On LP-Sasakian manifolds satisfying certain conditions on the concircular curvature tensor

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Abstract. In this paper we show that LP-Sasakian manifolds are Einstein manifolds if they satisfy the conditions $R(X,Y).S = 0$, $\tilde{C}(\xi, X).S = 0$ and $R.\tilde{C} = R.R$.

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1 Introduction

In 1989, K. Matsumoto [7] introduced the notion of Lorentzian Para Sasakian manifold. I. Mihai and R. Rosca [6] defined the same notion independently and thereafter many authors ([6], [14], [8]) studied LP-Sasakian manifolds. Cihan Özgür and U.C.De studied the same conditions on Para Sasakian manifolds. An LP-Sasakian manifold is called Ricci-semi-symmetric if $R(X,Y).S = 0$. In this paper we prove that an LP-Sasakian manifold is Ricci-semi symmetric if and only if it is an Einstein manifold. Also, we show that an LP-Sasakian manifold satisfying $\tilde{C}(\xi, X).S = 0$ is an Einstein manifold and manifold of constant scalar curvature $n(n-1)$, were $\tilde{C}$ is a concircular curvature tensor. Finally, we show that $R.\tilde{C} = R.R$.

2 Preliminaries

An $2n+1$- dimensional differentiable manifold M is called an LP-Sasakian manifold [7], [8] if it admits a $(1,1)$ tensor field φ, a contravariant vector field ξ, a $1-form\ \eta$ and Lorentzian metric g which satisfy

$$(2.1)\qquad \begin{cases} \varphi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \\ g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y), \\ (a)\ \ \nabla_X \xi = \varphi X, \quad (b)\ \ g(X,\xi) = \eta(X), \end{cases}$$

and

$$(\nabla_X \varphi)Y = g(X,Y)\xi + 2\eta(X)\eta(Y)\xi,$$

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where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. It can be easily seen that in an LP-Sasakian manifold, the following relations hold:

$$\varphi\xi = 0, \eta(\varphi X) = 0, \quad \text{rank } \varphi = 2n.$$

Again if we put

$$\Omega(X, Y) = g(X, \varphi Y),$$

for any vector fields X and Y, then the tensor field $\Omega(X, Y)$ is a symmetric $(0, 2)$ tensor field[7]. Also since the vector field η is closed in an LP-Sasakian manifold we have ([7], [11]):

$$(\nabla_X \eta)(Y) = \Omega(X, Y), \Omega(X, \xi) = 0,$$

for any vector fields X and Y. Also, an LP-Sasakian manifold M is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$$

for any vector fields X, Y where a, b are functions on M. Further, on such an LP-Sasakian manifold the following relations hold ([8], [11]):

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y),$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.2}$$

$$R(\xi, X)\xi = X + \eta(X)\xi,$$

$$S(X, \xi) = 2n\eta(X), \tag{2.3}$$

$$S(\varphi X, \varphi Y) = S(X, Y) + 2n\eta(X)\eta(Y),$$

for any vector fields X,Y,Z, where $R(X, Y)Z$ is the curvature tensor, and S is the Ricci tensor.

Definition 1. The concircular curvature tensor $\tilde{C}$ on LP-Sasakian manifold M of dimensional $2n + 1$ is defined by

$$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n + 1)}[g(Y, Z)X - g(X, Z)Y], \tag{2.4}$$

for any vector fields X,Y,Z, where R is the curvature tensor and r is the scalar curvature.

Definition 2. An $2n + 1$-dimensional LP-Sasakian manifold is said to be Ricci-semi-symmetric if

$$R(X, Y).S = 0, \tag{2.5}$$

where R is the curvature tensor and S is the Ricci tensor.

3 Main Results

In this section, we prove the following theorems:

Theorem 3.1. *Let M be an $2n+1$-dimensional LP-Sasakian manifold. Then M is Ricci-semi-symmetric if and only if it is an Einstein manifold.*

Proof. We know that every Einstein manifold is Ricci-semi symmetric but the converse is not true in general. Here, we prove that in an LP-Sasakian manifold $R(X,Y).S=0$ implies that the manifold is an Einstein manifold. It follows from (2.5) that

$$S(R(X,Y)U,V)+S(U,R(X,Y)V)=0. \tag{3.1}$$

Putting $X=\xi$ in (3.1) we get

$$S(R(\xi,Y)U,V)+S(U,R(\xi,Y)V)=0. \tag{3.2}$$

Using (2.2), (2.3), from (3.2) we get

$$\begin{aligned} &S(g(Y,U)\xi-\eta(U)Y,V)+S(U,g(Y,V)\xi-\eta(V)Y)=0,\\ &2n\eta(V)g(Y,U)-\eta(U)S(Y,V)+2n\eta(U)g(Y,V)-\eta(V)S(U,V)=0. \end{aligned} \tag{3.3}$$

Now, putting $U=\xi$ in (3.3), we obtain

$$\begin{aligned} &2n\eta(V)\eta(Y)+S(Y,V)-2ng(Y,V)-2n\eta(V)\eta(Y)=0,\\ &S(Y,V)=2ng(Y,V). \end{aligned}$$

Therefore, M is Einstein manifold. This the completes the proof of the theorem. □

Theorem 3.2. *Let M be an $2n+1$-dimensional LP-Sasakian manifold. Then M satisfies in condition*

$$\tilde{C}(\xi,X).S=0,$$

if and only if either M is Einstein manifold or M has scalar curvature $r=2n(2n+1)$.

Proof. Since $\tilde{C}(\xi,X).S=0$, we have

$$\tilde{C}(\xi,X).S(Y,\xi)=0.$$

This implies that

$$S(\tilde{C}(\xi,X)Y,\xi)+S(Y,\tilde{C}(\xi,X)\xi)=0. \tag{3.4}$$

In view of (2.3), (2.4) in (3.4) we infer

$$S\left(\left(1-\frac{r}{2n(2n+1)}\right)[g(Y,X)\xi-\eta(Y)X],\xi\right)+S\left(Y,\left(1-\frac{r}{2n(2n+1)}\right)[\eta(X)\xi+X]\right)=0,$$

$$(1-\frac{r}{2n(2n+1)})[g(X,Y)S(\xi,\xi)-\eta(Y)S(Y,\xi)+\eta(X)S(Y,\xi)+S(X,Y)]=0. \tag{3.5}$$

Using (2.3) in (3.5) we get

$$\left(1-\frac{r}{2n(2n+1)}\right)[2ng(X,Y)\eta(\xi)-2n\eta(Y)\eta(X)+2n\eta(X)\eta(Y)+S(X,Y)]=0. \tag{3.6}$$

In view of (2.1) in (3.6), it follows that

$$\left(1-\frac{r}{2n(2n+1)}\right)[-2ng(X,Y)+S(X,Y)]=0.$$

This implies $S(X,Y)=2ng(X,Y)$, or $r=2n(2n+1)$. Therefore M is an Einstein manifold with the scalar curvature $r=2n(2n+1)$. The converse is trivial. The proof is complete. □

Theorem 3.3. *Let M be an $2n+1$-dimensional LP-Sasakian manifold. Then $R.\tilde{C}=R.R$.*

Proof. We have

$$\begin{aligned}(R(X,Y).\tilde{C})(U,V,W) &= R(X,Y)\tilde{C}(U,V)W-\tilde{C}(R(X,Y)U,V)W\\ &\quad -\tilde{C}(U,R(X,Y)V)W-\tilde{C}(U,V)R(X,Y)W.\end{aligned} \tag{3.7}$$

In view of (2.4) in (3.7) we have

$$\begin{aligned}&(R(X,Y).\tilde{C})(U,V,W)=R(X,Y)[R(U,V)W-\tfrac{r}{2n(2n+1)}(g(V,W)U-g(U,W)V)]\\ &-R(R(X,Y)U,V)W+\tfrac{r}{2n(2n+1)}[g(V,W)R(X,Y)U-g(R(X,Y)U,W)V]\\ &-R(U,R(X,Y)V)W+\tfrac{r}{2n(2n+1)}[g(R(X,Y)V,W)U-g(U,W)R(X,Y)V]\\ &-R(U,V)R(X,Y)W+\tfrac{r}{2n(2n+1)}[g(V,R(X,Y)W)U-g(U,R(X,Y)W)V].\end{aligned}$$

We have

$$\begin{aligned}&(R(X,Y).\tilde{C})(U,V,W)=R(X,Y)R(U,V)W-R(R(X,Y)U,V)W\\ &-R(U,R(X,Y)V)W-R(U,V)R(X,Y)W\\ &+\tfrac{r}{2n(2n+1)}[g(R(X,Y)V,W)U+g(V,R(X,Y)W)U\\ &-g(R(X,Y)U,W)V-g(U,R(X,Y)W)V].\end{aligned}$$

Finally, we get

$$(R(X,Y).\tilde{C})(U,V.W)=(R(X,Y).R)(U.V.W).$$

Therefore $R.\tilde{C}=R.R$. This completes the proof of the theorem. □

Now, in view of Theorem 2.1 of [1] and Theorem 3.3, if $R(\xi,X).\tilde{C}=0$, then M is of constant curvature 1 and consequently we can state the following result

Theorem 3.4. *An n-dimensional LP-Sasakian manifold M satisfies $R(\xi,X).\tilde{C}=0$, if and only if M is locally isometric to the unit sphere $S^n(1)$.*

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