therefore $\|A\xi\| = 0$, which implies $\alpha = 0$.

Hence we can state:

Theorem 3.2. *On an $N(k)-$contact metric manifold with $k \neq 0$, there exists no nonzero second order skew-symmetric parallel tensor.*

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