

Surfaces at a constant distance from the edge of regression on a surface in E_1^3

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Abstract. Let M be a surface in E_1^3 . We define surfaces M^f at a constant distance from the edge of regression on a surface M . The correspondence of some theorems and properties which hold for parallel surfaces, which are a special case of M^f , are obtained for M^f .

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1 Introduction

Let M be a surface in E^3 (or E_1^3) with unit normal $N = (a_1, a_2, a_3)$, where each a_i is a real valued C^∞ function on M and $a_1^2 + a_2^2 + a_3^2 = 1$ (or $-a_1^2 + a_2^2 + a_3^2 = \pm 1$ in E_1^3). For any constant r in \mathbb{R} , let $M_r = \{P + rN_p : P \in M\}$. Thus if $P = (p_1, p_2, p_3)$ is on M , then $f(P) = P + rN_p = (p_1 + ra_1(p), p_2 + ra_2(p), p_3 + ra_3(p))$ defines a new surface M_r . The map f is called the natural map on M into M_r , and if f is univalent, then M_r is a parallel surface of M with unit normal N , i.e., $N_{f(p)} = N_p$ for all P in M .

In previous studies, parallel surfaces have been investigated in 3-dimensional and in higher dimensional Euclidean (or semi Euclidean) space by a number of differential geometers. In 1948 V. F. Kagan [8] studied the parallel surfaces in Euclidean space E^3 . U. Simon and A. Weinstein [10] generalized Kagan's work to hypersurfaces in E^n in 1969. A. Görgülü and C. Çöken [6] studied the Euler theorem for parallel hypersurfaces in E^{n+1} . The same authors [5] obtained the Dupin indicatrix for parallel pseudo-Euclidean hypersurfaces in E_1^{n+1} . On the other hand C. Çöken [3] found the Euler Theorem and Dupin indicatrix for parallel hypersurfaces in E_v^{n+1} . In 2005 H.H. Hacısalıhoğlu and Ö. Tarakçı ([11]) introduced surfaces at a constant distance from edge of regression on a surface in E^3 , and these surfaces are a generalization of parallel surfaces. They took any vector instead of normal vector and defined the following.

"Let M and M^f be two surfaces in E^3 Euclidean space and N_p be a unit normal vector of M at the point $P \in M$. Let $T_p(M)$ be the tangent space at $P \in M$ and let $\{X_p, Y_p\}$ be an orthonormal base of $T_p(M)$. Take a unit vector $Z_p = d_1X_p + d_2Y_p + d_3N_p$, where $d_1, d_2, d_3 \in \mathbb{R}$ are constant numbers and $d_1^2 + d_2^2 + d_3^2 = 1$. If a function f exists and satisfies the condition $f : M \rightarrow M^f$, $f(P) = P + rZ_p$, r constant, then

M^f is called *the surface at a constant distance from the edge of regression on M* , and M^f is denoted by the pair (M, M^f) . If $d_1 = d_2 = 0$, then we have $Z_p = N_p$ and $f(P) = P + rN_p$. In this case M and M^f are parallel surfaces.”

They proved the following:

”Let the pair (M, M^f) be given. Let the Gauss curvature and the mean curvature of M^f be denoted by K^f and H^f , respectively. Then

$$K^f = \frac{1}{A^4} (1 + \lambda_3 H + \lambda_3^2 K) \left\{ \left(\lambda_1 \lambda_2 \frac{\partial k_1}{\partial u} \frac{\partial k_2}{\partial v} + \lambda_1 k_2 \frac{\partial k_1}{\partial u} (\lambda_2^2 k_2^2 + (1 + \lambda_3 k_2)^2) \right. \right. \\ \left. \left. + \lambda_2 k_1 \frac{\partial k_2}{\partial v} (\lambda_1^2 k_1^2 + (1 + \lambda_3 k_1)^2) \right) + A^2 K \right\}$$

and

$$H^f = \frac{1}{A^{3/2}} \left\{ \lambda_1 \frac{\partial k_1}{\partial u} (1 + \lambda_3 k_2) (\lambda_2^2 k_2^2 + (1 + \lambda_3 k_2)^2) \right. \\ \left. + \lambda_2 \frac{\partial k_2}{\partial v} (1 + \lambda_3 k_1) (\lambda_1^2 k_1^2 + (1 + \lambda_3 k_1)^2) + A^{1/2} (H + 2\lambda_3 K) \right\}$$

where

$$A = \sqrt{\lambda_1^2 k_1^2 (1 + \lambda_3 k_2)^2 + \lambda_2^2 k_2^2 (1 + \lambda_3 k_1)^2 + (1 + \lambda_3 k_1)^2 (1 + \lambda_3 k_2)^2} \\ = \sqrt{\lambda_1^2 k_1^2 + \lambda_2^2 k_2^2 + 2\lambda_3 (\lambda_1^2 k_1 + \lambda_2^2 k_2) K + \lambda_3^2 (\lambda_1^2 + \lambda_2^2) K^2 + (1 + \lambda_3 H + \lambda_3^2 K)^2}.$$

In this paper we give the some theorems and properties for surfaces at a constant distance from edge of regression on a surface in Minkowski 3-space E_1^3 which were given in [11] for E^3 .

2 Preliminaries

Let E_1^3 be a Minkowski 3-space with linear coordinates x_1, x_2, x_3 and let $\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2$ be the standard flat Lorentzian metric tensor. In order to get an orthonormal moving frame, we define the cross product $X \times Y$ of vectors $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ in E_1^3 , via

$$\langle X \times Y, Z \rangle = \det(X, Y, Z) \quad \text{for all } Z \in E_1^3$$

Explicitly if X, Y belong to E_1^3 , then

$$\langle X, Y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3 \\ X \times Y = (-x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1) \\ \langle A \times B, X \times Y \rangle = - \begin{vmatrix} \langle A, X \rangle & \langle B, X \rangle \\ \langle A, Y \rangle & \langle B, Y \rangle \end{vmatrix},$$

where $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ in E_1^3 (the Lagrange identity in E_1^3).

Furthermore, a vector X in E_1^3 is called spacelike if $X = 0$, $\langle X, X \rangle > 0$, timelike if $\langle X, X \rangle < 0$, and lightlike if $\langle X, X \rangle = 0$.

Let $e_1, e_2 \in E_1^3$ be such that $\langle e_i, e_i \rangle = \pm 1$ and $\langle e_1, e_2 \rangle = 0$ and let $e_3 = e_1 \times e_2$. Then these three vectors form an orthonormal frame. If $\langle e_1, e_1 \rangle = \varepsilon_1$ and $\langle e_2, e_2 \rangle = \varepsilon_2$, where $\varepsilon_1, \varepsilon_2 = \pm 1$, it follows from the Lagrange identity that $\langle e_3, e_3 \rangle = -\varepsilon_1 \varepsilon_2$. Each vector $X \in E_1^3$ can be written uniquely in terms of e_1, e_2, e_3 by

$$X = \varepsilon_1 \langle X, e_1 \rangle e_1 + \varepsilon_2 \langle X, e_2 \rangle e_2 - \varepsilon_1 \varepsilon_2 \langle X, e_3 \rangle e_3.$$

A surface M in E_1^3 is called spacelike $\forall p \in M$, $\langle \cdot, \cdot \rangle_p = \langle \cdot, \cdot \rangle_{T_p M}$ is positive definite, timelike if $\forall p \in M$, $\langle \cdot, \cdot \rangle_p = \langle \cdot, \cdot \rangle_{T_p M}$ is nondegenerate of index 1 and lightlike if $\forall p \in M$, $\langle \cdot, \cdot \rangle_p = \langle \cdot, \cdot \rangle_{T_p M}$ is degenerate.

Definition 2.1. Let M be a pseudo-Euclidean surface in E_1^3 and D be the Levi-Civita connection on E_1^3 . Then,

$$S : \chi(M) \rightarrow \chi(M), \quad X \rightarrow S(X) = D_X N$$

is called the shape operator, where N is the unit normal vector on M ([5]).

Definition 2.2. Let M be a pseudo-Euclidean surface in E_1^3 and let S be the shape operator of M , for $p \in M$; K denotes Gauss curvature of M and is defined as

$$\begin{aligned} K : M &\rightarrow R \\ p &\rightarrow K(p) = \varepsilon \det(S_p) \end{aligned}$$

where $\varepsilon = \langle N, N \rangle = \pm 1$ and N is the unit normal vector field on M ([9]).

Definition 2.3. Let M be a pseudo-Euclidean surface in E_1^3 and let H be the mean curvature vector field of M , defined as

$$H = \varepsilon \text{trace}(S_p)$$

where $\varepsilon = \langle N, N \rangle = \pm 1$ and N is the unit normal vector field on M ([9]).

3 Parallel surfaces in E_1^3

Let M be a pseudo-Euclidean surface with $N = (a_1, a_2, a_3)$ where each a_i is a real valued C^∞ function on M and $-a_1^2 + a_2^2 + a_3^2 = \pm 1$. For any constant r in \mathbb{R} , let $M_r = \{P + rN_p : P \in M\}$. Thus if $P = (p_1, p_2, p_3)$ is on M , then $f(P) = P + rN_p = (p_1 + ra_1(p), p_2 + ra_2(p), p_3 + ra_3(p))$ defines a new surface M_r . The map f is called the natural map on M into M_r , and if f is univalent, then M_r is a parallel surface of M with unit normal N , i.e., $N_{f(p)} = N_p$ for all P in M .

Theorem 3.1. Let M and M_r be two parallel pseudo-Euclidean surface in E_1^3 and let S_r be the Weingarten map on M_r . Let

$$f : M \rightarrow M_r$$

be a parallelization function. Then for $X \in X(M)$,

1. $f_*(X) = X + r\overline{S(X)}$

$$2. S_r(f_*(X)) = \overline{S(X)}$$

3. f preserves principal directions of curvature, that is

$$S_r(f_*(X)) = \frac{k}{1+rk} f_*(X)$$

where S_r is the shape operator on M_r , and k is the principal curvature of M at p in direction X ([5]).

Theorem 3.2. Let M and M_r be two parallel pseudo-Euclidean surface in E_1^3 . Then we have

$$K_r = \frac{K}{1+r\varepsilon H+r^2\varepsilon K} \quad \text{and} \quad H_r = \frac{H+2rK}{1+r\varepsilon H+r^2\varepsilon K}$$

where $\langle N_r, N_r \rangle = \varepsilon$ and Gaussian curvature and mean curvature of M (and M_r) be denoted by K (and K_r) and H (and H_r).

Proof. Let k_1 and k_2 be the principal curvatures of the Weingarten map on M . From their definition, we infer

$$S(X_i) = k_i X_i \quad 1 \leq i \leq 2$$

where X_i , $1 \leq i \leq 2$ are the principal directions on M . The unit normal vector N_r of M_r , satisfies

$$N_r|_{f(p)} = N_p$$

and is given by

$$\langle N_r, N_r \rangle|_{f(p)} = \langle N_r|_{f(p)}, N_r|_{f(p)} \rangle = \langle N_p, N_p \rangle = \varepsilon.$$

The shape operator matrix of M with respect to the basis $\{X_1, X_2\}$ can be written as

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

Then we get

$$K = \varepsilon \det(S) = \varepsilon k_1 k_2,$$

and

$$H = \varepsilon \operatorname{trace}(S) = \varepsilon(k_1 + k_2).$$

On the other hand, by the Theorem 3.1, $f_*(X_i)$, $1 \leq i \leq 2$, are the principal directions on M_r , too. The shape operator matrix of M_r with respect to the basis $\{f_*(X_1), f_*(X_2)\}$ can be written as

$$S_r = \begin{bmatrix} \frac{k_1}{1+rk_1} & 0 \\ 0 & \frac{k_2}{1+rk_2} \end{bmatrix}$$

Since $\langle N_r, N_r \rangle = \varepsilon$, we obtain

$$\begin{aligned}
 K_r &= \varepsilon \det(S_r) = \varepsilon \left(\frac{k_1}{1 + rk_1} \right) \left(\frac{k_2}{1 + rk_2} \right) \\
 (3.1) \quad &= \varepsilon \left(\frac{k_1 k_2}{(1 + rk_1)(1 + rk_2)} \right) = \varepsilon \left(\frac{k_1 k_2}{1 + r(k_1 + k_2) + r^2 k_1 k_2} \right) \\
 &= \varepsilon \left(\frac{\varepsilon K}{1 + r\varepsilon H + r^2 \varepsilon K} \right) = \left(\frac{K}{1 + r\varepsilon H + r^2 \varepsilon K} \right)
 \end{aligned}$$

and

$$\begin{aligned}
 H_r &= \varepsilon \text{trace}(S_r) = \varepsilon \left(\frac{k_1}{1 + rk_1} + \frac{k_2}{1 + rk_2} \right) = \varepsilon \left(\frac{(k_1 + k_2) + 2r(k_1 k_2)}{1 + r(k_1 + k_2) + r^2 k_1 k_2} \right) \\
 (3.2) \quad &= \varepsilon \left(\frac{\varepsilon H + 2r\varepsilon K}{1 + r(k_1 + k_2) + r^2 k_1 k_2} \right) = \left(\frac{H + 2rK}{1 + r\varepsilon H + r^2 \varepsilon K} \right)
 \end{aligned}$$

□

4 Surfaces at a constant distance from edge of regression on M in E_1^3

Definition 4.1. Let M and M^f be two surfaces in E_1^3 and N_p be a unit normal vector of M at the point $P \in M$. Let $T_p(M)$ be tangent space at $P \in M$ and $\{X_p, Y_p\}$ be an orthonormal bases of $T_p(M)$. Let $Z_p = d_1 X_p + d_2 Y_p + d_3 N_p$ be a unit vector, where $d_1, d_2, d_3 \in \mathbb{R}$ are constant numbers and $\varepsilon_1 d_1^2 + \varepsilon_2 d_2^2 - \varepsilon_1 \varepsilon_2 d_3^2 = \pm 1$. If a function f exists and satisfies the condition $f : M \rightarrow M^f$, $f(P) = P + rZ_p$, r constant, M^f is called *the surface at a constant distance from the edge of regression on M* , and M^f denotes the pair (M, M^f) . If $d_1 = d_2 = 0$, then we have $Z_p = N_p$ and $f(P) = P + rN_p$. In this case M and M^f are parallel surfaces.

Theorem 4.2. Let the pair (M, M^f) be given in E_1^3 . For any $W \in \chi(M)$, we have $f_*(W) = \overline{W} + r\overline{D_W Z}$, where $W = \sum_{i=1}^3 w_i \frac{\partial}{\partial x_i}$, $\overline{W} = \sum_{i=1}^3 \overline{w}_i \frac{\partial}{\partial x_i}$ and $\forall P \in M$, $w_i(P) = \overline{w}_i(f(p))$, $1 \leq i \leq 3$.

Proof. Similar to the proof of [11, Theorem 3]. □

5 Parametrization of surfaces at a constant distance from the edge of regression on M in E_1^3

Let (ϕ, U) be a parametrization of M , so we can write

$$\phi : U \underset{(u,v)}{\subset} E_1^3 \rightarrow \underset{P=\phi(u,v)}{M}.$$

In this case $\{\phi_u|_p, \phi_v|_p\}$ is a basis of $T_M(P)$. Let N_p be a unit normal vector at $P \in M$ and let $d_1, d_2, d_3 \in \mathbb{R}$ be constant numbers. Then we infer $Z_p = d_1 \phi_u|_p + d_2 \phi_v|_p + d_3 N_p$.

Since $M^f = \{f(P) \mid f(P) = P + rZ_p\}$, a parametric representation of M^f is $\psi(u, v) = \phi(u, v) + rZ(u, v)$. Thus we may write

$$M^f = \left\{ \psi(u, v) \mid \psi(u, v) = \phi(u, v) + r(d_1\phi_u(u, v) + d_2\phi_v(u, v) + d_3N(u, v)), \right. \\ \left. d_1, d_2, d_3, r \text{ are constants, } \varepsilon_1 d_1^2 + \varepsilon_2 d_2^2 - \varepsilon_1 \varepsilon_2 d_3^2 = \pm 1, \right\}$$

If we take $rd_1 = \lambda_1$, $rd_2 = \lambda_2$, $rd_3 = \lambda_3$ then we have

$$M^f = \{ \psi(u, v) \mid \psi(u, v) = \phi(u, v) + \lambda_1\phi_u(u, v) + \lambda_2\phi_v(u, v) + \lambda_3N(u, v), \lambda_1, \lambda_2, \lambda_3 \text{ are constants} \}.$$

Calculation of ψ_u and ψ_v gives us

$$(5.1) \quad \begin{aligned} \psi_u &= \phi_u + \lambda_1\phi_{uu} + \lambda_2\phi_{vu} + \lambda_3N_u, \\ \psi_v &= \phi_v + \lambda_1\phi_{uv} + \lambda_2\phi_{vv} + \lambda_3N_v. \end{aligned}$$

If we take $\langle \phi_u, \phi_u \rangle = \varepsilon_1$, $\langle \phi_v, \phi_v \rangle = \varepsilon_2$ and $\langle N, N \rangle = -\varepsilon_1\varepsilon_2$, in this case $\det(\phi_{uv}, \phi_u, \phi_v) = 0$ and $\langle \phi_u, \phi_v \rangle = 0$. Then $\phi_{uu}, \phi_{uv}, \phi_{vv}, N_u, N_v$ are equal to

$$(5.2) \quad \begin{aligned} \phi_{uu} &= \varepsilon_1 \langle \phi_{uu}, \phi_u \rangle \phi_u + \varepsilon_2 \langle \phi_{uu}, \phi_v \rangle \phi_v - \varepsilon_1 \varepsilon_2 \det(\phi_{uu}, \phi_u, \phi_v) N, \\ \phi_{uv} &= \phi_{vu} = \varepsilon_1 \langle \phi_{uv}, \phi_u \rangle \phi_u + \varepsilon_2 \langle \phi_{uv}, \phi_v \rangle \phi_v - \varepsilon_1 \varepsilon_2 \det(\phi_{uv}, \phi_u, \phi_v) N, \\ \phi_{vv} &= \varepsilon_1 \langle \phi_{vv}, \phi_u \rangle \phi_u + \varepsilon_2 \langle \phi_{vv}, \phi_v \rangle \phi_v - \varepsilon_1 \varepsilon_2 \det(\phi_{vv}, \phi_u, \phi_v) N, \\ N_u &= -\varepsilon_1 \det(\phi_{uu}, \phi_u, \phi_v) \phi_u - \varepsilon_2 \det(\phi_{uv}, \phi_u, \phi_v) \phi_v, \\ N_v &= -\varepsilon_1 \det(\phi_{uv}, \phi_u, \phi_v) \phi_u - \varepsilon_2 \det(\phi_{vv}, \phi_u, \phi_v) \phi_v. \end{aligned}$$

Moreover, the principal curvatures k_1 and k_2 of M are

$$\begin{aligned} S(\phi_u) &= k_1 \phi_u \Rightarrow k_1 = \varepsilon_1 \langle S(\phi_u), \phi_u \rangle \\ S(\phi_v) &= k_2 \phi_v \Rightarrow k_2 = \varepsilon_2 \langle S(\phi_v), \phi_v \rangle \end{aligned}$$

From the definition of the shape operator

$$\begin{aligned} \langle S(\phi_u), \phi_u \rangle &= \langle D_{\phi_u} N, \phi_u \rangle \\ &= \langle N_u, \phi_u \rangle \\ &= -\langle N, \phi_{uu} \rangle \\ &= -\det(\phi_{uu}, \phi_u, \phi_v) \end{aligned}$$

and

$$\begin{aligned} \langle S(\phi_v), \phi_v \rangle &= \langle D_{\phi_v} N, \phi_v \rangle \\ &= \langle N_v, \phi_v \rangle \\ &= -\langle N, \phi_{vv} \rangle \\ &= -\det(\phi_{vv}, \phi_u, \phi_v) \end{aligned}$$

are found.

After all these calculations we obtain

$$(5.3) \quad \begin{aligned} k_1 &= -\varepsilon_1 \det(\phi_{uu}, \phi_u, \phi_v) \\ k_2 &= -\varepsilon_2 \det(\phi_{vv}, \phi_u, \phi_v) \end{aligned}$$

and

$$(5.4) \quad \begin{aligned} \phi_{uu} &= \varepsilon_2 k_1 N \\ \phi_{vv} &= \varepsilon_1 k_2 N \\ N_u &= k_1 \phi_u \\ N_v &= k_2 \phi_v. \end{aligned}$$

From (5.1) and (5.4), we find the basis $\{\psi_u, \psi_v\}$ of $\chi(M^f)$

$$(5.5) \quad \begin{aligned} \psi_u &= (1 + \lambda_3 k_1) \phi_u + \varepsilon_2 \lambda_1 k_1 N, \\ \psi_v &= (1 + \lambda_3 k_2) \phi_v + \varepsilon_1 \lambda_2 k_2 N. \end{aligned}$$

Let us define the following quantities useful in differential geometric computations:

$$E = \langle \psi_u, \psi_u \rangle, \quad F = \langle \psi_u, \psi_v \rangle, \quad G = \langle \psi_v, \psi_v \rangle.$$

The unit normal vector field N^f of M^f is

$$N^f = \frac{\psi_u \wedge \psi_v}{\|\psi_u \wedge \psi_v\|} = \frac{\psi_u \wedge \psi_v}{\sqrt{\varepsilon(F^2 - EG)}}$$

where $\varepsilon = \langle N^f, N^f \rangle$. Therefore we obtain

$$(5.6) \quad N^f = \frac{\varepsilon_2 \lambda_1 k_1 (1 + \lambda_3 k_2) \phi_u - \varepsilon_1 \lambda_2 k_2 (1 + \lambda_3 k_1) \phi_v + (1 + \lambda_3 k_1)(1 + \lambda_3 k_2) N}{\sqrt{\varepsilon(\varepsilon_1 \lambda_1^2 k_1^2 (1 + \lambda_3 k_2)^2 + \varepsilon_2 \lambda_2^2 k_2^2 (1 + \lambda_3 k_1)^2 - \varepsilon_1 \varepsilon_2 (1 + \lambda_3 k_1)^2 (1 + \lambda_3 k_2)^2)}}.$$

In (5.6), if we write

$$\begin{aligned} A &= \sqrt{\varepsilon(\varepsilon_1 \lambda_1^2 k_1^2 (1 + \lambda_3 k_2)^2 + \varepsilon_2 \lambda_2^2 k_2^2 (1 + \lambda_3 k_1)^2 - \varepsilon_1 \varepsilon_2 (1 + \lambda_3 k_1)^2 (1 + \lambda_3 k_2)^2)} \\ &= (\varepsilon(\varepsilon_1 \lambda_1^2 k_1^2 + \varepsilon_2 \lambda_2^2 k_2^2 - 2\varepsilon_1 \varepsilon_2 \lambda_3 (\varepsilon_1 \lambda_1^2 k_1 + \varepsilon_2 \lambda_2^2 k_2) K \\ &\quad + \lambda_3^2 (\varepsilon_1 \lambda_1^2 + \varepsilon_2 \lambda_2^2) K^2 - \varepsilon_1 \varepsilon_2 (1 - \varepsilon_1 \varepsilon_2 \lambda_3 H - \varepsilon_1 \varepsilon_2 \lambda_3^2 K)^2))^{1/2}, \end{aligned}$$

then

$$N^f = \frac{\varepsilon_2 \lambda_1 k_1 (1 + \lambda_3 k_2)}{A} \phi_u - \frac{\varepsilon_1 \lambda_2 k_2 (1 + \lambda_3 k_1)}{A} \phi_v + \frac{(1 + \lambda_3 k_1)(1 + \lambda_3 k_2)}{A} N$$

or

$$N^f = \frac{\lambda_1 (\varepsilon_2 k_1 - \varepsilon_1 \lambda_3 K)}{A} \phi_u - \frac{\lambda_2 (\varepsilon_1 k_2 - \varepsilon_2 \lambda_3 K)}{A} \phi_v + \frac{(1 - \varepsilon_1 \varepsilon_2 \lambda_3 H - \varepsilon_1 \varepsilon_2 \lambda_3^2 K)}{A} N.$$

If $k_1 = k_2$ and $\lambda_3 = -1/k_1 = -1/k_2$, since ψ_u and ψ_v are not linear independent, then M^f is not regular surface. We will not consider this case.

Corollary 5.1. $\lambda_1 = \lambda_2 = 0$ if and only if the pair (M, M^f) is a parallel surface.

Proof. Since $1 + \lambda_3 k_1 \neq 0$ and $1 + \lambda_3 k_2 \neq 0$ and from (5.6), we have $N^f \in Sp\{N\}$ if and only if $\lambda_1 = \lambda_2 = 0$. In addition $A = (1 - \varepsilon_1 \varepsilon_2 \lambda_3 H - \varepsilon_1 \varepsilon_2 \lambda_3^2 K)$ and $N^f = N$. \square

Remark 5.2. M and M^f may not have to be the same casual character. To be more clear if M is a spacelike or (timelike) surface then M^f may not be spacelike or (timelike) surface respectively.

Example 5.3. Let M be a hyperbolic cylinder in E_1^3 with parametrization

$$\phi(u, v) = (\sinh u, \cosh u, v).$$

It is clear that M is a timelike surface ($\varepsilon_1 = \langle \phi_u, \phi_u \rangle = -1 < 0$ and $\varepsilon_2 = \langle \phi_v, \phi_v \rangle = 1$).

According to the definition of M^f ,

$$\psi(u, v) = ((1 - \lambda_3) \sinh u + \lambda_1 \cosh u, (1 - \lambda_3) \cosh u + \lambda_1 \sinh u, v + \lambda_2).$$

For $\lambda_1 = 3$, $\lambda_2 = 3$ and $\lambda_3 = 2$, M^f is a spacelike surface. Because

$$\psi(u, v) = (-\sinh u + 3 \cosh u, -\cosh u + 3 \sinh u, v + 3)$$

and $\langle \Psi_u, \Psi_u \rangle = 8 > 0$, $\langle \Psi_v, \Psi_v \rangle = 1$.

On the other hand for $\lambda_1 = 1$, $\lambda_2 = -1$ and $\lambda_3 = 3$, M^f is a timelike surface, since

$$\psi(u, v) = (-2 \sinh u + \cosh u, -2 \cosh u + \sinh u, v + 3)$$

and $\langle \Psi_u, \Psi_u \rangle = -3 < 0$, $\langle \Psi_v, \Psi_v \rangle = 1$.

Theorem 5.4. Let N^f be a unit normal vector field of M^f . M^f is spacelike (timelike) surface if and only if

$$\varepsilon_1 \left(\frac{\lambda_1 k_1}{1 + \lambda_3 k_1} \right)^2 + \varepsilon_2 \left(\frac{\lambda_2 k_2}{1 + \lambda_3 k_2} \right)^2 - \varepsilon_1 \varepsilon_2 < 0 \quad (> 0).$$

Proof. A surface M is a spacelike (timelike) surface if and only if $\langle N, N \rangle < 0$ ($\langle N, N \rangle > 0$). So that the proof is clear from $\langle N^f, N^f \rangle$. \square

6 Shape operator of surfaces at a constant distance from edge of regression on M

Let S and S^f be the shape operators of M and M^f , respectively. We find S^f with respect to the basis $\{\psi_u = (1 + \lambda_3 k_1)\phi_u + \varepsilon_2 \lambda_1 k_1 N, \psi_v = (1 + \lambda_3 k_2)\phi_v + \varepsilon_1 \lambda_2 k_2 N\}$ of $\chi(M^f)$.

$\{\phi_u, \phi_v, N\}$ is a basis of $\chi(E_1^3)$, where N is the unit normal vector field on M , $\{\phi_u, \phi_v\}$ (orthonormal and principal vectors field on M) is a basis of $\chi(M)$ and k_1, k_2 are principal curvatures of M . We note that the curvature of E_1^3 , denoted by R , is zero ([9]). Thus

$$\begin{aligned} R(\phi_u, \phi_v)N &= 0 \\ D_{\phi_u}(D_{\phi_v}N) - D_{\phi_v}(D_{\phi_u}N) - D_{[\phi_u, \phi_v]}N &= 0 \\ D_{\phi_u}(k_2 \phi_v) - D_{\phi_v}(k_1 \phi_u) - D_{[D_{\phi_u} \phi_v, D_{\phi_v} \phi_u]}N &= 0. \end{aligned}$$

Since $D_{\phi_u}\phi_v - D_{\phi_v}\phi_u = \phi_{vu} - \phi_{uv} = 0$

$$\begin{aligned} D_{\phi_u}(k_2\phi_v) - D_{\phi_v}(k_1\phi_u) &= 0 \\ \frac{\partial k_2}{\partial u}\phi_v + k_2\phi_{vu} - \frac{\partial k_1}{\partial v}\phi_u - k_1\phi_{uv} &= 0 \\ \frac{\partial k_2}{\partial u}\phi_v - \frac{\partial k_1}{\partial v}\phi_u &= 0 \end{aligned}$$

and $\{\phi_u, \phi_v\}$ is linear independent, we have

$$(6.1) \quad \frac{\partial k_1}{\partial v} = 0 \quad \text{and} \quad \frac{\partial k_2}{\partial u} = 0.$$

We will use this equalities below.

$$\begin{aligned} S^f(\psi_u) = D_{\psi_u}N^f &= \frac{\partial}{\partial u} \left(\frac{\varepsilon_2\lambda_1k_1(1+\lambda_3k_2)}{A} \right) \phi_u + \frac{\varepsilon_2\lambda_1k_1(1+\lambda_3k_2)}{A} \underbrace{\phi_{uu}}_{\varepsilon_2k_1N} \\ &+ \frac{\partial}{\partial u} \left(\frac{-\varepsilon_1\lambda_2k_2(1+\lambda_3k_1)}{A} \right) \phi_v - \frac{\varepsilon_1\lambda_2k_2(1+\lambda_3k_1)}{A} \underbrace{\phi_{uv}}_0 \\ &+ \frac{\partial}{\partial u} \left(\frac{(1+\lambda_3k_1)(1+\lambda_3k_2)}{A} \right) N + \frac{(1+\lambda_3k_1)(1+\lambda_3k_2)}{A} \underbrace{N_u}_{k_1\phi_u}, \end{aligned}$$

and thus

$$\begin{aligned} (6.2) \quad S^f(\psi_u) &= \left\{ \frac{\partial}{\partial u} \left(\frac{\varepsilon_2\lambda_1k_1(1+\lambda_3k_2)}{A} \right) + \frac{k_1(1+\lambda_3k_1)(1+\lambda_3k_2)}{A} \right\} \phi_u \\ &+ \frac{\partial}{\partial u} \left(\frac{-\varepsilon_1\lambda_2k_2(1+\lambda_3k_1)}{A} \right) \phi_v \\ &+ \left\{ \frac{\partial}{\partial u} \left(\frac{(1+\lambda_3k_1)(1+\lambda_3k_2)}{A} \right) + \frac{\lambda_1k_1^2(1+\lambda_3k_2)}{A} \right\} N. \end{aligned}$$

Now we obtain the coefficients of ϕ_u, ϕ_v and N , hence

$$\begin{aligned} (6.3) \quad \frac{\partial A}{\partial u} &= \frac{\varepsilon}{2A} \left(2\varepsilon_1\lambda_1^2k_1 \frac{\partial k_1}{\partial u} (1+\lambda_3k_2)^2 + 2\varepsilon_2\lambda_2^2\lambda_3k_2^2 \frac{\partial k_1}{\partial u} (1+\lambda_3k_1) \right. \\ &\quad \left. - 2\varepsilon_1\varepsilon_2\lambda_3 \frac{\partial k_1}{\partial u} (1+\lambda_3k_1)(1+\lambda_3k_2)^2 \right) \\ &= \frac{\varepsilon}{A} \frac{\partial k_1}{\partial u} \left\{ \varepsilon_1\lambda_1^2k_1(1+\lambda_3k_2)^2 + \varepsilon_2\lambda_3(1+\lambda_3k_1)(\lambda_2^2k_2^2 - \varepsilon_1(1+\lambda_3k_2)^2) \right\}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial u} \left(\frac{\varepsilon_2\lambda_1k_1(1+\lambda_3k_2)}{A} \right) &= \frac{\frac{\partial}{\partial u}(\varepsilon_2\lambda_1k_1(1+\lambda_3k_2))A - \frac{\partial A}{\partial u}(\varepsilon_2\lambda_1k_1(1+\lambda_3k_2))}{A^2} \\ &= \frac{\varepsilon\varepsilon_2}{A^3} \frac{\partial k_1}{\partial u} \lambda_1(1+\lambda_3k_2) \left\{ A^2 - \varepsilon k_1 \left(\varepsilon_1\lambda_1^2k_1(1+\lambda_3k_2)^2 \right. \right. \\ &\quad \left. \left. + \varepsilon_2\lambda_3(1+\lambda_3k_1)(\lambda_2^2k_2^2 - \varepsilon_1(1+\lambda_3k_2)^2) \right) \right\} \\ &= \frac{\varepsilon}{A^3} \frac{\partial k_1}{\partial u} \lambda_1(1+\lambda_3k_1)(1+\lambda_3k_2)(\lambda_2^2k_2^2 - \varepsilon_1(1+\lambda_3k_2)^2). \end{aligned}$$

Then we have

$$(6.4) \quad \begin{aligned} & \frac{\partial}{\partial u} \left(\frac{\varepsilon_2 \lambda_1 k_1 (1 + \lambda_3 k_2)}{A} \right) + \frac{k_1 (1 + \lambda_3 k_1) (1 + \lambda_3 k_2)}{A} \\ &= \frac{(1 + \lambda_3 k_1) (1 + \lambda_3 k_2)}{A^3} \left\{ \varepsilon \lambda_1 \frac{\partial k_1}{\partial u} (\lambda_2^2 k_2^2 - \varepsilon_1 (1 + \lambda_3 k_2)^2) + k_1 A^2 \right\}, \end{aligned}$$

$$(6.5) \quad \frac{\partial}{\partial u} \left(\frac{-\varepsilon_1 \lambda_2 k_2 (1 + \lambda_3 k_1)}{A} \right) = \frac{\varepsilon \lambda_1^2 \lambda_2 k_1 k_2}{A^3} \frac{\partial k_1}{\partial u} (1 + \lambda_3 k_2)^2$$

and

$$(6.6) \quad \begin{aligned} \frac{\partial}{\partial u} \left(\frac{(1 + \lambda_3 k_1) (1 + \lambda_3 k_2)}{A} \right) &= \frac{\frac{\partial}{\partial u} ((1 + \lambda_3 k_1) (1 + \lambda_3 k_2)) A - \frac{\partial A}{\partial u} ((1 + \lambda_3 k_1) (1 + \lambda_3 k_2))}{A^2} \\ &= \frac{\varepsilon \frac{\partial k_1}{\partial u} (1 + \lambda_3 k_2)}{A^3} \left\{ \lambda_3 A^2 - \varepsilon (1 + \lambda_3 k_1) (\varepsilon_1 \lambda_1^2 k_1 (1 + \lambda_3 k_2)^2 \right. \\ &\quad \left. + \varepsilon_2 \lambda_3 (1 + \lambda_3 k_1) (\lambda_2^2 k_2^2 - \varepsilon_1 (1 + \lambda_3 k_2)^2) \right\} \\ &= \frac{-\varepsilon \varepsilon_1 \lambda_1^2 k_1}{A^3} \frac{\partial k_1}{\partial u} (1 + \lambda_3 k_2)^3. \end{aligned}$$

Hence, we get

$$(6.7) \quad \begin{aligned} & \frac{\partial}{\partial u} \left(\frac{(1 + \lambda_3 k_1) (1 + \lambda_3 k_2)}{A} \right) + \frac{\lambda_1 k_1^2 (1 + \lambda_3 k_2)}{A} \\ &= \frac{\lambda_1 k_1 (1 + \lambda_3 k_2)}{A^3} \left\{ -\varepsilon \varepsilon_1 \lambda_1 \frac{\partial k_1}{\partial u} (1 + \lambda_3 k_2)^2 + k_1 A^2 \right\}. \end{aligned}$$

From (6.2), (6.4), (6.5) and (6.7) we obtain

$$(6.8) \quad \begin{aligned} S^f(\psi_u) &= \frac{(1 + \lambda_3 k_1) (1 + \lambda_3 k_2)}{A^3} \left\{ \varepsilon \lambda_1 \frac{\partial k_1}{\partial u} (\lambda_2^2 k_2^2 - \varepsilon_1 (1 + \lambda_3 k_2)^2) + k_1 A^2 \right\} \phi_u \\ &+ \frac{\varepsilon \lambda_1^2 \lambda_2 k_1 k_2}{A^3} \frac{\partial k_1}{\partial u} (1 + \lambda_3 k_2)^2 \phi_v \\ &+ \frac{\lambda_1 k_1 (1 + \lambda_3 k_2)}{A^3} \left\{ -\varepsilon \varepsilon_1 \lambda_1 \frac{\partial k_1}{\partial u} (1 + \lambda_3 k_2)^2 + k_1 A^2 \right\} N, \end{aligned}$$

$$\begin{aligned} S^f(\psi_v) &= D_{\psi_v} N^f = \frac{\partial}{\partial v} \left(\frac{\varepsilon_2 \lambda_1 k_1 (1 + \lambda_3 k_2)}{A} \right) \phi_u + \frac{\varepsilon_2 \lambda_1 k_1 (1 + \lambda_3 k_2)}{A} \underbrace{\phi_{uv}}_0 \\ &+ \frac{\partial}{\partial v} \left(\frac{-\varepsilon_1 \lambda_2 k_2 (1 + \lambda_3 k_1)}{A} \right) \phi_v - \frac{\varepsilon_1 \lambda_2 k_2 (1 + \lambda_3 k_1)}{A} \underbrace{\phi_{vv}}_{\varepsilon_1 k_2 N} \\ &+ \frac{\partial}{\partial v} \left(\frac{(1 + \lambda_3 k_1) (1 + \lambda_3 k_2)}{A} \right) N + \frac{(1 + \lambda_3 k_1) (1 + \lambda_3 k_2)}{A} \underbrace{N_v}_{k_2 \phi_v}, \end{aligned}$$

$$\begin{aligned}
(6.9) \quad S^f(\psi_v) &= \frac{\partial}{\partial v} \left(\frac{\varepsilon_2 \lambda_1 k_1 (1 + \lambda_3 k_2)}{A} \right) \phi_u \\
&+ \left\{ \frac{\partial}{\partial v} \left(\frac{-\varepsilon_1 \lambda_2 k_2 (1 + \lambda_3 k_1)}{A} \right) + \frac{k_2 (1 + \lambda_3 k_1) (1 + \lambda_3 k_2)}{A} \right\} \phi_v \\
&+ \left\{ \frac{\partial}{\partial v} \left(\frac{(1 + \lambda_3 k_1) (1 + \lambda_3 k_2)}{A} \right) - \frac{\lambda_2 k_2^2 (1 + \lambda_3 k_1)}{A} \right\} N.
\end{aligned}$$

Calculations similar to the previous ones yield

$$\begin{aligned}
(6.10) \quad S^f(\psi_u) &= \frac{-\varepsilon \lambda_1 \lambda_2^2 k_1 k_2 (1 + \lambda_3 k_1)^2}{A^3} \frac{\partial k_2}{\partial v} \phi_u \\
&+ \frac{(1 + \lambda_3 k_1) (1 + \lambda_3 k_2)}{A^3} \left\{ -\varepsilon \lambda_2 \frac{\partial k_2}{\partial v} (\lambda_1^2 k_1^2 - \varepsilon_2 (1 + \lambda_3 k_1)^2) + k_2 A^2 \right\} \phi_v \\
&+ \frac{\lambda_2 k_2 (1 + \lambda_3 k_1)}{A^3} \left\{ -\varepsilon \varepsilon_2 \lambda_2 \frac{\partial k_2}{\partial v} (1 + \lambda_3 k_1)^2 - k_2 A^2 \right\} N.
\end{aligned}$$

We find $S^f(\psi_u)$ and $S^f(\psi_v)$ with respect to the basis $\{\phi_u, \phi_v, N\}$. Further, we obtain $S^f(\psi_u)$ and $S^f(\psi_v)$ with respect to ψ_u, ψ_v :

$$\begin{aligned}
(6.11) \quad S^f(\psi_u) &= \mu_1 \psi_u + \mu_2 \psi_v, \\
S^f(\psi_v) &= \mu_3 \psi_u + \mu_4 \psi_v.
\end{aligned}$$

From (5.5) and (6.11), we get

$$(6.12) \quad S^f(\psi_u) = \mu_1 (1 + \lambda_3 k_1) \phi_u + \mu_2 (1 + \lambda_3 k_2) \phi_v + (\mu_1 \varepsilon_2 \lambda_1 k_1 + \mu_2 \varepsilon_1 \lambda_2 k_2) N$$

From (6.8) and (6.12) we obtain

$$\begin{aligned}
\mu_1 &= \frac{(1 + \lambda_3 k_2)}{A^3} \left\{ \varepsilon \lambda_1 \frac{\partial k_1}{\partial u} (\lambda_2^2 k_2^2 - \varepsilon_1 (1 + \lambda_3 k_2)^2) + k_1 A^2 \right\} \\
\mu_2 &= \frac{\varepsilon \lambda_1^2 \lambda_2 k_1 k_2 (1 + \lambda_3 k_2)}{A^3} \frac{\partial k_1}{\partial u}.
\end{aligned}$$

Using (5.5),

$$(6.13) \quad S^f(\psi_v) = \mu_3 (1 + \lambda_3 k_1) \phi_u + \mu_4 (1 + \lambda_3 k_2) \phi_v + (\mu_3 \varepsilon_2 \lambda_1 k_1 + \mu_4 \varepsilon_1 \lambda_2 k_2) N$$

and from (6.10) and (6.13) we have

$$\begin{aligned}
\mu_3 &= \frac{-\varepsilon \lambda_1 \lambda_2^2 k_1 k_2 (1 + \lambda_3 k_1)}{A^3} \frac{\partial k_2}{\partial v} \\
\mu_4 &= \frac{(1 + \lambda_3 k_1)}{A^3} \left\{ -\varepsilon \lambda_2 \frac{\partial k_2}{\partial v} (\lambda_1^2 k_1^2 - \varepsilon_2 (1 + \lambda_3 k_1)^2) + k_2 A^2 \right\}.
\end{aligned}$$

Thus the matrix of shape operator S^f of M^f with respect to the basis $\{\psi_u, \psi_v\}$ of $\chi(M^f)$ is

$$(6.14) \quad S^f = \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{bmatrix},$$

where

$$\begin{cases} \mu_1 = \frac{1}{A^3} t_2 \left\{ \varepsilon \lambda_1 \frac{\partial k_1}{\partial u} (\lambda_2^2 k_2^2 - \varepsilon_1 (1 + \lambda_3 k_2)^2) + k_1 A^2 \right\} \\ \mu_2 = \frac{1}{A^3} \varepsilon \lambda_1^2 \lambda_2 k_1 k_2 (1 + \lambda_3 k_2) \frac{\partial k_1}{\partial u} \\ \mu_3 = -\frac{1}{A^3} \varepsilon_1 \lambda_1 \lambda_2^2 k_1 k_2 (1 + \lambda_3 k_1) \frac{\partial k_2}{\partial v} \\ \mu_4 = \frac{1}{A^3} t_1 \left\{ -\varepsilon \lambda_2 \frac{\partial k_2}{\partial v} (\lambda_1^2 k_1^2 - \varepsilon_2 (1 + \lambda_3 k_1)^2) + k_2 A^2 \right\} \end{cases}$$

and $t_1 = (1 + \lambda_3 k_1)$ and $t_2 = (1 + \lambda_3 k_2)$.

Theorem 6.1. *Let the pair (M, M^f) be given. Consider $\{\phi_u, \phi_v\}$ (orthonormal and principal vector fields on M) be basis of $\chi(M)$ and k_1, k_2 be principal curvatures of M . The matrix of the shape operator of M^f with respect to the basis $\{\psi_u = (1 + \lambda_3 k_1)\phi_u + \varepsilon_2 \lambda_1 k_1 N, \psi_v = (1 + \lambda_3 k_2)\phi_v + \varepsilon_1 \lambda_2 k_2 N\}$ of $\chi(M^f)$ is (6.14), where $t_1 = (1 + \lambda_3 k_1)$ and $t_2 = (1 + \lambda_3 k_2)$.*

Theorem 6.2. *Let the pair (M, M^f) be given. Let the Gauss curvature and the mean curvature of M^f be denoted by K^f and H^f , respectively. Then*

$$K^f = \frac{\varepsilon(1 + \lambda_3 k_1)(1 + \lambda_3 k_2)}{A} \left\{ \varepsilon \left(\lambda_1 \lambda_2 \frac{\partial k_1}{\partial u} \frac{\partial k_2}{\partial v} + \lambda_1 k_2 \frac{\partial k_1}{\partial u} (\lambda_2^2 k_2^2 - \varepsilon_1 (1 + \lambda_3 k_2)^2) - \lambda_2 k_1 \frac{\partial k_2}{\partial v} (\lambda_1^2 k_1^2 - \varepsilon_2 (1 + \lambda_3 k_1)^2) \right) + k_1 k_2 A^2 \right\}$$

or with $k_1 k_2 = -\varepsilon_1 \varepsilon_2 K$ and $k_1 + k_2 = -\varepsilon_1 \varepsilon_2 H$,

$$(6.15) \quad \begin{aligned} K^f &= \frac{1}{A^4} \varepsilon (1 - \varepsilon_1 \varepsilon_2 \lambda_3 H - \varepsilon_1 \varepsilon_2 \lambda_3^2 K) \\ &\times \left\{ \varepsilon \left(\lambda_1 \lambda_2 \frac{\partial k_1}{\partial u} \frac{\partial k_2}{\partial v} + \lambda_1 k_2 \frac{\partial k_1}{\partial u} (\lambda_2^2 k_2^2 - \varepsilon_1 (1 + \lambda_3 k_2)^2) - \lambda_2 k_1 \frac{\partial k_2}{\partial v} (\lambda_1^2 k_1^2 - \varepsilon_2 (1 + \lambda_3 k_1)^2) \right) - A^2 \varepsilon_1 \varepsilon_2 K \right\} \end{aligned}$$

and

$$H^f = \frac{\varepsilon}{A^3} \left\{ (1 + \lambda_3 k_2) \left(\varepsilon \lambda_1 \frac{\partial k_1}{\partial u} (\lambda_2^2 k_2^2 - \varepsilon_1 (1 + \lambda_3 k_2)^2) + k_1 A^2 \right) + (1 + \lambda_3 k_1) \left(-\varepsilon \lambda_2 \frac{\partial k_2}{\partial v} (\lambda_1^2 k_1^2 - \varepsilon_2 (1 + \lambda_3 k_1)^2) + k_2 A^2 \right) \right\}$$

or with $k_1 k_2 = -\varepsilon_1 \varepsilon_2 K$ and $k_1 + k_2 = -\varepsilon_1 \varepsilon_2 H$,

$$(6.16) \quad \begin{aligned} H^f &= \frac{\varepsilon}{A^3} \left\{ \varepsilon \lambda_1 \frac{\partial k_1}{\partial u} (1 + \lambda_3 k_2) (\lambda_2^2 k_2^2 - \varepsilon_1 (1 + \lambda_3 k_2)^2) - \varepsilon \lambda_2 \frac{\partial k_2}{\partial v} (1 + \lambda_3 k_1) (\lambda_1^2 k_1^2 - \varepsilon_2 (1 + \lambda_3 k_1)^2) - A^2 (\varepsilon_1 \varepsilon_2 H + 2\varepsilon_1 \varepsilon_2 \lambda_3 K) \right\} \end{aligned}$$

where

$$A = \sqrt{\varepsilon(\varepsilon_1\lambda_1^2k_1^2(1 + \lambda_3k_2)^2 + \varepsilon_2\lambda_2^2k_2^2(1 + \lambda_3k_1)^2 - \varepsilon_1\varepsilon_2(1 + \lambda_3k_1)^2(1 + \lambda_3k_2)^2)} \\ = \left(\varepsilon(\varepsilon_1\lambda_1^2k_1^2 + \varepsilon_2\lambda_2^2k_2^2 - 2\varepsilon_1\varepsilon_2\lambda_3(\varepsilon_1\lambda_1^2k_1 + \varepsilon_2\lambda_2^2k_2)K \right. \\ \left. + \lambda_3^2(\varepsilon_1\lambda_1^2 + \varepsilon_2\lambda_2^2)K^2 - \varepsilon_1\varepsilon_2(1 - \varepsilon_1\varepsilon_2\lambda_3H - \varepsilon_1\varepsilon_2\lambda_3^2K^2) \right)^{1/2}.$$

Proof. Since $K^f = \varepsilon \det(S^f)$ and $H^f = \varepsilon \text{trace}(S^f)$ and from (6.14), one infers the above values of K^f and H^f . \square

As special case, if we take $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = r = \text{constant}$ and $\varepsilon = -\varepsilon_1\varepsilon_2$ then $\psi(u, v) = \phi(u, v) + rN|_{\phi(u, v)}$. Then M and M^f are parallel surfaces. From (6.15) and (6.16) we obtain

$$K^f = \frac{K}{1 + r\varepsilon H + r^2\varepsilon K} \quad \text{and} \quad H^f = \frac{H + 2rK}{1 + r\varepsilon H + r^2\varepsilon K}.$$

On the contrary, if $K^f = \frac{K}{1 + r\varepsilon H + r^2\varepsilon K}$ and $H^f = \frac{H + 2rK}{1 + r\varepsilon H + r^2\varepsilon K}$, then from (6.15) and (6.16) $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = r = \text{constant}$ and $\varepsilon = -\varepsilon_1\varepsilon_2$ are determined. Thus M and M^f are parallel surfaces.

Theorem 6.3. M and M^f are parallel surfaces if and only if $K^f = \frac{K}{1 + r\varepsilon H + r^2\varepsilon K}$ and $H^f = \frac{H + 2rK}{1 + r\varepsilon H + r^2\varepsilon K}$ where K (and K^f) and H (and H^f) are the Gauss curvature and the mean curvature of M (and M^f), respectively.

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