Projective motion in bi-recurrent Finsler space

Surendra Pratap Singh

Abstract. The existence of projective motion in a symmetric Finsler space was developed by F.M.Meher ([1]). The present author has studied affine and projective motions with contra field in the same space ([4]). Recently the author has discussed projective motions in Finsler spaces ([5]). The purpose of this paper is to study projective motion in bi-recurrent Finsler space. The notations of H.Rund [3] are used in the sequel.

M.S.C. 2000: 53B40.
Key words: Bi-recurrent Finsler space; recurrent tensor field; contra field; affine motion; projective motion.

1 Introduction

Let us consider an $n-$dimensional Finsler space F_n with the connection parameters $G^i_{jk}(x,\dot{x})$ which are positively homogeneous functions of degree zero in the directional arguments, where we briefly denote the line-element $(x,\dot{x}^i$ by $(x,\dot{x})$. By Euler's law, we have

$$\dot{x}^h G^i_{jkh} = \dot{x}^h \dot{\partial}_h G^i_{jk} = 0. \tag{1.1}$$

The covariant derivative, in the sense of Berwald [3], of a tensor field X^i, is given by

$$X^i_{(h)} = \partial_h X^i - \dot{\partial}_k X^i \dot{\partial}_h G^k + X^k G^i_{kh}, \tag{1.2}$$

where $G^i(x,\dot{x})$ is positively homogeneous function of degree two in $\dot{x}^i$ and $\partial_h = \frac{\partial}{\partial x^h}, \dot{\partial}_k = \frac{\partial}{\partial \dot{x}^k}$. The corresponding curvature tensor field H^i_{jkh} of F_n is expressed in the form

$$H^i_{hjk} = \partial_k G^i_{hj} - \partial_j G^i_{hk} + G^r_{hj} G^i_{rk} - G^r_{hk} G^i_{jr} + G^i_{rhk} \dot{\partial}_j G^r - G^i_{rhj} \dot{\partial}_k G^r. \tag{1.3}$$

The curvature tensor H^i_{jkh} is skew-symmetric in the last two lower indices k, h and is homogeneous function of degree zero in $\dot{x}^i$, hence we have

$$H^i_{jkh} = -H^i_{jhk}, \tag{1.4}$$

$$\dot{\partial}_l H^i_{jkh} \dot{x}^l = 0 \tag{1.5}$$

DIFFERENTIAL GEOMETRY - DYNAMICAL SYSTEMS, Vol.12, 2010, pp. 221-227.

and

$$H^i_{hjk}\dot{x}^h = H^i_{jk}. \tag{1.6}$$

The commutation formulae involving the above curvature tensors are given as

$$2T_{[(h)(k)]} = -\dot{\partial}_i T H^i_{hk} \tag{1.7}$$

and

$$2T^i_{j[(h)(k)]} = -\dot{\partial}_r T^i_j H^r_{hk} - T^i_r H^r_{jhk} + T^r_j H^i_{rhk}, \tag{1.8}$$

where $[hk]$ represents skew-symmetric part. In a non-flat Finsler space F_n if there exists a non-zero vector field K_l whose components are positively homogeneous functions of degree zero in $\dot{x}^i$ such that the curvature tensor H^i_{jkh} satisfies the relation

$$H^i_{jkh(l)} = K_l H^i_{jkh}, \tag{1.9}$$

then it is called a recurrent Finsler space ([2], [6]) Transvection of (1.9) yields

$$H^i_{kh(l)} = K_l H^i_{kh}. \tag{1.10}$$

Also in a non-flat Finsler space F_n if there exists a non-zero tensor field A_{lm} such that the curvature tensor H^i_{jkh} satisfied the relation

$$H^i_{jkh(l)(m)} = A_{lm} H^i_{jkh}, \tag{1.11}$$

where

$$A_{lm} = K_{l(m)} + K_l K_m, \tag{1.12}$$

then it is called a bi-recurrent Finsler space ([7]). The tensor A_{lm} defined by (1.12) is called recurrence tensor field whereas the non-zero vector K_l is called recurrence vector field. We denote such a bi-recurrent Finsler space by $\overline{F}_n$.

The successive transvections of (1.11) by $\dot{x}$ yield

$$H^i_{kh(l)(m)} = A_{lm} H^i_{kh}, \tag{1.13}$$

$$H^i_{h(l)(m)} = A_{lm} H^i_h. \tag{1.14}$$

The Lie-derivatives of a tensor T^i_j and the connection coefficients G^i_{jk} defined by the infinitesimal transformation

$$\overline{x}^i = x^i + \epsilon v^i(x^j) \tag{1.15}$$

are characterised by [8]

$$L_v T^i_j = v^h T^i_{j(h)} - T^h_j v^i_{(h)} + T^i_h v^h_{(j)} + (\dot{\partial}_h T^i_j) v^h_{(s)} \dot{x}^s \tag{1.16}$$

and

$$L_v G^i_{jk} = v^i_{(j)(k)} + H^i_{jkh} v^h + G^i_{jkh} v^h_{(s)} \dot{x}^s, \tag{1.17}$$

respectively. The commutation formulae with respect to Lie-derivative and other derivatives for any tensor T^i_{jk} are given by

$$(L_v T^i_{jk(l)}) - (L_v T^i_{jk})_{(l)} = L_v G^i_{rl} T^r_{jk} - L_v G^r_{jl} T^i_{rk} - L_v G^r_{kl} T^i_{jr} - (L_v G^r_{lp}) \dot{x}^p \dot{\partial}_r T^i_{jk} \tag{1.18}$$

and

$$\dot{\partial}_l (L_v T^i_{jk}) - L_v (\dot{\partial}_l T^i_{jk}) = 0. \tag{1.19}$$

The Lie-derivative of the curvature tensor H^i_{jkh} is given in the form

$$(L_v G^i_{jh})_{(k)} - (L_v G^i_{kh})_{(j)} = L_v H^i_{hjk} + (L_v G^r_{kl}) \dot{x}^l G^i_{rjh} - (L_v G^r_{jl}) \dot{x}^l G^i_{rkh}. \tag{1.20}$$

2 Projective motion in a bi-recurrent Finsler space

The infinitesimal transformation (1.15) defines a projective motion if it transforms a system of geodesics of F_n into geodesics.A necessary and sufficient condition that the infinitesimal transformation (1.15) defines a projective motion is that [1]

$$L_v G^i_{jk} = 2\delta^i_{(j} p_{k)} + \dot{x}^i p_{jk}, \tag{2.1}$$ [1]

where

$$p_k = \dot{\partial}_k p \quad , p_{jk} = \dot{\partial}_j p_k, \tag{2.2}$$

for some homogeneous scalar function $p(x, \dot{x})$ of degree one in $\dot{x}^i$. For the homogeneity of p_k and p_{jk}, they satisfy

$$p_k \dot{x}^k = p, \quad p_{jk} \dot{x}^k = 0. \tag{2.3}$$

Definition 2.1. A bi-recurrent Finsler space F_n, in which the infinitesimal transformation (1.15) defines a projective motion is called projective bi-recurrent Finsler space and briefly denoted by $P\overline{F}_n$.

The present author [5] has established that if the infinitesimal transformation (1.15) defines a projective motion then in view of (2.3) and the homogeneity property of the connection coefficients G^i_{jk}, the Lie-derivative of the curvature tensor H^i_{jkh} satisfied the relation

$$L_v H^i_{hjk} = 2\delta^i_h p_{[j(k)]} + 2\delta^i_{[j} p_{|h|(k)]} + 2\dot{x}^i p_{[j|h|(k)]}, \tag{2.4}$$

where the index with two parallel bars is unaffected when we consider skew-symmetric part.
Transvecting (2.4) by $\dot{x}^h$ and using the fact $\dot{x}^k$ remains invariant under covariant and Lie-differentiations, we find

$$L_v H^i_{jk} = 2\dot{x}^i p_{[j(k)]} + 2\delta^i_{[j} p_{(k)]}, \tag{2.5}$$

[1] (jk) represents symmetric part.

in view of (1.6) and (2.3).
Applying Lie-operator to (1.11) and observing (2.4), we obtain

$$L_v H^i_{jkh(l)(m)} = L_v A_{lm} H^i_{jkh} + A_{lm}[2\delta^i_j p_{[k(h)]} + 2\delta^i_{[k} p_{|j|(h)]} + 2\dot{x}^i p_{[k|j|(h)]}] \tag{2.6}$$

Similarly, the Lie-derivative of (1.13) in view of (2.5) yields

$$L_v H^i_{kh(l)(m)} = L_v A_{lm} H^i_{kh} + A_{lm}[2\dot{x}^i p_{[k(h)]} + 2\delta^i_{[k} p_{(h)]}]. \tag{2.7}$$

Hence we state

Theorem 2.1. *In a bi-recurrent Finsler space $P\overline{F}_n$, which admits projective motion, the relations* (2.6) *and* (2.7) *hold good.*

When the projective motion becomes an affine motion, the condition $L_v G^i_{jk} = 0$ is satisfied. If we apply this condition in (2.1), it yields

$$\delta^i_j p_k + \delta^i_k p_j + \dot{x}^i p_{jk} = 0. \tag{2.8}$$

Setting the indices $i = k$ in the above equation, we get

$$(n+1)p_j + \dot{x}^k p_{jk} = 0. \tag{2.9}$$

Application of (2.3) in the above equation gives

$$(n+1)p_j = 0. \tag{2.10}$$

which implies

$$p_j = 0. \tag{2.11}$$

Conversely, if (2.11) is true, the equation (2.1) reduces to $L_v G^i_{jk} = 0$, in view of (2.3). It means $p_j = 0$ is the necessary and sufficient condition for the infinitesimal transformation (1.15) which defines projective motion to be an affine motion. In that case from the equation (1.6) and (1.20), we get $L_v H^i_{jkh} = 0, L_v H^i_{jk} = 0$ and the equations (2.6) and (2.7) assume the forms

$$L_v H^i_{jkh(l)(m)} = L_v A_{lm} H^i_{jkh} \tag{2.12}$$

and

$$L_v H^i_{kh(l)(m)} = L_v A_{lm} H^i_{kh}, \tag{2.13}$$

respectively. Hence we have

Theorem 2.2. *In a projective bi-recurrent Finsler space $P\overline{F}_n$, if the projective motion becomes an affine motion, the relations* (2.12) *and* (2.13) *are necessarily true.*

Applying the commutation formula (1.19) for H^i_{jkh}, we find

$$\dot{\partial}_l (L_v H^i_{jkh}) = L_v (\dot{\partial}_l H^i_{jkh}). \tag{2.14}$$

If the projective motion (1.15) becomes an affine motion the equation (2.14) reduces to

$$L_v(\dot{\partial}_l H^i_{jkh}) = 0. \tag{2.15}$$

Using the commutation formula (1.8) for the curvature tensor H^i_{jkh}, we obtain

$$\begin{aligned} 2H^i_{jkh[(l)(m)]} = -\dot{\partial}_r H^i_{jkh} H^r_{lm} + H^r_{jkh} H^i_{rlm} - H^i_{rkh} H^r_{jlm} \\ -H^i_{jrh} H^r_{klm} - H^i_{jkr} H^r_{hlm} \end{aligned} \tag{2.16}$$

Applying Lie-operator to both sides of (2.16) and using (2.15), we get

$$L_v H^i_{jkh[(l)(m)]} = 0, \tag{2.17}$$

since the projective motion becomes an affine motion. In a bi-recurrent Finsler space $\overline{F}_n$, in view of (1.11), the equation (2.17) takes the form

$$L_v A_{[lm]} = 0, \tag{2.18}$$

since $\overline{F}_n$ is non-flat space. Accordingly we have

Theorem 2.3. *In a projective bi-recurrent Finsler space $P\overline{F}_n$, if the projective motion becomes an affine motion, the recurrence tensor field A_{lm} satisfies the identity* (2.18).

We notice that in a recurrent Finsler space $\overline{F}_n$, if an affine motion is admitted, then the recurrence vector is Lie-invariant, that is $L_v K_l = 0$. Commuting (1.13) with respect to the indices l, m and noting commutation formula (1.8), we obtain

$$(A_{lm} - A_{ml}) H^i_{kh} = H^i_{kh} H^i_{rlm} - H^r_{lm} H^i_{rkh} - H^i_{rk} H^r_{hlm} - H^i_{hr} H^r_{klm}. \tag{2.19}$$

Covariant differentiation of (2.19) with respect to x^n yields

$$A_{[lm](n)} = K_n A_{[lm]}, \tag{2.20}$$

in view of (1.9), (1.10) and (2.19). If we observe the fact that $L_v K_l = 0$ and Theorem 2.3, in the Lie-derivative of (2.20), we obtain

$$L_v A_{[lm](n)} = 0, \tag{2.21}$$

which gives the identity

$$L_v A_{[lm](n)} + L_v A_{[mn](l)} + L_v A_{[nl](m)} = 0. \tag{2.22}$$

Hence we state

Theorem 2.4. *In a projective bi-recurrent Finsler space $P\overline{F}_n$, if the projective motion becomes an affine motion, the recurrence tensor field A_{lm} satisfies the identity* (2.22).

3 Further discussion

In a projective bi-recurrent Finsler space $P\overline{F}_n$, if the vector field $v^i(x)$ satisfies the relation

$$v^i_{(j)} = 0, \tag{3.1}$$

then the vector field $v^i(x)$ determines a contra field. In this case, we consider a special infinitesimal transformation

$$\bar{x}^i = x^i + \epsilon v^i(x^j), \quad v^i_{(j)} = 0, \tag{3.2}$$

which defines projective motion in $P\overline{F}_n$. Applying equations (2.1) and (3.1) in the equation (1.17), we find

$$H^i_{jkh} v^h = 2\delta^i_{(j} p_{k)} + \dot{x}^i p_{jk}. \tag{3.3}$$

The covariant differentiation of (3.3) twice with respect to x^l and x^m yields

$$H^i_{jkh(l)(m)} v^h = \delta^i_j p_{k(l)(m)} + \delta^i_k p_{j(l)(m)} + \dot{x}^i p_{jk(l)(m)} \tag{3.4}$$

in view of (3.1) and the fact that $\dot{x}^i$ remains invariant under covariant differentiation. From equations (3.3) and (3.4), we conclude

$$\delta^i_j [p_{k(l)(m)} - A_{lm} p_k] + \delta^i_k [p_{j(l)(m)} - A_{lm} p_j] + \dot{x}^i [p_{jk(l)(m)} - A_{lm} p_{jk}] = 0 \tag{3.5}$$

by using (1.11). The above equation is only true if

$$(a) p_{k(l)(m)} = A_{lm} p_k, \quad (b) \ \ p_{jk(l)(m)} = A_{lm} p_{jk}, \tag{3.6}$$

which implies that the scalar function $p(x, \dot{x})$ is bi-recurrent in $P\overline{F}_n$. Hence we have

Theorem 3.1. *In a projective bi-recurrent Finsler space $P\overline{F}_n$, which admits projective motion, if the vector field $v^i(x)$ spans a contra field, then the scalar function $p(x, \dot{x})$ is bi-recurrent.*

References

[1] F.M. Meher, *Projective motion in a symmetric Finsler space*, Tensor N.S. 23 (1972), 275-278.

[2] A. Moor, *Untersuchungen uber Finsler raume von rekurrenter krummung*, Tensor N.S. 13 (1963), 1-18.

[3] H. Rund, *The Differential Geometry of Finsler Spaces*, Springer-Verlag, 1959.

[4] S.P. Singh, *Motion with contra field in Finsler space*, Quest. Mathem. 22 (1999), 137-141.

[5] S.P. Singh, *On projective motion in Finsler spaces*, Prog. of Mathem. 36 (2002), 151-158.

[6] B.B. Sinha and S.P. Singh, *On recurrent Finsler spaces*, Roum. Mat. Pures Appl. 16 (1971), 977-986.

[7] B.B. Sinha and S.P. Singh, *Recurrent Finsler space of second order*, Yokohama Math. Jour. 19 (1971), 79-85.
[8] K. Yano, *The Theory of Lie Derivatives and Its Applications*, North Holland, 1957.

Author's address:

Surendra Pratap Singh
Department of Mathematics, Kenyatta University,
P. O. Box 43844-00100, Nairobi, Kenya.
E-mail: profspskenya551@yahoo.com