Extremal problems and related Dirichlet problems

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Abstract. The paper firstly considers extremal problems related to the Dirichlet problem. One gives the solution of the Dirichlet problem on an arbitrary compact in terms of the Fourier coefficients of the restriction-function. One proves some related uniform convergence results for the solution. We show that the exact value of the minimum of the considered functional equals the area of the image of the compact which we are working on, through the related holomorphic function. In particular, this method can be used for problems involving holomorphic functions on the unit disc, for which the coefficients of the power series around the origin are known and the series converges. An example is given. Maximizing some convex operators on sets of subharmonic and respectively convex nonsmooth functions is also a goal of this work. Here the solutions are extreme functions of the set. Finally, minimizing problems for some convex operators on a finite dimensional simplex are considered. The main idea is to point up the extremal qualities of some "points" or lines on a surface.

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1 Introduction

The problem of minimizing some classical functionals ([7], [6], [19], [17]) continues to be difficult. Recently, variational, operatorial and approximation-methods have been applied to solve such problems ([10], [8], [12], [15], [17], [19], [22]). We solve a minimization problem of a well-known functional, which leads to the Dirichlet problem. The exact minimum value equals the area of the image of the domain through the holomorphic function which has as real part the solution. When the restriction condition is given by the restriction of a harmonic function, the problem is easy, but some computational inconveniences may occur. An example when the method works is given. When the domain is the open unit disc, this area can be computed by means of the coefficients of the power series of the holomorphic function mentioned above (see [21]). If the restriction-function is not harmonic, one can use its Fourier coefficients to solve the Dirichlet problem and to find the minimum value of the functional (see

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Theorems 2.2, 2.3). Maximizing some convex operators on subsets of subharmonic (respectively convex non-smooth) functions is also a goal of the paper. Here the solutions are not searched as critical points, but as extreme points of the related convex set of functions. Vector optimization appears naturally as scalar optimization over several criteria. Some earlier basic results were developed in [2]. Other problems lead to construction of the "global inverse" of a monotone convex operator. A problem related to a convex set can be sometimes reduced to several problems on simplexes. A similar decomposition of a compact in particular other compacts is used in Theorem 2.3. of the present work. The role of the simplexes is played by the compacts D_j, each of which have a "center", without being necessarily a convex subset. Finally, we give a characterization of the origin as being the minimum point of a convex operator, on a special simplex and we show that the method works for any vertex of any simplex, via a suitable affine transformation. Some other interesting results related to convexity and partially to the present work were published in [2], [3], [4], [5], [6], [8], [11], [13], [14], [16], [17], [18], [20].

The rest of the paper is organized as follows. The first part of Section 2 is concerned with minimization problems related to boundary conditions. The associated Dirichlet problem is also solved. The extremals of some convex (respectively concave) functionals on subsets with extreme "points" is also considered. The problem of finding the minimum of some convex operators on a finite dimensional simplex is studied in the second part of Section 3. In the end, some conclusions are mentioned.

2 Extremal problems. The exact minimum value of some functionals

Let $\bar{D}$ be a compact subset in $\mathbb{R}^2$, such that the interior of $\bar{D}$ is a simply connected domain. All the functions will be assumed to be C^2 functions, up to the boundary of $\bar{D}$, which is supposed to be a Jordan curve.

Theorem 2.1 *Assume that*

$$\alpha = \min J[u] = \min \iint_D \left(u_x^2 + u_y^2\right) dxdy,\ u = h \text{ on } \partial D, \quad h \in C(\partial D). \tag{2.1}$$

Then u is harmonic, $u = \operatorname{Re} F$, where F is holomorphic in the interior of D, continuous up to ∂D and we have:

$$\alpha = Area\,(F(D)). \tag{2.2}$$

Proof. We note that the necessary extremum condition for J leads, via Euler-Gauss-Ostrogradski equation, to $\Delta u = 0$ in D. Using the boundary condition and the harmonicity of the function u in D, as well as the assumptions on the continuity of partial derivatives up to ∂D, it easily follows that the harmonic function u is uniquely determined by the boundary condition. Observe also that J is a convex functional for which u is a critical point, hence a minimum point. To find α, we note that:

$$\alpha = \iint_D \left(u_x^2 + u_y^2\right) dxdy = \iint_D \frac{\partial\left(\operatorname{Re} F, \operatorname{Im} F\right)}{\partial(x,y)} dxdy = \iint_{F(D)} dudv = Area\,(F(D)),$$

which completes the proof. □

Corollary 2.1 *(a) Consider that h is the restriction to ∂D of a harmonic function on the interior of $\bar{D}$. Let $h = \operatorname{Re} F$, where F is holomorphic. Then the solution of the problem (2.1), where we changed the assumptions on h, is given by:*

$$u = \operatorname{Re} F \text{ in } D,\ \alpha = Area(F(D)). \tag{2.3}$$

(b) If we additionally assume that $\bar{D} = \overline{U} = \{|z| \le 1\}$, then we have

$$\alpha = \pi \cdot \sum_{n=1}^{\infty} n\,|c_n|^2\,, \text{ where } F(z) = \sum_{n=0}^{\infty} c_n z^n,\ z \in U. \tag{2.4}$$

Proof. From Theorem 2.1, we know that u must be harmonic and that $\alpha = Area(F(D))$. From the assumptions on u and h, and using the uniqueness of the solution of such Dirichlet problems, we have the conclusion mentioned at (2.3), so that the assertion at (a) is proved. In the particular case when $\bar{D}$ is the closure of the open unit disc U, by using derivation term by term of F, and then integrating on U, one obtains the formula

$$Area\ (F(U)) = \pi \cdot \sum_{n=1}^{\infty} n\,|c_n|^2$$

after passing to polar coordinates (see [21], pp. 239, exercise 6). □

Corollary 2.2 *Consider the problem: find α, where $\alpha = \min \iint_U \left(u_x^2 + u_y^2\right) dxdy$, such that $u\left(e^{i\theta}\right) = \ln\left|1 - \rho e^{i\theta}\right|$, $\theta \in [-\pi, \pi]$, and determine the solution u, where $\rho < 1$ is fixed. Then we have:*

$$u(z) = \ln|1 - \rho z| = \operatorname{Re}\left[\ln(1 - \rho z)\right],\ z \in U,\ \alpha = -\pi \cdot \ln(1 - \rho^2).$$

Proof. One applies the preceding Corollary 2.1 to $D = U$, $h = \operatorname{Re}\ln(1 - \rho z) = \operatorname{Re} F$, where $F(z) = \ln(1-\rho z)$ is holomorphic on the open disc centered at the origin, of radius $1/\rho > 1$, whence the solution u is given by (2.3), while $\alpha = Area(F(U))$ is given by (2.4). Because of the well known expansion: $F(z) = -\sum_{n\ge1} \frac{\rho^n z^n}{n}$, hence $c_n = -\frac{\rho^n}{n}$, we can use the formula of the area of $F(U)$ in terms of the coefficients (see [21]). One obtains:

$$\alpha = Area(F(U)) = \pi\cdot\sum_{n\ge1} n\,|c_n|^2 = \pi\cdot\sum_{n\ge1} \frac{\rho^{2n}}{n} = \pi\cdot\int_{[0,\rho^2]} \left(\sum_{n\ge0} t^n\right) dt = -\pi\cdot\ln(1-\rho^2),$$

which ends the proof. □

Theorem 2.2 *Consider the problem (2.1), where $D = \overline{U}$, h is a continuous and piecewise C^1 function on ∂U, and let*

$$h\left(e^{i\theta}\right) = \sum_{n\in\mathbb{Z}} a_n \exp\left(in\theta\right),\ \theta \in [0, 2\pi],$$

be the Fourier expansion of the function h *on* ∂U*. Then* $\sum_{n=0}^{\infty} a_n z^n$ *has convergence radius* $R \geq 1$*, and the solution for the problem* (2.1) *is given by*:

$$\alpha = 2\pi \cdot \sum_{n\in\mathbb{Z}} n a_n a_{-n} = 2\pi \cdot \sum_{n\in\mathbb{Z}} n\left|a_n\right|^2, F(z) = 2\left(\sum_{n=0}^{\infty} a_n z^n\right), \; u = \operatorname{Re} F. \tag{2.5}$$

The convergence of the power series of F *to its sum is uniform on* $\bar{D}$*, and we also have:*

$$\|F - s_n\|_2^2 = 4\left(\sum_{j=n}^{\infty} |a_j|^2\right),$$

where the norm is the usual one in the space $L^2(\partial U)$*.*

Proof. From Theorem 2.1, we have seen that u must be harmonic. Let F be holomorphic, such that $\operatorname{Re} F = u$. We have:

$$\begin{aligned}
F(e^{i\theta}) &= \sum_{n=0}^{\infty} c_n \cdot e^{in\theta} = \sum_{n=0}^{\infty} (\lambda_n + i\mu_n)\cdot(\cos(n\theta) + i\sin(n\theta)) \\
&= \sum_{n=0}^{\infty} (\lambda_n \cos(n\theta) - \mu_n \sin(n\theta)) + i\cdot\left[\sum_{n=0}^{\infty} (\lambda_n \sin(n\theta) + \mu_n\cos(n\theta))\right] \\
&= u\left(e^{i\theta}\right) + iv\left(e^{i\theta}\right)
\end{aligned}$$

By Euler's formulas, and separating the real and the imaginary parts, the preceding relations yield:

$$\begin{aligned}
u(e^{i\theta}) &= \tfrac{1}{2}\textstyle\sum_{n\in\mathbb{N}}[(\lambda_n + i\mu_n)e^{in\theta} + (\lambda_n - i\mu_n)e^{-in\theta}] \Rightarrow \\
a_n &= \tfrac{1}{2}\left(\lambda_n + i\mu_n\right), a_{-n} = \overline{a_n}, n\in\mathbb{N},\; n\geq 1 \Rightarrow \\
c_n &= 2a_n,\; n\geq 1 \Rightarrow \alpha = \pi\textstyle\sum_{n=1}^{\infty} n\left|c_n\right|^2 = 4\pi\textstyle\sum_{n=1}^{\infty} n\left|a_n\right|^2,
\end{aligned}$$

where λ_n, $\mu_n \in \mathbb{R}$. For any $z \in U$, we have:

$$\max_{|z|\leq 1} |F(z)| \leq M < \infty \Rightarrow R \geq 1,$$

where R is the convergence radius of the power series which defines F in formula (2.5). To prove the last assertions of the theorem, we observe that the continuity and the piecewise C^1 smoothness of h on the boundary leads to uniform convergence of its Fourier trigonometric series to its sum h, on the boundary. Then, by using the maximum modulus principle, one obtains the conclusion. The relation concerning the norm of the error follows from Parseval's equality in $L^2(\partial U)$, applied to $F - s_n$. The proof is complete. □

Remark 2.1 All the preceding results can be adapted when we work on $\bar{D} = \{r \leq |z| \leq 1\}$, $\forall r > 0$. We now show that the method from Theorem 2.2 can be used for the Dirichlet problem on very general compacts D with nonempty interior. Let

$$\bar{D} = \bigcup_{j=1}^{m} \bar{D}_j, \tag{2.6}$$

be such that for each fixed j, $\exists\ z_j^* \in D_j$, with respect to which we have:

$$z = z_j^* + \rho_j(\theta) \cdot e^{i\cdot\theta},\ \forall z \in \partial D_j,\ \rho_j(\theta) \geq d\left(z_j^*, \partial D_j\right) > 0,$$

$$\rho_j(\theta) := \max\{\rho; z_j^* + \rho e^{i\theta} \in \bar{D}_j\}$$

where ρ_j is a periodic function of θ, with period 2π, while the interiors of the compacts $\bar{D}_j$ are pairwise disjoint.

In other words, any reasonable compact can be decomposed as joint of compacts $\bar{D}_j$, such that for each j, int D_j is simply connected and there exists a "center" $z_j^* \in$ int D_j, with the quality that for any point z on the boundary ∂D_j, the line segment connecting z with z_j^* meets the closed curve ∂D_j only at z, and when $\theta \in [0, 2\pi]$ the whole boundary is covered n_j times, $n_j \in \mathbb{N}$, $n_j \geq 1$. Simple examples show that there are compacts D with simply connected interior, for which such a center point z^* for the whole boundary ∂D does not exist. In such cases, we use a suitable conformal (regular) transformation w, which modifies the boundary such that $w(D)$ to have a "center" point. We observe that from our notations and definitions, it follows $\rho_j(\theta) \leq \max\left\{\left|z - z_j^*\right|; z \in \partial D_j\right\} = R_j = \left|z_{\max} - z_j^*\right|$. Other method is to use a decomposition (2.6).

Theorem 2.3 *Let D be a compact written as in relation* (2.6) *and let $h \in C(\partial D)$ be a given real continuous piecewise C^1 function. For each $j \in \{1, \ldots, m\}$, let*

$$\varphi_j(\theta) = H_j(z_j^* + \rho_j(\theta)e^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n^{(j)} e^{in\theta},\ \theta \in [0, 2\pi[\tag{2.7}$$

be the Fourier expansion of the restriction H_j of h to ∂D_j. Then the solution for the Dirichlet problem is given by:

$$F(z) = 2\sum_{n=0}^{\infty} a_n^{(j)} \left(z - z_j^*\right)^n,\ u(z) := \operatorname{Re} F(z),\ z \in D_j \tag{2.8}$$

The convergence of the power series from relation (2.8) *to its sum $F_j = F$ is uniform on $\bar{D}_j$, $j = 1, \ldots, m$ and we have:*

$$\| F_j - s_{n,j}\|_{2,j}^2 = 4 \cdot \sum_{k=n}^{\infty} \left|a_k^{(j)}\right|^2,$$

where $\|\cdot\|_{2,j}$ is the norm of the space $L^2(\partial D_j)$.

Proof. Due to our assumptions on the boundaries ∂D_j, for each $j \in \{1, \ldots, m\}$, the function $\varphi_j^*(\theta) = H_j(z_j^* + \rho_j(\theta)e^{i\theta})$ is continuous and piecewise C^1 on $\mathbb{R}$, and periodic, with period 2π. From the complex Stone-Weierstrass Theorem, the function φ_j is the uniform limit of a sequence of trigonometric polynomials. Moreover, the

Fourier trigonometric series attached to φ_j converges uniformly to φ_j. From the proof of Theorem 2.2, if we are looking for a holomorphic function F_j on the interior of D_j, $F_j(z) = \sum_{n=0}^{\infty} c_n^j (z - z_j^*)^n$, such that $F_j = \varphi_j$ on ∂D_j, we must have:

$$c_n^j = 2 \cdot a_n^{(j)},\ n \in \mathbb{N}.$$

The maximum modulus principle and a direct computation and a straighford computation show that:

$$\sup_{z \in D_j} |F_j(z)| < \infty \Rightarrow R_j^* \geq R_j > 0$$

where R_j^* is the convergence radius of the power series which defines F_j. Hence F_j is holomorphic in the interior of D_j and equals H_j on the boundary, by the proof of Theorem 2.2. This is true for any $j \in \{1, \ldots, m\}$. Using the properties of the decomposition $D = \bigcup_{j=1}^{m} D_j$, for any $z \in D$, there exists a unique j such that $z \in \operatorname{int} D_j$, so that F from the relation (2.8) is well defined and holomorphic in int D. The boundary condition is also accomplished continuous function. If h is a real continuous function, then $u = \operatorname{Re} F = h$ on ∂D. From the maximum modulus principle, we have:

$$\sup_{z \in D_j} |F_j(z) - s_{n,j}(z)| \leq \varepsilon_{n,j} \to 0,\ n \to \infty$$

because of the uniform convergence of the Fourier trigonometric series of the continuous periodic function φ_j. The expression of the error in $L^2(\partial D_j)$ follows by using Parseval equality. The proof is complete. □

The next result shows that in some cases, the maximum value of some convex operators and the minimum value of some concave operators can be founded directly, on subsets with only one or two extreme elements. Such results remain valid for convex sets of nonsmooth subharmonic functions, with extreme elements, on which the functional J has sense, is convex, bounded from above, and attains its maximum.

Theorem 2.4 *Let J be as above, h be a continuous real positive given function on $\partial\bar{D}$, and S the class of all C^2 subharmonic functions s on D ($\triangle s \geq 0$), such that $s \leq h$ on $\partial\bar{D}$. Then the problem: find*

$$\alpha = \max_{s \in S} J[s] \tag{2.9}$$

has the solution

$$\alpha = J[u],$$

where u is the unique solution of the Dirichlet problem (2.1). *A similar assertion remains valid for any convex continuous functional J which is bounded from above, and attains its maximum value on S.*

Proof. It is known (see [9]) that if a subharmonic function is dominated by a harmonic one on the boundary of a compact in $\mathbb{R}^2$, the same relation on the whole compact holds. It follows that $s \leq u$, $\forall s \in S$. Let us observe that $u \in S$ and, by the preceding relations, it is the unique extreme point of the convex set $S = u - C$, where C is the convex cone of all C^2 superharmonic ($\triangle v \leq 0$) nonnegative functions v on

D. Since we assumed that J attains its maximum value at a function from S, by the maximum principle for convex functionals, this function must be an extreme "point" of S. Hence it must be u, since there are no other extreme functions in S. □

Example 2.1 We have:

$$\max \iint_D \left(s_x^{2k} + s_y^{2k}\right) dxdy = \iint_D \left(u_x^{2k} + u_y^{2k}\right) dxdy$$

and for $k = 1$, the maximum value can be computed as in Theorem 2.2, formula (2.5).

3 Optimization related to finite dimensional simplexes

In the following statements, Y will be a Banach lattice. In some cases, as a model for applications, we may consider Y to be the commutative Banach algebra of self adjoint operators (which is also an order complete Banach lattice) constructed in [2] and generalized in [11]. The order completeness is used to apply extension results of linear operators to some classical operator valued one dimensional and multidimensional moment problems ([11], [20]). The same condition is used in many other applications.

Proposition 3.1 *Let S_n be a simplex in $\mathbb{R}^n$, $S_n = co\{e_1, \ldots, e_{n+1}\}$, where the vertices e_j are affine independent. Let $y_1, \ldots, y_{n+1}$ be arbitrary given positive elements of Y_+. Define the affine operator $A : S_n \to Y$, $A(x) = A\left(\sum \lambda_j e_j\right) = \sum \lambda_j y_j$ where $\lambda_1, \ldots, \lambda_{n+1}$ are the barycentric coordinates of x. It is known that A has a unique affine extension to $\mathbb{R}^n$. We denote by K the convex set of all convex continuous nonnegative operators $f : S_n \to Y$, which satisfy the following restriction related to their values at vertices of the simplex: $f(e_j) \le y_j$, $j = 1, \ldots, n+1$. Let $P : K \to Y$ be a convex operator, such that at each element f of K, there exists a nonnegative subgradient $\nabla P(f)$ of P at f. Then we have:*

$$\max P(K) = P(A), \quad \min P(K) = P(0).$$

The corresponding statement for the minimum of a concave operator Q on the set G of all concave continuous operators g, which satisfy $g(e_j) \ge y_j$, $j = 1, \ldots, n+1$ holds, assuming that Q attains its minimum on G. In this case, no restriction on the supragradient of Q is required. Under these hypothesis, we have:

$$\min Q(G) = Q(A).$$

Proof. For f in K and for any $j \in \{1, \ldots, n+1\}$, we have $(f - A)(e_j) \le 0$, and $f - A$ is convex (both these assertions being given in the hypothesis). Due to the fact that the set of vertices $\{e_j\}$ is exactly the set of all extreme points of S_n, it follows that $f - A \le 0$ on the whole simplex S_n. This is true for all $f \in K$. Since obviously A is an element of K, we conclude that $\max K = A$, where the order relation on K is the pointwise (usual) one. It follows that A is the greatest function from of K and an extremal function for K, while 0 is the smallest function of and, obviously an extreme "point" of K. Because of the assumptions on P, one obtains: $P(A) - P(f) \ge \nabla P(f)(A - f) \ge 0$, where we choose a nonnegative subgradient at f. Hence $P(A) \ge P(f)$, $f \in K$. Since $f = 0 = \min K$, a similar argument leads to $\inf P(K) = \min P(K) = P(0)$. For concave operators Q on G, one repeats a similar argument to show that $A = \min G$, so that it is sufficient to observe that Q, which is assumed to attain its minimum on G, must attain its minimum on G at the unique extremal point of G, which is $A = \min G$. The proof is complete. □

Corollary 3.1 *For* $Y = \mathbb{R}$, *we have:*

(a) for all $p \geq 1$, the following relations hold true:

$$\max_{f\in K}\left\{\max\left(\left(\int_{S_n}(f(x))^p\,dx\right)^{1/p},\int_{S_n}\exp(f(x))\,dx\right)\right\}$$

$$=\max\left\{\left[\int_{S_n}(A(x))^p dx\right]^{1/p},\int_{S_n}\exp(A(x))dx\right\}$$

$$\in\left[\exp\left(\frac{1}{n+1}\sum_{j=1}^{n+1}y_j\right)\cdot Vol(S_n),\frac{1}{n+1}\sum_{j=1}^{n+1}\exp(y_j)\cdot Vol(S_n)\right],$$

(b) for all p, with $0 < p < 1$, we have:

$$\min_{g\in G}\left\{\min\left[\int_{S_n}\ln(1+g(x))\,dx,\left[\int_{S_n}(g(x))^p dx\right]^{1/p}\right]\right\}$$

$$=\min\left(\int_{S_n}\ln(1+(A(x))\,dx,\left[\int_{S_n}(A(x))^p\,dx\right]^{1/p}\right)\leq\ln\left(\sum_{j=1}^{n+1}\frac{1+y_j}{n+1}\right)\cdot Vol(S_n).$$

An earlier result asserts that any convex operator P defined on a bounded subset K of $\mathbb{R}^n$, (the target space being any order complete vector lattice Y with strong order unit), is bounded from below. A natural problem is to find (under some additional assumptions) the minimal value $\inf(P(K))$. The next result gives an answer to this problem, under some additional assumptions. One proves that a natural condition is sufficient and necessary for origin O to be a minimum point for a convex differentiable operator, on a particular simplex. Since any vertex of any other simplex can be reduced with the aid of an affine transformation to this particular case (see the point (c) from below), the assertion can be used in the general case. In the next result, Y is assumed to be an order complete Banach algebra, P convex on a convex neighborhood of $\hat{S}_n$.

Proposition 3.2 *(a) Additionally assume that* $\hat{S}_n = co\{O, f_1, \ldots, f_n\}$, *where* $\{f_1, \ldots, f_n\}$*is the canonical base in* $\mathbb{R}^n$, P*is subdifferentiable at* $O = (0, \ldots, 0)$, $\nabla P(O) \geq 0$, (i.e., $\nabla P(O)(\mathbb{R}^n_+) \subset Y_+$), ∇P*is continuous around* O; *then we have*:

$$\min P(\hat{S}_n) = P(O).$$

An approximation sequence of $P(O)$ *is given by*:

$$\|P(x_n) - P(O)\| \leq \|\nabla P(x_n) - \nabla P(O)\| \cdot \|x_n\|,$$

where $x_n \to O$ *is an arbitrary sequence of elements from the simplex. This estimate is good enough when* ∇P *is continuous around* O. *We can choose a preferential fixed* *direction x_0 and then take $x_n = \lambda_n x_0$, where $\lambda_n \to 0$ rapidly, by positive values. Conversely, if P is differentiable at O, which is a minimum point for P, then $\nabla P(O) \geq 0$;*

(b) Let $K = [0,1]^n$ be the unit hypercube and $P : K \to \mathbb{R}^n$ be as at point (a), where $Y = \mathbb{R}^n$. Assume that the Jacoby matrix $JP(x) = P'(x)$ has no zero elements. Then we have:

$$\inf_{j=1,\dots,n} P_j(K) \geq \min_{i,j} P_j(V_i) \text{ where } V_i,\ i = 1,\dots,2^n \text{ are the vertices of } K.$$

If additionally we assume that $\dfrac{\partial P_j}{\partial x_k} > 0$ *on K for all j,k, then we have:*

$$\min P(K) = P(O),\ O = (0,\dots,0) \text{ and } O \text{ is a strict minimum point for } P.$$

(c) for any simplex S_n given as in Proposition 3.1, there exists a linear isomorphism $T = U \circ A = A_1 \circ U_1 \in L(\mathbb{R}^n)$, A, A_1 symmetric positive definite matrices, U, U_1 orthogonal matrices, such that $T(f_j) = e_{j+1} - e_1$, $j = 1,\dots,n$, where $\{f_1,\dots,f_n\}$ is the canonical orthonormal base of $\mathbb{R}^n$. Hence any such simplex S_n can be obtained by means of such a affine transformation, from a simplex $\hat{S}_n$ which has n faces contained in the different coordinates hyperplanes. In particular, we have: $\hat{S}_n \subset [0,1]^n$.

Proof. For any $x \in \hat{S}_n$, we can use the convexity of P and the hypothesis (a). These lead to:

$$P(x) \geq P(O) + \nabla P(O)(x) \geq P(O),$$

since $\hat{S}_n \subset \mathbb{R}^n_+$. Obviously, $O \in S_n$, so that a first conclusion at point (a) follows. For the approximation of $P(O)$, we use the inequalities:

$$\nabla P(O)(x_n) \leq P(x_n) - P(O) \leq \nabla P(x_n)(x_n).$$

Note that convexity implies the existence of subgradient $\nabla P(x)$, $\forall x \in \hat{S}_n$. From this relation, we obtain:

$$|P(x_n) - P(O)| \leq |(\nabla P(x_n) - \nabla P(O))(x_n)|$$

and finally, by using the fact that Y is a Banach lattice (the norm on Y is monotone: $|u| \leq |v| \Rightarrow \|u\| \leq \|v\|$), the first conclusion of (a) follows. Assume now that P is differentiable at O, which is assumed to be a minimum point for P. We prove that $\nabla P(O) \geq 0$. Assume the contrary; then there should exist $x_+ \in (S_n)_+ \subset \mathbb{R}^n_+$ such that $\nabla P(O)(x_+) \notin Y_+$. Since Y_+ is closed, by separation theorem, there exists a linear continuous positive functional h on Y such that $h(\nabla P(O))(x_+) < 0$. But O was assumed to be a minimum point of P, so that we have $P(x_+) - P(O) \geq 0$, which leads (via positivity and linearity of h) to: $h(P(x_+)) - h(P(O)) \geq 0$. The differentiability of $h \circ P$ yields:

$$h(P(\lambda x_+)) - h(P(O)) = h(\nabla(P(O)))(\lambda x_+) + 0(\lambda x_+).$$

All these relations remain valid when we replace x_+ by λx_+, for arbitrary positive $\lambda \to 0$. Thus we have reached the expected contradiction: in the last equality, by multiplication with $1/\lambda$, the left side member is a nonnegative number, while the right side one converges to $h(\nabla(P(O)))(x_+) < 0$, when $\lambda \to 0$. The assertions at point (c) shows that any simplex S_n is linearly homeomorphic to the special simplex considered at (a). In the particular case $Y = \mathbb{R}^n$, a more general result is stated at point (b). Its proof is based on the method of minimizing convex functionals P_j which attain their minimum on the boundary. If the minimum point would be in the "relative interior" of any of the faces, then the gradient ∇P_j at that point should have some zero-components. This contradicts the hypothesis from (b). Hence, for any P_j, its minimum point must be a vertex. This proves (b). The proof of the assertion at (c) follows from general results on linear operators in Euclidean spaces. We define $T(f_j) = e_{j+1} - e_1$, $j = 1, \ldots, n$ and extend it by linearity to $\mathbb{R}^n$. One obtains a linear isomorphism $T \in L(\mathbb{R}^n, \mathbb{R}^n)$, which can be written as a composition of a unitary operator and a self-adjoint operator. The proof is complete. □

In the proof of the preceding proposition, we have used the fact that a linear one to one mapping applies extreme points into extreme points. The same remark works for geodesics on arbitrary surfaces.

In particular, the ellipses obtained as images of the diameter-circles on a sphere are geodesics of the corresponding ellipsoid. The conclusion is that the notion of an extreme point is useful in optimization. It can be generalized to that of extremal subset. Both qualities are maintained when applying linear isomorphisms.

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