On quasi-conformally flat quasi-Einstein spaces

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Abstract. The present paper deals with a study of quasi-conformally flat quasi-Einstein spaces and obtained several necessary and sufficient conditions for such a space to be of semi-symmetric (resp. projectively semi-symmetric, concircularly semi-symmetric and conformally semi-symmetric). The existence of such space is ensured with a proper example.

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1 Introduction

The notion of quasi-Einstein spaces arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces. For instance, the Robertson-Walker spacetimes are quasi-Einstein spaces. It is well known that a connected Riemannian space $(M^n, g)(n > 2)$ is Einstein if its Ricci tensor R_{ij} of type (0, 2) is of the form $R_{ij} = \alpha g_{ij}$, where α is a constant, which turns into $R_{ij} = \frac{R}{n} g_{ij}$, R being the scalar curvature (constant) of the space. Let $(M^n, g)(n > 2)$ be a connected Riemannian space. Let $U = \{x \in M : R_{ij} \neq \frac{R}{n} g_{ij}$ at $x\}$. Then (M^n, g) is said to be a quasi-Einstein space ([6], [16],[17],[18],[19],[21], [20],[23],[22],[24],[26],[27]) if on $U \subset M$, the relation

$$R_{ij} - \alpha g_{ij} = \beta A_i A_j \tag{1.1}$$

holds, where A_i is a unit covariant vector on U and α, β are some scalars on U. It is obvious that the covariant vectorA_i as well as the function β are non-zero at every point on U. Also every Einstein space is quasi-Einstein but not conversely (see, Example 1 of section 5). Especially, every Ricci-flat space (e.g. Schwarzschild spacetime is quasi-Einstein). The scalars α, β are known as the associated scalars of the space and the unit covariant vector A_i is called the generator of the space. Such an n-dimensional quasi-Einstein space is denoted by $(QE)_n$. $(QE)_n$ is also studied by U. C. De and G. C. Ghosh [9], C. Özgür and S. Sular [29], A. A. Shaikh et. al ([32]).

The notion of $(QE)_n$ has been generalized by many authors in several ways (see, [4], [1], [2], [3], [5], [10], [11], [15], [28], [30], [31]).

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In 1968 Yano and Sawaki [34] introduced the notion of a new curvature tensor called quasi-conformal curvature tensor which includes both the conformal and concircular curvature tensor as special cases.
The quasi-conformal curvature tensor W^h_{ijk} of type (1,3) of a Riemannian space of dimension $n(>3)$ [This condition is assumed throughout the paper as for $n = 3$, the conformal curvature tensor vanishes] is defined by

$$W^h_{ijk} = -(n-2)bC^h_{ijk} + [a + (n-2)b]\widetilde{C}^h_{ijk}, \tag{1.2}$$

where a, b are arbitrary constants not simultaneously zero, C^h_{ijk} and $\widetilde{C}^h_{ijk}$ are conformal and concircular curvature tensor of type $(1,3)$ respectively.

When $a = 1$ and $b = -\frac{1}{n-2}$, then the quasi-conformal curvature tensor reduces to conformal curvature tensor. Again, if $a = 1$ and $b = 0$, then the quasi-conformal curvature tensor reduces to concircular curvature tensor. The conformal curvature tensor and concircular curvature tensor $C^h_{ijk}, \widetilde{C}^h_{ijk}$ are respectively given by

$$\begin{aligned} C^h_{ijk} &= R^h_{ijk} - \frac{1}{(n-2)}[\delta^h_k R_{ij} - \delta^h_j R_{ik} + R^h_k g_{ij} - R^h_j g_{ik}] \\ &+ \frac{R}{(n-1)(n-2)}[\delta^h_k g_{ij} - \delta^h_j g_{ik}] \end{aligned} \tag{1.3}$$

and

$$\widetilde{C}^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)}[\delta^h_k g_{ij} - \delta^h_j g_{ik}], \tag{1.4}$$

where R denotes the scalar curvature of the space. Using (1.3) and (1.4) in (1.2) we get

$$\begin{aligned} W^h_{ijk} &= aR^h_{ijk} + b[\delta^h_k R_{ij} - \delta^h_j R_{ik} + R^h_k g_{ij} - R^h_j g_{ik}] \\ &- \frac{R}{n}\left(\frac{a}{n-1} + 2b\right)[\delta^h_k g_{ij} - \delta^h_j g_{ik}]. \end{aligned} \tag{1.5}$$

A Rimennian space of dimension $n(>3)$ is said to be quasi-conformally flat if its quasi-conformal curvature tensor vanishes identically. The object of the present paper is to study a quasi-conformally flat $(QE)_n$. In section 2, it is shown that a quasi-conformally flat $(QE)_n$ is of quasi-constant curvature. Then we find a necessary and sufficient condition for a quasi-conformally flat $(QE)_n$ to be of semi-symmetric. Section 3 deal with several necessary and sufficient conditions for a quasi-conformally flat $(QE)_n$ to be of concircularly, projectively and conformally semi-symmetric respectively. Section 4 is devoted to the study of totally umbilical hypersurface ([7],p.43)of a quasi-conformally flat $(QE)_n$. The last section deals with a proper example of a $(QE)_n$.

2 Conditions for a quasi-conformally flat $(QE)_n$ to be semi-symmetric

Let us consider a $(QE)_n$ which is quasi-conformally flat. Then from (1.5), it follows that

$$R^h_{ijk} = X[\delta^h_k R_{ij} - \delta^h_j R_{ik} + R^h_k g_{ij} - R^h_j g_{ik}] + YR[\delta^h_k g_{ij} - \delta^h_j g_{ik}], \tag{2.1}$$

where

$$X = -\frac{b}{a} \text{ and } Y = \frac{1}{an(n-1)}[a + (n-1)2b] \text{ provided that } a \neq 0. \tag{2.2}$$

Since the space under consideration is $(QE)_n$, its Ricci tensor R_{ij} of type (0,2) can be written as (1.1). Transvecting with g^{ij}, from (1.1) we get

$$R = n\alpha + \beta. \tag{2.3}$$

Using (2.2), (1.1) and (2.3) in (2.1), we get

$$R^h_{ijk} = P[\delta^h_k g_{ij} - \delta^h_j g_{ik}] + Q[\delta^h_k A_i A_j - \delta^h_j A_i A_k + g_{ij} A^h A_k - g_{ik} A^h A_j], \tag{2.4}$$

where $P = \frac{n\alpha+\beta}{n(n-1)} + \frac{2b\beta}{an}$, $Q = -\frac{b}{a}\beta$ are scalars and $g^{ij}A_i = A^j$. Generalizing the notion of a space of constant curvature, in 1972, Chen and Yano[8] introduced the notion of a space of quasi-constant curvature defined as follows:

Definition 2.1. A Riemannian space (M^n, g) $(n > 3)$ is said to be of quasi-constant curvature if it is conformally flat and its curvature tensor R^h_{ijk} of type (1, 3) has the form

$$R^h_{ijk} = U[\delta^h_k g_{ij} - \delta^h_j g_{ik}] + V[\delta^h_k A_i A_j - \delta^h_j A_i A_k + g_{ij} A^h A_k - g_{ik} A^h A_j],$$

where A_i is a covariant vector, and U; V are scalars of which $V \neq 0$.

Hence from (2.4) it follows that a quasi-conformally flat $(QE)_n$ is of quasi-constant curvature provided that $a \neq 0$ and $b \neq 0$. Hence we can state the following:

Theorem 2.2. *A quasi-conformally flat $(QE)_n$ with $a \neq 0$ and $b \neq 0$ is a space of quasi-constant curvature.*

Corollary 2.3. *[14] A conformally flat $(QE)_n$ is of quasi-constant curvature.*

A Riemannian space (M^n, g) is said to be semi-symmetric [33] if its curvature tensor R^h_{ijk} of type (1, 3) satisfies the condition

$$R^h_{ijk,\,lm} - R^h_{ijk,\,ml} = 0, \tag{2.5}$$

where ',' denotes the covariant differentiation with respect to the coordinates. From (2.5), it follows that

$$R_{ij,\,lm} = R_{ij,\,ml}. \tag{2.6}$$

We first suppose that a quasi-conformally flat $(QE)_n$ is semi-symmetric. Then it satisfies both the condition (1.1) and (2.6). From (1.1), it follows that

$$R_{ij,\, lm} - R_{ij,\, ml} = \beta[(A_{i,\, lm} - A_{i,\, ml})A_j + A_i(A_{j,\, lm} - A_{j,\, ml})]. \tag{2.7}$$

Using (2.6) in (2.7), we get

$$(A_{i,\, lm} - A_{i,\, ml})A_j + A_i(A_{j,\, lm} - A_{j,\, ml}) = 0, \tag{2.8}$$

since $\beta \neq 0$. Now using Ricci identity ([12], p 234) we get from (2.8)

$$(A_h R^h_{ilm})A_j + A_i(A_h R^h_{jlm}) = 0. \tag{2.9}$$

Transvecting with A^j we get

$$A_h R^h_{ilm} + A_i A_h A^j R^h_{jlm} = 0. \tag{2.10}$$

Since $A_i A_h A^j R^h_{jlm} = 0$, we have $A_h R^h_{ilm} = 0$ if and only if

$$A_{i,\, lm} = A_{i,\, ml}. \tag{2.11}$$

Hence we obtain that if a quasi-conformally flat$(QE)_n$ is semi-symmetric, then the generator of the space satisfies the condition (2.11) provided that $a \neq 0$.

Conversely, let us assume that the generator of a quasi-conformally flat $(QE)_n$ satisfies the condition (2.11). Now, from (2.4) we get

$$\begin{aligned} R^h_{ijk,\, lm} &= P_{,\, lm}[\delta^h_k g_{ij} - \delta^h_j g_{ik}] \\ &+Q_{,\, lm}[\delta^h_k A_i A_j - \delta^h_j A_i A_k + g_{ij} A^h A_k - g_{ik} A^h A_j] \\ &+Q[\delta^h_k (A_{i,\, lm} A_j + A_i A_{j,\, lm}) - \delta^h_j (A_{i,\, lm} A_k + A_i A_{k,\, lm})] \\ &+Q[g_{ij}(A^h_{,\, lm} A_k + A^h A_{k,\, lm}) - g_{ik}(A^h_{,\, lm} A_j + A^h A_{j,\, lm})]. \end{aligned} \tag{2.12}$$

This gives,

$$R^h_{ijk,\, lm} - R^h_{ijk,\, ml} = 0 \tag{2.13}$$

i.e., the space is semi-symmetric. Hence we can state the following:

Theorem 2.4. *A quasi-conformally flat $(QE)_n$ with $a \neq 0$ is semi-symmetric if and only if the generator A_i of the space satisfies the relation (2.11).*

Corollary 2.5. *A quasi-conformally flat $(QE)_n$ with $a \neq 0$ is semi-symmetric if the generator satisfies any one of the following:*
(i) A^i is a parallel vector i.e., $A^i_{,\, j} = 0$;
(ii)A^i is a concurrent vector, i.e., $A^i_{,\, j} = c\delta^i_j$, where c is a constant.

Proof. Let the generator A^i be a parallel vector. So,

$$A^i_{,\, l} = 0. \tag{2.14}$$

Then from (2.14) we get

$$A^i_{,\ lm} = A^i_{,\ ml}$$

i.e.,

$$A_{i,\ lm} = A_{i,\ ml}$$

since $A^i = g^{ij}A_j$. By Theorem 2.4, the space is semi-symmetric, which proves (i). Again we suppose that the generator A^i of the space is concurrent. Then

$$A^i_{,\ l} = d\delta^i_j, \tag{2.15}$$

where d is a constant. By virtue of (2.15) we get

$$A^i_{,\ lm} = A^i_{,\ ml}$$

i.e.,

$$A_{i,\ lm} = A_{i,\ ml}.$$

Hence by virtue of Theorem 2.4, the result is proved.

Corollary 2.6. *[14] A conformally flat $(QE)_n$ is semi-symmetric if and only if the generator A_i of the space satisfies*

$$A_{i,\ lm} = A_{i,\ ml}.$$

Now, the condition (2.11) is equivalent to

$$A_h R^h_{ilm} = 0.$$

i.e.,

$$A_h R^h_{ijk} = 0.$$

Expressing this with respect to P and Q we get from (2.4)

$$(P + Q)(A_k g_{ij} - A_j g_{ik}) = 0, \tag{2.16}$$

which yields

$$(P + Q)(n - 1) = 0.$$

For $n > 3$ we have $P + Q = 0$. This gives

$$\alpha + T\beta = 0, \quad \text{where} \quad T = \frac{1}{an}[a - b(n - 1)(n - 2)]. \tag{2.17}$$

Conversely, if the condition (2.17) holds, then $P + Q = 0$ and hence it can easily be shown that the space under consideration is semi-symmetric. Thus we can state the following:

Theorem 2.7. *A quasi-conformally flat $(QE)_n$ with $a \neq 0$ is semi-symmetric if and only if the associated scalars satisfies the condition (2.17).*

3 Conditions for a quasi-conformally flat $(QE)_n$ to be concircularly, projectively and conformally semi-symmetric

We suppose that a Riemannian space (M^n, g) satisfies the condition

$$F^h_{ijk,\ lm} - F^h_{ijk,\ ml} = 0, \tag{3.1}$$

where F^h_{ijk} is a tensor of type $(1,3)$. Then the Riemannian space is said to be concircularly (resp., projectively, conformally) semi-symmetric if $F^h_{ijk} = \widetilde{C}^h_{ijk}$ (resp., P^h_{ijk}, C^h_{ijk}). Let us consider a quasi-conformally flat quasi-Einstein space which is concircularly semi-symmetric. Then from (1.4) we get

$$\widetilde{C}^h_{ijk,\ lm} = R^h_{ijk,\ lm} - \frac{R_{,\ lm}}{n(n-1)}[\delta^h_k g_{ij} - \delta^h_j g_{ik}]. \tag{3.2}$$

Using (3.1) (for $F^h_{ijk} = \widetilde{C}^h_{ijk}$) and (3.2) we have

$$R^h_{ijk,\ lm} - R^h_{ijk,\ ml} = 0. \tag{3.3}$$

This shows that the space is semi-symmetric. Then by Theorem 2.4, we obtain (2.11) provided that $a \neq 0$. Conversely, suppose that (2.11) holds. Then by Theorem 2.4, the space is semi-symmetric. Using (3.2) we get

$$\widetilde{C}^h_{ijk,\ lm} - \widetilde{C}^h_{ijk,\ ml} = 0$$

i.e., the space is concircularly semi-symmetric. Thus we can state the following:

Theorem 3.1. *A quasi-conformally flat $(QE)_n$ with $a \neq 0$ is concircularly semi-symmetric if and only if the generator A_i of the space satisfies the relation (2.11).*

Proceeding similarly as the proof of Theorem 2.7, we can state the following :

Theorem 3.2. *A quasi-conformally flat $(QE)_n$ with $a \neq 0$ is concircularly semi-symmetric if and only if its associated scalars satisfies the condition (2.17).*

The projective curvature tensor P^h_{ijk} of type $(1,3)$ of a Riemannian space (M^n, g) is given by

$$P^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}[R_{ij}\delta^h_k - R_{ik}\delta^h_j]. \tag{3.4}$$

We suppose that a quasi-conformally flat $(QE)_n$ is projectively semi-symmetric. Then from (3.4) we get

$$P^h_{ijk,\ lm} = R^h_{ijk,\ lm} - \frac{1}{n-1}[\delta^h_k R_{ij,\ lm} - \delta^h_j R_{ik,\ lm}]. \tag{3.5}$$

Using (2.7), (2.12) and (3.5) in (3.1) (for$F^h_{ijk} = P^h_{ijk}$) we have

$$
\begin{aligned}
(3.6)\qquad & Q \quad [\delta^h_k\{(A_{i,\,lm} - A_{i,\,ml})A_j + A_i(A_{j,\,lm} - A_{j,\,ml})\} \\
& - \quad \delta^h_j\{(A_{i,\,lm} - A_{i,\,ml})A_k + A_i(A_{k,\,lm} - A_{k,\,ml})\} \\
& + \quad g_{ij}\{A_k(A^h_{,\,lm} - A^h_{,\,ml}) + A^h(A_{k,\,lm} - A_{k,\,ml})\} \\
& - \quad g_{ik}\{A_j(A^h_{,\,lm} - A^h_{,\,ml}) + A^h(A_{j,\,lm} - A_{j,\,ml})\}] \\
& - \quad \frac{\beta}{n-1}[(A_j\delta^h_k - A_k\delta^h_j)(A_{i,\,lm} - A_{i,\,ml}) \\
& + \quad A_i\delta^h_k(A_{j,\,lm} - A_{j,\,ml}) - \delta^h_j A_i(A_{k,\,lm} - A_{k,\,ml})] = 0.
\end{aligned}
$$

Contracting h and k we obtain

$$
(3.7)\qquad [(n-2)Q - \beta][(A_{i,\,lm} - A_{i,\,ml})A_j + A_i(A_{j,\,lm} - A_{j,\,ml})] = 0,
$$

which yields

$$
\begin{aligned}
(3.8)\qquad & (A_{i,\,lm} - A_{i,\,ml})A_j + A_i(A_{j,\,lm} - A_{j,\,ml}) = 0, \\
& \text{provided that } [a + (n-2)b] \neq 0 \ \text{ and } \ a \neq 0.
\end{aligned}
$$

Continuing in a similar way as in section 2, we obtain the relation (2.11). Hence we find that if a quasi-conformally flat $(QE)_n$ is projectively semi-symmetric, then the generator A_i of the space satisfies the condition (2.11) provided that $[a+(n-2)b] \neq 0$ and $a \neq 0$.

Conversely, we assume that the generator of a quasi-conformally flat $(QE)_n$ satisfies the condition (2.11). By virtue of (2.12) we obtain (2.5) and (2.6), and hence from (3.5) it follows that

$$
P^h_{ijk,\,lm} - P^h_{ijk,\,ml} = 0,
$$

i.e., the space is projectively semi-symmetric. Hence we can state the following:

Theorem 3.3. *A quasi-conformally flat $(QE)_n$ with $[a+(n-2)b] \neq 0$ and $a \neq 0$ is projectively semi-symmetric if and only if the generator A_i of the space satisfies the relation (2.11).*

Also we can state the following:

Theorem 3.4. *A quasi-conformally flat $(QE)_n$ with $[a+(n-2)b] \neq 0$ and $a \neq 0$ is projectively semi-symmetric if and only if the associated scalars satisfies the condition (2.17).*

Let us consider a quasi-conformally flat $(QE)_n$ which is conformally semi-symmetric. Then from (1.3) we get

$$
\begin{aligned}
(3.9)\qquad & C^h_{ijk,\ lm} = R^h_{ijk,\ lm} \\
& - \frac{1}{n-2}[\delta^h_k R_{ij,\ lm} - \delta^h_j R_{ik,\ lm} + R^h_{k,\ lm} g_{ij} \\
& - R^h_{j,\ lm} g_{ik}] + \frac{R_{,\ lm}}{(n-1)(n-2)}[\delta^h_k g_{ij} - \delta^h_j g_{ik}].
\end{aligned}
$$

Using (2.7), (2.12) and (3.9) in the relation $C^h_{ijk,\ lm} - C^h_{ijk,\ ml} = 0$, and then contracting h and k we have

$$[(n-2)Q - \beta][(A_{i,\,lm} - A_{i,\,ml})A_j + A_i(A_{j,\,lm} - A_{j,\,ml})] = 0, \tag{3.10}$$

which yields

$$(A_{i,\,lm} - A_{i,\,ml})A_j + A_i(A_{j,\,lm} - A_{j,\,ml}) = 0, \tag{3.11}$$

provided that $[a + (n-2)b] \neq 0$ and $a \neq 0$.

Continuing in a similar way as in the previous case we obtain the relation (2.11). Hence we find that if a quasi-conformally flat $(QE)_n$, is conformally semi-symmetric then the generator A_i of the space satisfies the condition (2.11) provided that $[a + (n-2)b] \neq 0$ and $a \neq 0$.

Conversely, we assume that the generator of a quasi-conformally flat $(QE)_n$ satisfies the condition (2.11). By virtue of (2.12) we obtain (2.5) and (2.6). Using (2.5) and (2.6) in (3.9) we get

$$C^h_{ijk,\,lm} - C^h_{ijk,\,ml} = 0.$$

Hence we can state the following:

Theorem 3.5. *A quasi-conformally flat $(QE)_n$ with $[a + (n-2)b] \neq 0$ and $a \neq 0$ is conformally semi-symmetric if and only if the generator A_i of the space satisfies the relation (2.11).*

Also we can state the following:

Theorem 3.6. *A quasi-conformally flat $(QE)_n$ with $[a+(n-2)b] \neq 0$ and $a \neq 0$ is conformally semi-symmetric if and only if the associated scalars satisfies the condition (2.17).*

4 Totally umbilical hypersurface of a quasi-conformally flat $(QE)_n$

Let M^n be a quasi-conformally flat $(QE)_n$ of dimension n and M^{n-1} be a Riemannian space of dimension $(n-1)$ immersed in M^n by a differentiable immersion $i : M^{n-1} \to M^n$. We identify $i(M^{n-1})$ with M^{n-1} and call it as a hypersurface ([25], p.8) of M^n. The Gauss equation ([25], p.149) relates the curvature tensor of type $(0,4)$ as

$$K_{hijk} = R_{\mu\nu\lambda\eta} B^\mu_h B^\nu_i B^\lambda_j B^\eta_k + H_{ij}H_{hk} - H_{ik}H_{jh}, \tag{4.1}$$

where H_{ij} is the second fundamental tensor and

$$B^\mu_h = \frac{\partial x^\mu}{\partial x^h}. \tag{4.2}$$

If on the hypersurface M^{n-1} there exist two functions L and N and a unit vector v_i such that

$$H_{ij} = Lg_{ij} + Nv_iv_j, \tag{4.3}$$

then M^{n-1} is said to be quasi-umbilical [8]. In particular, if $N = 0$, then M^{n-1} is said to be totally umbilical. Again if $L = N = 0$, then M^{n-1} is said to be totally geodesic.

We suppose that M^n is a quasi-conformally flat $(QE)_n$ and M^{n-1} is a totally umbilical hypersurface of M^n. Since M^n is a quasi-conformally flat $(QE)_n$, so it is the space of quasi-constant curvature. Then by virtue of (2.4), (4.1), (4.2), (4.3) we have

$$K_{hijk} = (P+L^2)(g_{hk}g_{ij}-g_{hj}g_{ik})+Q(g_{hk}A_iA_j-g_{jh}A_iA_k+g_{ij}A_hA_k-g_{ik}A_hA_j) \tag{4.4}$$

In particular, if the generator vector A_i of M^n is orthogonal to M^{n-1}, then from (4.4) we get

$$K_{hijk} = (P + L^2)(g_{hk}g_{ij} - g_{hj}g_{ik}) \tag{4.5}$$

Thus we have the following:

Theorem 4.1. *If the generator of a quasi-conformally flat $(QE)_n$ with $a \neq 0$ is orthogonal to totally umbilical hypersurface, then the space is of constant curvature.*

Corollary 4.2. *[14] If the generator of a conformally flat $(QE)_n$ is orthogonal to totally umbilical hypersurface, then the space is of constant curvature.*

Now we suppose that a quasi-conformally flat $(QE)_n$ with $a \neq 0$ is semi-symmetric. Then from Theorem 2.2, we obtain (2.17).

Similarly, a totally umbilical hypersurface of the quasi-conformally flat $(QE)_n$ is semi-symmetric if and only if

$$\alpha + T\beta + L^2 = 0 \text{ and } a \neq 0, \tag{4.6}$$

where T is given in (2.17). Now from (2.17) and (4.6) we get $L = 0$, and hence the hypersurface M^{n-1} is totally geodesic. Thus we can state the following:

Theorem 4.3. *Let a quasi-conformally flat $(QE)_n$ with $a \neq 0$ be semi-symmetric. Then a totally umbilical hypersurface of the space is semi-symmetric if and only if it is totally geodesic.*

5 A proper example of a $(QE)_n$

Example 1. We define a Riemannian metric g on the 4-dimensional real number space $\mathbb{R}^4$ by the formula

$$ds^2 = g_{ij}dx^idx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2, \tag{5.1}$$

$(i, j = 1, 2, ..., 4)$, where $0 < x^4 < \infty$; $x^1, ..., x^4$ are the standard coordinates of $\mathbb{R}^4$. Then the only non-vanishing components of the Christoffel symbols [13] and the curvature tensor are

$$\Gamma^4_{11} = -\frac{2}{3}(x^4)^{\frac{1}{3}},\ \Gamma^1_{14} = \frac{2}{3}\frac{1}{(x^4)},\ \Gamma^4_{33} = -\frac{2}{3}(x^4)^{\frac{1}{3}}$$

$$\Gamma^4_{22} = -\frac{2}{3}(x^4)^{\frac{1}{3}},\ \Gamma^3_{34} = \frac{2}{3}\frac{1}{(x^4)},\ \Gamma^2_{42} = \frac{2}{3}\frac{1}{(x^4)}$$

$$R_{1441} = R_{2442} = R_{4334} = -\frac{2}{9}(x^4)^{-\frac{2}{3}},\ R_{2112} = R_{3113} = R_{2332} = \frac{4}{9}(x^4)^{\frac{2}{3}}$$

and the components which can be obtained from these by the symmetry properties. Using the above relations, we can find the non-vanishing components of the Ricci tensor, the scalar curvature and the quasi-conformal curvature tensor as follows:

$$R_{11} = R_{22} = R_{33} = \frac{2}{3}(x^4)^{-\frac{2}{3}}, \;\; R_{44} = -\frac{2}{3}(x^4)^{-2}, \;\; R = \frac{4}{3}(x^4)^{-2}.$$

$$W_{1441} = W_{2442} = W_{4334} = -\frac{1}{3}(x^4)^{-\frac{2}{3}}[a+2b],$$

$$W_{2112} = W_{3113} = W_{2332} = \frac{1}{3}(x^4)^{\frac{2}{3}}[a+2b].$$

We see that the scalar curvature of the space is non-zero. Therefore $\mathbb{R}^4$ with the considered metric is a Riemannian space (M^4, g) of non-vanishing scalar curvature. We shall now show that this M^4 is a $(QE)_4$.
Let us now consider the associated scalars, and the components of the covariant vector A_i as follows:

$$\alpha = \frac{2}{3}(x^4)^{-2}, \; \beta = -\frac{4}{3}(x^4)^{-2}, \tag{5.2}$$

$$\begin{aligned} A_i(x) &= 1 \quad \text{for} \quad i = 4, \\ &= 0 \quad \text{otherwise}, \end{aligned} \tag{5.3}$$

at any point $x \in M$. In our M^4, (1.1) reduces with these associated scalars and the covariant vector to the following equations:
(i) $R_{11} = \alpha g_{11} + \beta A_1 A_1$
(ii) $R_{22} = \alpha g_{22} + \beta A_2 A_2$,
(iii) $R_{33} = \alpha g_{33} + \beta A_3 A_3$,
(iv) $R_{44} = \alpha g_{44} + \beta A_4 A_4$
since for the cases other than (i)-(iv) the components of each term of (1.1) vanishes identically and the relation holds trivially. Now from (5.2) and (5.3) we get the following relations for the right hand side (R.H.S.) and left hand side (L.H.S.) of (i):
R.H.S. of (i) $= \alpha g_{11} + \beta A_1 A_1 = \frac{2}{3}(x^4)^{-\frac{2}{3}} = R_{11}$=L.H.S. of (i).
Again R.H.S. of (ii) $= \alpha g_{22} + \beta A_2 A_2 = \frac{2}{3}(x^4)^{-\frac{2}{3}} = R_{22}$=L.H.S. of (ii).
By a similar argument as in (i) and (ii) it can be shown that the relations (iii)-(iv) are true. Therefore, (M^4, g) is a $(QE)_4$ which is neither quasi-conformally flat nor quasi-conformally symmetric. Hence we can state the following:

Theorem 5.1. *Let (M^4, g) be a Riemannian space endowed with the metric given by (5.1). Then (M^4, g) is a $(QE)_4$ of non vanishing scalar curvature which is neither quasi-conformally flat nor quasi-conformally symmetric.*

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