Ordinary differential equations on trivial vector bundles and a splitting of double tangent bundle

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Abstract. This article illustrates how to correspond to a linear connection of a trivial vector bundle an ordinary differential equation, such that the derivative principal part of the vertical global section of the connection is a solution for the ordinary differential equation. In the process of continuation, the article introduces a new splitting of a double tangent bundle using the correspondence to sprays.

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1 Introduction

The study of differential equations on trivial vector bundles of infinite dimensional manifolds have received an increasing interest by its interaction with Theoretical Physics (see for example [5], [17]).

The main problems related to the manifolds modelled on non-Banach spaces and specially on Fréchet spaces, are the pathological structure of general linear group $GL(\mathbb{F})$ and the lack of a general solvability for differential equations (for full details see [12], [15], [16]). These obstacles can be overcome if we restrict ourselves to the category of Fréchet manifolds obtained as projective limits of Banach manifolds (see [1], [2], [7-10]). Hence, we apply our technique for Fréchet manifolds and in particular Lie group.

We express of [4] that second order vector fields may be used for describing a class of autoparallel curves on infinite-dimensional manifolds, including geodesics in the Riemannian case, which allows us for establish a correspondence between this discussion of sprays and this local condition of vector fields of second order. Hence, we introduce a new splitting of a double tangent bundle by using the fact that for any second order vector field there is a unique corresponding spray, such that the integral curves of the vector field are geodesic curves with respect to the spray.

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2 Preliminaries

In this section we introduce some of basic notions which are used in this paper. Let $\mathcal{M}$ be a manifold and $\mathcal{V}$ a vector space on Banach space $\mathbb{E}$. Now suppose that the triple $(\mathcal{M} \times \mathcal{V}, \pi, \mathcal{M})$ is a vector bundle, where π is the projection map of $\mathcal{M} \times \mathcal{V}$ into $\mathcal{M}$. If the trivializing covering of the vector bundle only contains $\mathcal{M}$ and its trivializing map is defined as follows:

$$\varphi : \mathcal{M} \times \mathcal{V} \to \mathcal{M} \times \mathbb{E} : (b, x) \mapsto (b, \alpha(x)), \quad b \in \mathcal{M}, \quad x \in \mathcal{V},$$

where $\alpha : \mathcal{V} \to \mathbb{E}$ is the coordinate isomorphism given by:

$$x = \sum_i a^i x^i \mapsto (a^1, \ldots, a^n).$$

Then the vector bundle and any vector bundle isomorphism to it is called trivial vector bundle of order n.

Let $\mathcal{M}$ be a smooth manifold modelled on the Banach space $\mathbb{E}$ with the atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ and let $\{(\pi^{-1}(U_\alpha), \Psi_\alpha)\}_{\alpha \in I}$ and $\{(\pi_{T\mathcal{M}}^{-1}(\pi_{\mathcal{M}}^{-1}(U_\alpha)), \tilde{\Psi}_\alpha)\}_{\alpha \in I}$ be the corresponding trivialization for $T\mathcal{M}$ and $T(T\mathcal{M})$ respectively. Following e.g. Vilms [19], a connection on $\mathcal{M}$ is a vector bundle morphism $\nabla : T(T\mathcal{M}) \to T\mathcal{M}$ with the local forms $\omega_\alpha : \psi_\alpha(U_\alpha) \times \mathbb{E} \to \mathcal{L}(\mathbb{E}, \mathbb{E})$. The local representation of ∇ is as follows:

$$\nabla_\alpha : \psi_\alpha(U_\alpha) \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \to \psi_\alpha(U_\alpha) \times \mathbb{E},$$

with $\nabla_\alpha := \Psi_\alpha \circ \nabla \circ \tilde{\Psi}_\alpha^{-1}$ for $\alpha \in I$, and the equation:

$$\nabla_\alpha(y, u, v, w) = (y, w + \omega_\alpha(y, u).v),$$

is satisfied. Furthermore ∇ is a linear connection iff $\{\omega_\alpha\}_{\alpha \in I}$ are linear with respect to their second variables. This connection ∇ is completely determined by Christoffel symbols $\{\Gamma_\alpha\}_{\alpha \in I}$ which are smooth and defined by:

$$\Gamma_\alpha : \psi_\alpha(U_\alpha) \to \mathcal{L}(\mathbb{E}, \mathcal{L}(\mathbb{E}.\mathbb{E})) \equiv \mathcal{L}^2(\mathbb{E} \times \mathbb{E}, \mathbb{E}),$$

where $\Gamma_\alpha(y)(u) = \omega_\alpha(y, u)$ for any $(y, u) \in \psi_\alpha(U_\alpha)$. We define the equivalence relation $\approx_2$ as follows:

$$f \approx_2 g \Longleftrightarrow f'(0) = g'(0) \textit{ and } f''(0) = g''(0),$$

for $f, g \in \mathcal{C}_x$. So $\mathcal{T}^2\mathcal{M}_x = \mathcal{C}_x / \approx_2$ and $\mathcal{T}^2\mathcal{M} = \bigcup_{x \in \mathcal{M}} \mathcal{T}^2\mathcal{M}_x$. Here $\mathcal{T}^2\mathcal{M}$ is a vector bundle over $\mathcal{M}$ with fibres of $\mathbb{E} \times \mathbb{E}$ (see [6]) and the structure group $\mathcal{GL}(\mathbb{E} \times \mathbb{E})$ by the trivialization as follows:

$$\begin{aligned}\Phi_\alpha : (\pi^2_{\mathcal{M}})^{-1}(U_\alpha) &\to (U_\alpha) \times \mathbb{E} \times \mathbb{E} \\ [(c, \alpha)]_2 &\mapsto \Big(x, (\phi_\alpha \circ c)'(0), (\phi_\alpha \circ c)''(0) \\ &\quad + \Gamma_\alpha(\phi_\alpha(x))((\phi_\alpha \circ c)'(0), (\phi_\alpha \circ c)'(0))\Big).\end{aligned}$$

3 The Banach case

Definition 3.1. A section $\zeta : \mathcal{M} \to \mathbb{E}$ of a vector bundle with the linear connection ∇, i.e a smooth map satisfying:

$$\pi_{\mathcal{M}} \circ \zeta = id_{\mathcal{M}}, \quad and \quad \nabla(\dot{\zeta}(t)) = 0, \quad \forall t \in \mathcal{M},$$

where $id_{\mathcal{M}}$ denote the identity map on $\mathcal{M}$, is called a horizontal global section of the vector bundle.

Definition 3.2. the linear connections ∇ *and* ∇' on the vector bundles $L = (\mathbb{E}, \pi, \mathcal{M})$ and $L' = (\mathbb{E}', \pi', \mathcal{M}')$, are called $(\mathcal{F}, h) - related$ if,

$$\mathcal{T}\mathcal{F} \circ \nabla = \nabla' \circ (\mathcal{F} \times \mathcal{T}h),$$

or

$$\mathcal{F} \circ \nabla = \nabla' \circ \mathcal{T}\mathcal{F}.$$

Now, we have the following result

Theorem 3.3. ([*18*]). *Let ∇ and ∇' are linear connections on the vector bundles $L = (\mathbb{E}, \pi, \mathcal{M})$ and $L' = (\mathbb{E}', \pi', \mathcal{M}')$ respectively. ∇ and ∇' are $(\mathcal{F}, h) - related$ iff, for any pair chart $(\mathcal{U}, \varphi)$ and $(\mathcal{U}', \varphi')$, the Christoffel symbols satisfy the following compatibility condition:*

$$(\mathcal{F}_\varphi)^*{}_x[\Gamma_\varphi(x)(u, y)] = \Gamma'_{\varphi'}(h_\varphi(x))((\mathcal{F}_\varphi)^*{}_x(u), \mathcal{D}(h_\varphi)(x).u) + \mathcal{D}(\mathcal{F}_\varphi)^*(x).y.u,$$

where $(\mathcal{F}_\varphi)^*{}_x = (\Phi' \circ \mathcal{F} \circ \Phi^{-1})^*{}_x$ *and* $(y, u, v) \in \psi_\alpha(\mathcal{U}_\alpha) \times \mathbb{E} \times \mathcal{M}$.

Then by the theorem and by putting $(\mathcal{F}_\varphi)^* = d(\psi_\alpha \circ id_{\mathcal{B}} \circ {\psi_\beta}^{-1})$, we have

Corollary 3.1. *The necessary condition for a linear connection to be well defined on overlaps of the manifold $\mathcal{M}$ is that the Christoffel symbols satisfy the following compatibility condition:*

$$\Gamma_\alpha(\delta_{\alpha\beta}(y))(d\delta_{\alpha\beta}(y)(u))[d\delta_{\alpha\beta}(y)(v)] + (d^2\delta_{\alpha\beta}(y)(v))(u) = d\delta_{\alpha\beta}(y)((\Gamma_\beta(y)(u))(v)),$$

for any $(y, u, v) \in \psi_\alpha(\mathcal{U}_\alpha) \times \mathbb{E} \times \mathbb{E}$.

Now we consider the third $(\mathcal{M} \times \mathbb{E}, \mathcal{M}, \mathcal{P}r_1)$ as a trivial vector bundle on the manifold $\mathcal{M}$ with a identity element and the linear connection ∇. We want to correspond to ∇, an ordinary differential equation on the Banach space $\mathbb{E}$. For this we consider Christoffel symbols of the connection ∇, with respect to the atlas $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ as follows:

$$\Gamma_\alpha : \varphi_\alpha(\mathcal{U}_\alpha) \to \mathcal{L}^2(\mathcal{M} \times \mathbb{E}, \mathbb{E}).$$

Using the above notion, we may define the following map for any $t \in \varphi_\alpha(\mathcal{U}_\alpha)$:

$$\mathcal{A}_\alpha : \varphi_\alpha(\mathcal{U}_\alpha) \to \mathcal{L}(\mathbb{E}) : t \mapsto \Gamma_\alpha(t)(0, e), \tag{3.1}$$

where e is the identity element of $\mathcal{M}$. Now, we consider $\mathcal{A} : \mathcal{M} \to \mathcal{L}(\mathbb{E})$ such that $\mathcal{A}$ is in fact $\mathcal{A}_\alpha$ which is defined by (3.1) for a particular case where its atlas is $(\mathcal{M}, id_{\mathcal{M}})$

and its Christoffel symbol is Γ_1. Now suppose $\mathcal{U}_\alpha \bigcap \mathcal{U}_\beta \neq \varnothing$ and $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ are corresponding to the charts $(\mathcal{U}_\alpha, \varphi_\alpha)$ and $(\mathcal{U}_\beta, \varphi_\beta)$ respectively. Therefore choosing $y = t, v = e, u = 0$ in Corollary (3.1), the compatibility condition for $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ is given as follows:

$$\mathcal{A}_\alpha(t) = (\varphi_\alpha \circ \varphi_\beta^{-1})'(t).\mathcal{A}_\beta(\varphi_\alpha \circ \varphi_\beta^{-1}).$$

Theorem 3.4. *Any linear connection of trivial vector bundle* $(\mathcal{M} \times \mathbb{E}, \mathcal{M}, \mathcal{P}r_1)$ *is in one to one correspondence with the first order ordinary differential equation* $\frac{dx}{dt} = \mathcal{A}(t)x$.

Proof. Suppose linear connection ∇ is given. Since ∇ is well-defined, so its Christoffel symbols are well-defined as well. Therefore $\mathcal{A}_\alpha = \Gamma_\alpha(t)(0, e)$ and $\mathcal{A}_\beta = \Gamma_\beta(t)(0, e)$ satisfy in the compatibility condition, hence the existence of $\{\mathcal{A}_\alpha\}_{\alpha \in I}$ and particularly $\mathcal{A}_1$ is followed. Now one may define the following differential equation:

$$\frac{dx}{dt} = \mathcal{A}(t).x.$$

Conversely, since $\mathcal{A}$ is a smooth map in the differential equation $\frac{dx}{dt} = \mathcal{A}(t).x$, we can define smooth maps $\mathcal{A}_\beta$ as follows:

$$\mathcal{A}_\beta(t) : \varphi_\beta(\mathcal{U}_\beta) \to \mathcal{L}(\mathbb{E}) : t \mapsto (\varphi_\beta^{-1})'(t).\mathcal{A}(\varphi_\beta^{-1}(t)).$$

One should note that $\mathcal{A}_\beta$ is well-defined for any $\beta \in I$. in fact, we have:

$$\begin{aligned} \delta'_{\alpha\beta}(t).\mathcal{A}_\beta(\delta_{\alpha\beta}(t)) &= \delta'_{\alpha\beta}(\varphi^{-1})'(\delta_{\alpha\beta}(t)).\mathcal{A}(\varphi_\beta^{-1}(\delta_{\alpha\beta}(t))) \\ &= (\varphi_\beta^{-1} \circ \delta_{\alpha\beta})'(t).\mathcal{A}(\varphi_\beta^{-1}(\delta_{\alpha\beta(t)})) \\ &= (\varphi_\alpha^{-1})'(t).\mathcal{A}(\varphi_\alpha^{-1}(t)) \\ &= \mathcal{A}_\alpha(t). \end{aligned}$$

Now we define:

$$\Gamma_\alpha(t)(0, e) = \mathcal{A}_\alpha(t),$$

we simply get the following compatibility condition:

$$\Gamma_\beta(t)(0, e) = \delta'_{\alpha\beta}(t).\Gamma_\alpha(\delta_{\alpha\beta}(t))(0, e).$$

Therefore, the linear connection ∇ is well-defined on trivial vector bundle $(\mathcal{M} \times \mathbb{E}, \mathcal{M}, \mathcal{P}r_1)$. □

Example 3.5. (first order differential equations on Lie groups)
Let $\mathfrak{G}$ be a Lie group modelled on the Banach space $\mathbb{E}$ with the direct connection $\nabla_{\mathfrak{G}}$, which is $(\gamma, id_{\mathfrak{G}})$-related with the canonical flat connection $\nabla_{\mathfrak{F}}$ $(where \Gamma_{\mathfrak{F}} = 0)$ of the trivial vector bundle $(\mathfrak{G} \times \mathcal{G}, \mathfrak{G}, \mathcal{P}r_1)$ then by [9], where

$$\begin{aligned} \gamma : \mathfrak{G} \times \mathcal{G} &\to \mathcal{T}\mathfrak{G} \\ (g, X) &\mapsto \mathcal{T}_e\mathcal{L}_g(X) \end{aligned}$$

denotes the vector bundle isomorphism of $\mathfrak{G} \times \mathcal{G}$ such that $\mathcal{G}$ is the Lie algebra of $\mathfrak{G}$, in fact

$$\pi \circ \gamma(g, X) = \pi([\mathcal{L}_g \circ c, g]) = g = id_{\mathfrak{G}} \circ \mathcal{P}r_1(g, X),$$

and it is easy to verify the map $\gamma_x : \{x\} \times \mathcal{G} \cong \mathcal{G} \to \mathcal{T}_x\mathfrak{G}$ is linear and γ is one to one. Also γ is surjective, since for any X in $\mathcal{T}\mathfrak{G}$, here exists a unique x in $\mathfrak{G}$, which X_x is a vector on $\mathcal{T}_x\mathfrak{G}$, therefore we have a (x, X) in $\{x\} \times \mathcal{G} \subseteqq \mathfrak{G} \times \mathcal{G}$, such that $\gamma(x, X) = X$. Now since $(\mathfrak{G} \times \mathcal{G}, \mathfrak{G}, \mathcal{P}r_1)$ and $(\mathcal{T}\mathfrak{G}, \pi, \mathfrak{G})$ are vector bundles endowed with the similar base $\mathfrak{G}$ and the vector bundle isomorphism $(\gamma, id_{\mathfrak{G}})$, then by [18] the corresponding connections $\nabla_{\mathfrak{G}}$ and $\nabla_{\mathfrak{F}}$ are $(\gamma, id_{\mathfrak{G}})$-related iff, their Christoffel symbols satisfy the following local condition:

$$(\gamma_\varphi)^*_x[\Gamma_\varphi(x)(u, y)] = \Gamma'_{\varphi'}[(\gamma_\varphi)^*_x(u), y] + \mathcal{D}(\gamma_\varphi)^*(x).y.u,$$

therefore, by putting $(\gamma_\varphi)^*_x = F_\kappa(x)$, (where $F_\kappa(x)$ gives the local expression of the isomorphism $\gamma_x := \mathcal{T}_e\mathcal{L}_x : \mathcal{G} \cong \mathcal{T}_e\mathfrak{G} \to \mathcal{T}_x\mathfrak{G}$ with respect to the chart $(\mathcal{U}, \varphi)$). Hence, we have

$$\Gamma^{\mathfrak{G}}_\varphi(x)(u, F_\kappa(x).y) + \mathcal{D}F_\kappa(x)(u, y) = 0,$$

so

$$\Gamma^{\mathfrak{G}}_\varphi(x)(u, y) = -\mathcal{D}F_\kappa(x)(u, F_\kappa^{-1}(x).y).$$

As a result, if $\mathfrak{G}$ is a Lie group modelled on a Banach space, our Eqs reduces to

$$\frac{dx}{dt} = -\mathcal{D}F_\kappa(x)(0, F_\kappa^{-1}(x)).$$

Example 3.6. (a numerical solution for first order differential equations on Lie groups)
Let G be a Lie group as below

$$G = \{x \mid xx^t = constant\}.$$

Hence our Eqs changed to $\dot{x} = A(x)x$ which $A(x)$ is a skew- symmetric matrix orthogonal so, according to [11] this numerical solution of the Eqs lies on a closed curve $\frac{1}{2}(\frac{x_1^2}{2} + x_2^2 + \frac{3x_3^2}{2}) = constant$ and a sphere with the following numerical method

$$x_{n+1} = exp(h(A(x_n)))x_n.$$

Let $\zeta_p : \mathcal{M} \to \mathcal{M}$ be the principal part of ζ, thus, we have the following theorem.

Theorem 3.7. *For each $t. \in \mathcal{M}$, there exists a unique horizontal global section*

$$\zeta : \mathcal{M} \to \mathcal{M} \times \mathbb{E},$$

with $\zeta_p(t.) = f.$ az a constant value, such that ζ_p is a solution of the ordinary differential equation $\frac{dx}{dt} = \mathcal{A}(t).x$.

Proof. For any $t. \in \mathcal{M}$, we show that

$$[\mathcal{A}(t.)](\zeta_p(t.)) = \dot{\zeta}_p(t.).$$

Now, then by the definition of connection, we have:

$$\nabla_\alpha(t., \zeta_p(t.), e, 0) = (t, \Gamma_\alpha(t.)(\zeta_p(t.), e)),$$

hence, since ζ is a horizontal global section of the trivial vector bundle $(\mathcal{M}\times\mathbb{E}, \mathcal{M}, \mathcal{P}r_1)$ endowed with principal part ζ_p, where $\zeta_p(t.) = f.$, so

$$\nabla_\alpha(t., \zeta_p(t.), e, 0) = 0,$$

therefore

$$\mathcal{A}(t.) = \Gamma(t.)(\zeta_p(t., e)) = 0.$$

Also, by attention to $\dot{\zeta}_p(t.) = 0$, proof is complete. □

4 Fréchet case

Here we focus on those Fréchet manifolds which can be obtained as projective limits of Banach manifolds. (see [1], [7], [8], [10])

Let $\{M^i, \varphi^{ji}\}_{i,j\in\mathbb{N}}$ be a projective system of manifolds modelled on the projective system of Banach spaces $\{\mathbb{E}^i, \rho^{ji}\}_{i,j\in\mathbb{N}}$ respectively. Furthermore suppose that for $x = (x^i)_{i\in\mathbb{N}} \in \mathcal{M} = \varprojlim \mathcal{M}^i$ there exists a projective system of local charts $\{(U^i, \psi^i)\}_{i\in\mathbb{N}}$ such that $x^i \in U^i$ and $\varprojlim U^i$ is open in $\mathcal{M}$. Then we can endow $\mathcal{M} = \varprojlim \mathcal{M}^i$ with a Fréchet manifold structure modelled on the Fréchet space $\mathbb{F} = \varprojlim \mathbb{E}^i$.

Furthermore suppose that for each $i \in \mathbb{N}$, ∇^i be a linear connection on $\mathcal{M}^i$ such that $\nabla = \varprojlim \nabla^i$ exists. The intrinsic problems related to pathological structure of $GL(\mathbb{E})$ overcome if we replace it with the generalized Lie group.

Example 4.1. Consider $\mathbb{E}$ as a smooth manifold with the total chart $(\mathbb{E}, id_{\mathbb{E}})$. If we endow $\mathcal{M} = \mathbb{E}$ with the canonical flat connection i.e. the connection on $\mathcal{M}$ such that its Christoffel symbol vanishes everywhere, then the equations related to the Christoffel symbol reduce to;

$$\frac{d\theta}{dt}(t) = 0,$$

i.e. $\theta(t) = c$. As we mentioned earlier, the assigned equations are ordinary differential equations on the Fréchet space $\mathbb{F} = \varprojlim \mathbb{E}^i$ where the first one can be solved if we assume that the vector field $\xi := \varprojlim \xi^i$ is a projective limit vector field, and for the first equation if $\nabla := \varprojlim \nabla^i$.

Now here is another proof for this theorem which was mentioned in [3],

Theorem 4.2. *Let* $\nabla_\alpha := \varprojlim \nabla^i_\alpha$ *and* $\nabla'_\beta := \varprojlim \nabla'^i_\beta$ *be two linear connections over trivial vector bundle* $(\varprojlim \mathcal{M}^i \times \varprojlim \mathbb{E}^i, \varprojlim \mathcal{M}^i, \mathcal{P}r_1)$ *such that for any* $i \in \mathbb{N}$*, each* ∇^i_α *and* ∇'^i_β *be* $(g^i, id_{\mathcal{M}}) - related$*. Then their corresponding differential equations* $\frac{dx}{dt} = \mathcal{A}(t).x$ *and* $\frac{dy}{dt} = \mathcal{C}(t).y$ *are equivalent.*

Proof. Since ∇^i_α and ∇'^i_β are $(g^i, id_{\mathcal{M}}) - related$ over each trivial vector bundle $(\mathcal{M}^i \times \mathbb{E}^i, \mathcal{M}^i, \mathcal{P}r_1)$ and from corollary (3.1), for their corresponding Christoffel symbols we have

$$\mathcal{D}G^i(\varphi^i_\alpha(x))\Big(\Gamma^i_\alpha(\varphi^i_\alpha(x))(u)(y)\Big) = {\Gamma^i}'_\beta(G(\varphi^i_\alpha(x)))\Big(\mathcal{D}G^i(\varphi^i_\alpha(x))(u), y\Big) \\ -\mathcal{D}(\mathcal{D}G^i)(t)(u,e),$$

now notice that $\Gamma^i_\alpha(t)(u,e) = -\mathcal{A}^i_\alpha(t)(u)$ also by putting $\varphi^i_\alpha(x) = t^i$ and $y^i = 1$, for every $\alpha \in I$

$$\mathcal{D}G^i(t)\Big(\mathcal{A}^i_\alpha(t)(u)\Big) = \mathcal{D}(\mathcal{D}G^i)(t)(u) + \Big(\mathcal{A}^i_\beta(G^i(t))\Big)(\mathcal{D}G^i(t)(u)).$$

Now, we want show the existence of $\varprojlim \mathcal{B}^i(t)$. Before anything, from $\nabla_\alpha := \varprojlim \nabla^i_\alpha$ then by [7] we get $\mathcal{A}_\alpha(t) := \varprojlim \mathcal{A}^i_\alpha(t) = \varprojlim \Gamma^i_\alpha(t)(u,e)$. On the other hand, by attention to projective morphisms $\{\varphi^{ij}\}_{i,j\in\mathbb{N}}$ and $\{\rho^{ij}\}_{i,j\in\mathbb{N}}$ of the projective systems $\{\mathcal{M}^i\}_{i\in\mathbb{N}}$ and $\{\mathbb{E}^i\}_{i\in\mathbb{N}}$, we have:

$$\begin{aligned}
\rho^{ji}\circ[\mathcal{B}^j(t)] &= \rho^{ji}\circ[\mathcal{A}^j_\beta(G^j(t))]\\
&= \rho^{ji}\circ\mathcal{A}^j_\beta(\varphi^j_{\beta,g(x)}\circ g^j_x\circ\psi^j_{\alpha,x}(t))\\
&= [\mathcal{A}^i_\beta\circ\rho^{ji}](\varphi^j_{\beta,g(x)}\circ g^j_x\circ\psi^j_{\alpha,x}(t))\\
&= \mathcal{A}^i_\beta(\rho^{ji}\circ\varphi^j_{\beta,g(x)}\circ g^j_x\circ\psi^j_{\alpha,x}(t))\\
&= \mathcal{A}^i_\beta(\rho^{ji}\circ\varphi^j_{\beta,g(x)}\circ g^j_x\circ\psi^j_{\alpha,x}(t))\\
&= \mathcal{A}^i_\beta(\varphi^j_{\beta,g(x)}\circ\varphi^{ji}\circ g^j_x\circ\psi^j_{\alpha,x}(t))\\
&= \mathcal{A}^i_\beta(\varphi^j_{\beta,g(x)}\circ g^i_x\circ\varphi^{ji}\circ\psi^j_{\alpha,x}(t))\\
&= \mathcal{A}^i_\beta(\varphi^j_{\beta,g(x)}\circ g^i_x\circ\varphi^{ji}\circ\psi^j_{\alpha,x}\circ\rho^{ji}(t))\\
&= [\mathcal{A}^i_\beta](G^i\circ\rho^{ji}(t))\\
&= [\mathcal{B}^i(t)]\circ\rho^{ji}.
\end{aligned}$$

This concludes that

$$\varprojlim \mathcal{D}G^i(t)\Big(\varprojlim \mathcal{A}^i_\alpha(t)(u)\Big) = \varprojlim \mathcal{D}(\mathcal{D}G^i)(t)(u) + \Big(\varprojlim \mathcal{B}^i(t))\Big)(\varprojlim \mathcal{D}G^i(t)(u)),$$

whence

$$\varprojlim {\mathcal{Q}^i}^{-1}(t).\Big(\varprojlim \mathcal{A}^i(t)\circ\varprojlim \mathcal{Q}^i(t) - \varprojlim \dot{\mathcal{Q}}^i(t)\Big) = \varprojlim \mathcal{B}^i(t),$$

where $\mathcal{B}^i(t) = \mathcal{A}^i_\beta(G^i(t))$ and $\mathcal{A}^i(t) = \mathcal{A}^i_\alpha(t)$, for any $\alpha,\beta\in I$. In fact, we obtain:

$$\begin{aligned}
\varprojlim {\mathcal{Q}^i}^{-1}(t).\Big(\varprojlim \mathcal{A}^i(t)\circ\varprojlim \mathcal{Q}^i(t) - \varprojlim \dot{\mathcal{Q}}^i(t)\Big) &= \varprojlim \mathcal{B}^i(t)\\
\Leftrightarrow \varprojlim \mathcal{A}^i(t)\varprojlim \mathcal{Q}^i(t) &= \varprojlim \mathcal{B}^i(t)\varprojlim \mathcal{Q}^i(t) + \varprojlim \dot{\mathcal{Q}}^i(t)\\
\Leftrightarrow \varprojlim \mathcal{A}^i(t)\varprojlim \mathcal{Q}^i(t).\varprojlim y^i &= \varprojlim \dot{\mathcal{Q}}^i(t).\varprojlim y^i + \varprojlim \mathcal{Q}^i(t).\varprojlim \dot{y}^i\\
\Leftrightarrow \varprojlim \dot{x}^i &= \varprojlim \dot{\mathcal{Q}}^i(t).\varprojlim y^i + \varprojlim \mathcal{Q}^i(t).\varprojlim \dot{y}^i\\
\Leftrightarrow \varprojlim x^i &= \varprojlim \mathcal{Q}^i(t).\varprojlim y^i.
\end{aligned}$$

By using the previous notation, we conclude that the differential equations $\varprojlim \dot{x}^i = \varprojlim \mathcal{A}^i.\varprojlim x^i$ and $\varprojlim \dot{y}^i = \varprojlim \mathcal{C}^i.\varprojlim y^i$ are equivalent on the Fréchet space. □

5 Autoparallel curves and differential equations

Definition 5.1. A section $\zeta : \mathcal{M} \to \mathcal{T}^2\mathcal{M}$ of the second order vector bundle $\mathcal{T}^2\mathcal{M}$, i.e a smooth map satisfying:

$$\pi^2_{\mathcal{M}} \circ \zeta = id_{\mathcal{M}},$$

where $id_{\mathcal{M}}$ denote the identity map on $\mathcal{M}$, is called a second order vector field on the base manifold M (for more details [4]).

Definition 5.2. Let $\mathcal{M}$ be a smooth manifold with affine connection ∇. The correspondence of vector field $\mathcal{V}$ along smooth curve $\mathcal{C} : \mathcal{I} \subset \mathcal{R} \to \mathcal{M}$ to the vector field $\frac{\mathcal{DV}}{dt}$ is called covariant derivative of $\mathcal{V}$ along $\mathcal{C}$ iff we have:

$$i)\frac{\mathcal{DV}+\mathcal{W}}{dt} = \frac{\mathcal{DV}}{dt} + \frac{\mathcal{DW}}{dt},$$
$$ii)\frac{\mathcal{D}f\mathcal{V}}{dt} = f\frac{\mathcal{DV}}{dt} + \frac{\mathcal{D}f}{dt}\mathcal{V},$$
$$iii) if\ \mathcal{V}(t) = \mathcal{Y}(\mathcal{C}(t))\ for\ any\ \mathcal{Y} \in \chi(\mathcal{M}), then\ \frac{\mathcal{DV}}{dt} = \nabla_{\frac{d\mathcal{C}}{dt}}.$$

Definition 5.3. The curve $\gamma : \mathcal{I} \subset \mathcal{R} \to \mathcal{M}$ is called auto parallel, if satisfying:

$$\frac{d^2x_k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{dx_i}{dt}\frac{dx_j}{dt} = 0, \qquad k = 1, \cdots, n,$$

where $(\varphi_\alpha \gamma)(t) = (x_1(t), \cdots, x_n(t))$.

Now, according to [4] we have the following theorems

Theorem 5.4. *Any integral curve of a second order vector field induced by fix function is a autoparallel curve.*

Theorem 5.5. *Let ζ be a vector field of second order and also ϑ be a integral curve of ζ; then ϑ is a geodesic curve of $\mathcal{M}$ iff the vector field satisfies the following local condition:*

$$\Phi^2_\alpha \circ \zeta = \Phi^3_\alpha \circ \zeta.$$

6 Geodesic curves and sprays

Definition 6.1. A spray of smooth manifold $\mathcal{M}$ is a second order vector field $\mathcal{F}$ such that for each $s \in \mathcal{R}$ and $v \in \mathcal{TM}$ we have the following condition:

$$\mathcal{F}(sv) = \mathcal{T}s_{\mathcal{TM}}(s\mathcal{F}(v)),$$

where,

$$s_{\mathcal{TM}} : \mathcal{TM} \to \mathcal{TM}\ :\ v \mapsto sv.$$

According to [13] any spray is completely determined by symmetric bilinear map $\beta : \psi_0(U_\alpha) \to \mathcal{L}^2(E, E)$, whence then by [14] and by considering Γ_α in lieu of β we have:

Theorem 6.2. *Let ∇ be a linear connection on smooth manifold $\mathcal{M}$. therefore here exists a unique spray $\mathcal{F}$ such that any geodesic of ∇, is a geodesic with respect to $\mathcal{F}$.*

Theorem 6.3. *Let ζ be a vector field of second order with the local condition $\Phi^2_\alpha \circ \zeta = \Phi^3_\alpha \circ \zeta$, then there is a unique spray $\mathcal{F}$ such that any integral curve of ζ is a geodesic with respect to $\mathcal{F}$.*

Proof. Since each integral curve of a vector field ζ with locally condition $\Phi^2_\alpha \circ \zeta = \Phi^3_\alpha \circ \zeta$ is a geodesic curve with respect to Levi-Civita connection ∇, then by the above theorem, each geodesic curve with respect to ∇ is a geodesic curve with respect to spray $\mathcal{F}$, hence for any vector field of second order ζ with the local condition $\Phi^2_\alpha \circ \zeta = \Phi^3_\alpha \circ \zeta$, here exists a unique spray $\mathcal{F}$ such that each integral curve of a vector field ζ is a integral curve with respect to spray $\mathcal{F}$. □

Finally, we have a splitting of the double tangent bundle by the following theorem

Theorem 6.4. *Let $\mathcal{M}$ be a manifold with a vector field by local properties $\Phi^2_\alpha \circ \zeta = \Phi^3_\alpha \circ \zeta$. Then the map*

$$(\pi_{\mathcal{TM}}, \mathcal{S}_1, \mathcal{S}_2) : \mathcal{TTM} \to \mathcal{TM} \oplus \mathcal{TM} \oplus \mathcal{TM}$$

is a unique isomorphism of fiber bundles over $\mathcal{M}$, where

$$\begin{aligned}(\mathcal{S}_1)_{\varphi(U)} &: (\mathcal{TTM})_{\varphi(U)} \to (\mathcal{TM})_{\varphi(U)} \\ &: \Big((\varphi_\alpha \circ \gamma)(t), (\varphi_\alpha \circ \gamma)'(t), (\varphi_\alpha \circ \gamma)'(t), (\varphi_\alpha \circ \gamma)''(t)\Big) \mapsto \Big((\varphi_\alpha \circ \gamma)(t), (\varphi_\alpha \circ \gamma)'(t)\Big),\end{aligned}$$

and

$$\begin{aligned}(\mathcal{S}_2)_{\varphi(U)} &: (\mathcal{TTM})_{\varphi(U)} \to (\mathcal{TM})_{\varphi(U)} \\ &: \Big((\varphi_\alpha \circ \gamma)(t), (\varphi_\alpha \circ \gamma)'(t), (\varphi_\alpha \circ \gamma)'(t), (\varphi_\alpha \circ \gamma)''(t)\Big) \mapsto \Big((\varphi_\alpha \circ \gamma)(t), (\varphi_\alpha \circ \gamma)''(t) \\ &\quad -\Gamma_\alpha((\varphi_\alpha \circ \gamma)(t))[(\varphi_\alpha \circ \gamma)'(t), (\varphi_\alpha \circ \gamma)'(t)]\Big).\end{aligned}$$

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