Conharmonic curvature tensor and the Spacetime of General Relativity

Shah Alam Siddiqui and Zafar Ahsan

Abstract. The significance of conharmonic curvature tensor is very well known in the differential geometry of certain F-structures (e.g., complex, almost complex, Kähler, almost Kähler, Hermitian, almost Hermitian structures, etc.). In this paper, a study of conharmonic curvature tensor has been made on the four dimensional spacetime of general relativity. The spacetime satisfying Einstein field equations and having vanishing conharmonic tensor is considered and the existence of Killing and conformal Killing vectors on such spacetime have been established. Perfect fluid cosmological models have also been studied.

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1 Introduction

If V and $\tilde{V}$ are two Riemannian spaces with g_{ij} and $\tilde{g}_{ij}$ being their metric tensors related through equation

$$\tilde{g}_{ij} = e^{2\phi} g_{ij} \tag{1.1}$$

where ϕ is a real function of co-ordinates, then V and $\tilde{V}$ are called conformal spaces and the correspondence between V and $\tilde{V}$ is known as conformal transformation ([2]).

It is known that a harmonic function is defined as a function whose Laplacian vanishes. In general, a harmonic function does not transform into a harmonic function. The condition under which the harmonic functions remain invariant have been studied by Ishii [4] who introduced the conharmonic transformation as a subgroup of the conformal transformations (1.1) satisfying the condition

$$\phi^{i}_{;i} + \phi_{;i}\ \phi^{i}_{;} = 0 \tag{1.2}$$

A rank four tensor L^t_{ijk} that remains invariant under conharmonic transformation for a n-dimensional Riemannian differentiable manifold V_n, is given by

$$L^t_{ijk} = R^t_{ijk} - \frac{1}{n-2}\left(g_{ij}\ R^t_k - g_{ik}\ R^t_j + \delta^t_k\ R_{ij} - \delta^t_j\ R_{ik}\right) \tag{1.3}$$

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where R^t_{ijk} is the Riemann curvature tensor and R_{ij}, the Ricci tensor. In the index-free notation, equation (1.3) can be expressed as

$$L(X,Y,Z) = R(X,Y,Z) - \tfrac{1}{n-2}\,[g(Y,Z)\mathrm{R}(X) - g(X,Z)\mathrm{R}(Y) + X\mathrm{Ric}(Y,Z) - Y\mathrm{Ric}(X,Z)] \tag{1.4}$$

where $R(X,Y,Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$ (D being the Riemannian connection) is the Riemann curvature tensor and $\mathrm{Ric}(X,Y) = g(\mathrm{R}(X),Y)$ is the Ricci tensor. Also

$$L(X,Y,Z,W) = R(X,Y,Z,W) - \tfrac{1}{n-2}\,[g(Y,Z)\mathrm{Ric}(X,W) - g(X,Z)\mathrm{Ric}(Y,W) + g(X,W)\mathrm{Ric}(Y,Z) - g(Y,W)\mathrm{Ric}(X,Z)] \tag{1.5}$$

or, in local co-ordinates,

$$L_{hijk} = R_{hijk} - \frac{1}{n-2}\,(g_{ij}\, R_{hk} - g_{hj}\, R_{ik} + g_{hk}\, R_{ij} - g_{ik}\, R_{hj}) \tag{1.6}$$

where $R(X,Y,Z,W) = g(R(X,Y,Z),W)$ or, in local co-ordinates, $R_{hijk} = g_{ht}\, R^t_{ijk}$. It may be noted that the contraction of equation (1.3)

$$L_{ij} = -\frac{1}{n-2}\, R\, g_{ij} \tag{1.7}$$

is also invariant under condition (1.2) [4].

The curvature tensor defined by equations (1.3)/(1.4) or (1.5)/(1.6) is known as conharmonic curvature tensor. A manifold whose conharmonic curvature tensor vanishes at every point is called conharmonically-flat manifold. Thus, this tensor represents the deviation of the manifold from conharmonic flatness.

In the differential geometry of certain F-structures (for example, complex, almost complex, Kähler, almost Kähler, structures, etc.) the importance of conharmonic curvature tensor is very well known (cf.,[5],[10]). The relativistic significance of this tensor has been explored in this paper and it is seen that a conharmonically-flat spacetime is an Einstein space and consequently a space of constant curvature. The existence of Killing and conformal Killing vector fields for the spacetime of general relativity, satisfying Einstein field equations with a cosmological term and having vanishing conharmonic curvature tensor have been established. Perfect fluid cosmological models with vanishing conharmonic curvature tensor, along with some special cases (radiative perfect fluid and dust models), have also been studied.

2 Spacetime with vanishing conharmonic curvature tensor

Let V_4 be the spacetime of general relativity, then from equation (1.4) , we have

$$L(X,Y,Z) = R(X,Y,Z) - \tfrac{1}{2}\,[g(Y,Z)\mathrm{R}(X) - g(X,Z)\mathrm{R}(Y) + X\mathrm{Ric}(Y,Z) - Y\mathrm{Ric}(X,Z)] \tag{2.1}$$

If $L(X,Y,Z) = 0$, then equation (2.1) leads to

$$R(X,Y,Z) = \frac{1}{2}\,[g(Y,Z)\mathrm{R}(X) - g(X,Z)\mathrm{R}(Y) + X\mathrm{Ric}(Y,Z) - Y\mathrm{Ric}(X,Z)] \tag{2.2}$$

which, in local co-ordinates, can be expressed as

$$R^t_{ijk} = \frac{1}{2}\,(g_{ij}\,R^t_k - \delta^t_j\,R_{ik} + \delta^t_k\,R_{ij} - g_{ik}\,R^t_j) \tag{2.3}$$

The contraction of equation (2.3) yields

$$R_{ik} = -\frac{R}{4}\,g_{ik} \tag{2.4}$$

or, in index-free notation

$$\mathrm{Ric}(X,Z) = -\frac{R}{4}\,g(X,Z) \tag{2.5}$$

where R is the scalar curvature.

Thus, we have

Theorem 2.1. *A conharmonically-flat spacetime is an Einstein space.*

Also, from equation (2.3), we have

$$R_{hijk} = g_{ht}\,R^t_{ijk} = \frac{1}{2}\,(g_{ij}\,R_{hk} - g_{hj}\,R_{ik} + g_{hk}\,R_{ij} - g_{ik}\,R_{hj}) \tag{2.6}$$

which from equation (2.4) reduces to

$$R_{hijk} = \frac{R}{4}\,(g_{hj}\,g_{ik} - g_{ij}\,g_{hk}) \tag{2.7}$$

or,

$$R(X,Y,Z,W) = \frac{R}{4}\,[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] \tag{2.8}$$

We thus have

Theorem 2.2. *A conharmonically-flat spacetime is of constant curvature.*

Remarks. (i). The significance of the spaces of constant curvature is very well known in cosmology. The simplest cosmological model of the universe is obtained by making the assumption that the universe is isotropic and homogeneous. This is known as cosmological principle. By isotropy we mean that all spatial directions are equivalent, while homogeneity means that it is impossible to distinguish one place in the universe from the other. That is, in the rest system of matter there is no preferred point and no preferred direction, the three dimensional space being constituted in the same way everywhere. This cosmological principle, when translated into the language of Riemannian geometry, asserts that the three dimensional position space is a space of maximal symmetry [a space is of maximal symmetry if it has maximum number of Killing vector fields; and the maximum number of Killing vector fields in a Riemannian space of dimension n is $\frac{1}{2}n(n+1)$], that is, a space of constant curvature whose curvature depends upon time.

The cosmological solutions of Einstein equations which contain a three dimensional space-like surface of a constant curvature are the Robertson-Walker metrics, while a four dimensional space of constant curvature is the de Sitter model of the universe. de Sitter model possess a three dimensional space of constant curvature and thus belongs to Robertson-Walker metrics (for further details, see [6]).

These discussions clearly point towards the relativistic significance of the conharmonic curvature tensor and conharmonal flatness. Moreover, from Theorem 2.2, the conharmonic curvature tensor represents the deviation of the spacetime manifold V_4 from the constant curvature. The deviation from the conharmonal flatness is measured by

$$L = sup_A \, \frac{|L_{hijk}A^{hi}A^{jk}|}{A_{ij}A^{ij}}$$

where $A^{hi} = Y^h X^i - Y^i X^h$, X, Y being two mutually orthogonal unit vectors. Also

$$A_{ij} = -A_{ji}$$

(ii) Consider the covariant derivative of equation (2.4), we get

$$R_{ik;j} = -\frac{1}{4}\ (R\ g_{ik})_{;j} = -\frac{1}{4}\ g_{ik}\ R_{;j}$$

Multiplying this equation by g^{kh}, we have

$$R^h_{i;j} = -\frac{1}{4}\ R_{;j}\ \delta^h_i$$

Contraction with respect to indices h and i leads to

$$R_{;j} = 0 \tag{2.9}$$

which shows that the scalar curvature of the spacetime is constant.

In the general theory of relativity, the curvature tensor describing the gravitational field consists of two parts, viz., the matter part and the free gravitational part. The interaction between these two parts is described through the Bianchi identities. For a given distribution of matter, the construction of gravitation potential satisfying Einstein field equations is the principal aim of all investigations in gravitational physics; and this has often be achieved by imposing symmetries on the geometry compatible with the dynamics of the chosen distribution of matter. The geometrical symmetries of the spacetime are expressible through the equation of the form

$$\mathcal{L}_\xi A - 2\ \Omega\ A = 0 \tag{2.10}$$

where A represents a geometrical/physical quantity, $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the vector field ξ^a (this vector field may be time-like, space-like or null), and Ω is a scalar.

One of the most simple and widely used example is the metric inheritance symmetry for which $A = g_{ij}$ in equation (2.10); and for this case ξ^a is Killing vector (KV) if Ω is zero ([7, 8]) (for a comprehensive review of symmetry inheritance and related results, see [1]). In what follows, we shall explore the role of such symmetry inheritance for a spacetime V_4 with vanishing conharmonic curvature tensor.

From equation (1.6), it is evident that, for a V_4, the first part of the conharmonic curvature tensor represents the curvature of the spacetime and the second part the distribution and motion of the matter. The conharmonic curvature tensor has the following properties [from equation (1.5)]:

(i) $L(X, Y, Z, W) = -L(Y, X, Z, W) = -L(X, Y, W, Z) = L(Z, W, X, Y)$
(ii) $L(X, Y, Z, W) + L(X, Z, W, Y) + L(X, W, Y, Z) = 0$
(iii) If $\mathrm{Ric}(X, Y) = 0$, then $L(X, Y, Z, W) = R(X, Y, Z, W)$
pas Let $\mathrm{Ric}(X, Y) \neq 0$ and the distribution and motion of the matter be related through the Einstein field equations with a cosmological term

$$R_{ij} - \frac{1}{2} R\, g_{ij} + \Lambda\, g_{ij} = k\, T_{ij} \tag{2.11}$$

where k is a constant, Λ is the cosmological constant and T_{ij} is the energy-momentum tensor.

From equation (2.4), equation (2.11) reduces to

$$(\Lambda - \frac{3}{4} R)\, g_{ij} = k\, T_{ij} \tag{2.12}$$

Since R is constant by virtue of equation (2.9), taking the Lie derivative of both sides of equation (2.12), we get

$$(\Lambda - \frac{3}{4} R)\, \mathcal{L}_\xi g_{ij} = k\, \mathcal{L}_\xi T_{ij} \tag{2.13}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the vector ξ. If ξ is a Killing vector, then

$$\mathcal{L}_\xi g_{ij} = 0 \tag{2.14}$$

and thus equation (2.13) leads to

$$\mathcal{L}_\xi T_{ij} = 0 \tag{2.15}$$

Conversely, if equation (2.15) holds and since k, Λ and R are constants then from equation (2.13) $\mathcal{L}_\xi g_{ij} = 0$. Hence ξ is a Killing vector field. We can thus state the following:

Theorem 2.3. *For a conharmonically-flat spacetime admitting the Einstein field equations with a cosmological term, there exists a Killing vector field ξ if and only if the Lie derivative of the energy-momentum tensor with respect to ξ vanishes.*

If a point transformation in a Riemannian manifold does not change the angle between two arbitrary vectors of the manifold, the transformation is known to be a conformal motion or a conformal transformation (cf., equation (1.1)). For a 1-parameter group of transformations generated by the vector field $\xi = \xi^i\, \partial_i$ to admit a group of conformal transformation, it is necessary and sufficient that ξ satisfy the equation (cf., [9])

$$\mathcal{L}_\xi g_{ij} = 2\ \Omega\ g_{ij} \tag{2.16}$$

where Ω is a scalar function. A vector field ξ satisfying equation (2.16) is called a conformal Killing vector field (CKV). Since conharmonic curvature tensor is invariant under special type of conformal transformations, it is worthwhile to investigate the role of CKV for a conharmonically-flat spacetime.

Using equation (2.16) in equation (2.13), we have

$$2(\Lambda - \frac{3}{4}\ R)\ \Omega\ g_{ij} = k\ \mathcal{L}_\xi\ T_{ij} \tag{2.17}$$

which from equation (2.12) reduces to

$$\mathcal{L}_\xi T_{ij} = 2\ \Omega\ T_{ij} \tag{2.18}$$

From equation (2.18) we say that the energy-momentum tensor has the symmetry inheritance property. Conversely, if equation (2.18) holds then it follows that equation (2.16) holds. Thus, we have the following:

Theorem 2.4. *A conharmonically-flat spacetime satisfying the Einstein field equations with a cosmological term admits a conformal Killing vector field if and only if the energy-momentum tensor has the symmetry inheritance property.*

3 Cosmological models with vanishing conharmonic curvature tensor

Soon after obtaining the field equations of general relativity, Einstein applied these equations to find a model of universe. The universe on a large scale shows isotropy and homogeneity and the matter contents of the universe (stars, galaxies, nebulas, etc.) can be assumed to be that of a perfect fluid.

We shall now consider perfect fluid spacetimes with vanishing conharmonic curvature tensor. The energy-momentum tensor T_{ij} of a perfect fluid is given by

$$T_{ij} = (\mu + p)\ u_i\ u_j + p\ g_{ij} \tag{3.1}$$

where μ is the energy density, p the isotropic pressure and u_i the fluid flow velocity vector such that $u_i\ u^i = -1$. We have seen in the previous section that the Einstein field equations (2.11) with a vanishing conharmonic curvature tensor leads to equation (2.12). That is

$$(\Lambda - \frac{3}{4}\ R)\ g_{ij} = k\ T_{ij}$$

From equation (3.1) this equation leads to

$$(\Lambda - \frac{3}{4}\ R - k\ p)\ g_{ij} = k\ (\mu + p)\ u_i\ u_j \tag{3.2}$$

Multiplying this equation by g^{li}, we get

$$(\Lambda - \frac{3}{4}\ R - k\ p)\ \delta_j^l = k\ (\mu + p)\ u^l\ u_j$$

which on contracting with l and j yields

$$3\ R = 4\ \Lambda + k\ (\mu - 3\ p) \tag{3.3}$$

Moreover, multiplying equation (3.2) by u^i and using $u_i\ u^i = -1$, we get

$$3\ R = 4\ \Lambda + 4\ k\ \mu \tag{3.4}$$

Combining equation (3.3) and (3.4), we get

$$\mu + p = 0 \tag{3.5}$$

which is not possible due to our assumption. Thus we can state the following:

Theorem 3.1. *A spacetime with vanishing conharmonic curvature tensor and satisfying Einstein field equation with a cosmological term is a perfect fluid spacetime if* $\mu + p \neq 0$.

Special cases

(i) It is known that [3] for a radiative perfect fluid spacetime (when $\mu = 3p$) the resulting universe must be isotropic and homogeneous. Thus, if we take $\mu = 3p$ and proceed as above, then Theorem 3.1 leads to the following:

Corollary 3.1. *A spacetime with vanishing conharmonic curvature tensor and satisfying Einstein field equations with a cosmological term is an isotropic and homogeneous spacetime if the energy density of the fluid does not vanish.*

(ii) Cosmological models with dust are known to be as Einstein-de Sitter, the closed elliptic and open hyperbolic model (for details, see [6]). The energy-momentum tensor for a dust model is

$$T_{ij} = \mu\ u_i\ u_j \tag{3.6}$$

Now, replacing equation (3.1) with equation (3.6) and proceeding in the same manner as we have done in obtaining Theorem 3.1, we can state the following:

Corollary 3.2. *A spacetime satisfying Einstein field equations and having vanishing conharmonic curvature tensor represent a dust cosmological model if the energy-density does not vanish.*

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Authors' address:

Shah Alam Siddiqui and Zafar Ahsan
Department of Mathematics, Aligarh Muslim University,
Aligarh-202002, India.
Emails: shah_alam78@rediffmail.com, zafar.ahsan@rediffmail.com