Harmonic maps on Riemannian manifolds

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Abstract. The aim of this article is to underline several properties of harmonic maps on Riemannian manifolds. First we define and give some examples of harmonic maps on compact Riemannian manifolds. Then we prove the first and the second variation formula, we write the Euler-Lagrange equation and we discuss the stability of harmonic maps.

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1 Definition and examples

Let us consider two compact Riemannian manifolds (M,g), (N,h), $\dim M = m$, $\dim N = n$ and $\Phi : M \to N$ a C^∞-morfism. We define the integral of energy

$$E(\Phi) = \frac{1}{2}\int_M \sum_{i=1}^m (\Phi^* h)(e_i, e_i) v_g,$$

where $\Phi_* : T_x M \to T_{\Phi(x)}N$ is the differential of Φ, $(e_i)_{i=\overline{1,m}}$ is an orthonormal frame in $T_x M$. A critical point for the energy is called a *harmonic map*. We write $\mathrm{Har}(M,N) = \{\Phi : M \to N | \Phi \text{ harmonic map}\}$ (in general, this is not a manifold).

The C^∞-application $F : (-\varepsilon,\varepsilon) \times M \to N$, $F(0,x) = \Phi(x)$, $F(t,x) = \Phi_t(x)$, $\forall x \in M$, $\forall t \in (-\varepsilon,\varepsilon)$ induces a variational vector field $V \in \Gamma(\Phi_*^{-1}TN)$,

$$V(x) = \left.\frac{d}{dt}\right|_{t=0} \Phi_t(x) = F_*\left(\frac{\partial}{\partial t}\right)_{(0,x)} \in T_{\Phi(x)}N$$

and a covariant derivative $\tilde\nabla_X \in \Gamma(\Phi_*^{-1}TN)$ with respect to the Riemannian metric h, satisfying

$$(1.1)\quad \tilde\nabla_X(\Phi_* Y) - \tilde\nabla_Y(\Phi_* X) - \Phi_*([X,Y]) = 0,\ \forall X,Y \in \chi(M),\ \tilde\nabla_X V = {}^N\nabla_{\Phi_* X} V.$$

Theorem 1.1. *(the first variational formula). The function Φ is a harmonic map if and only if satisfies the Euler-Lagrange equation $\tau(\Phi) = 0$, where*

$$\tau(\Phi)(x) = \sum_{i=1}^m (\tilde\nabla_{e_i}\Phi_* e_i - \Phi_* \nabla_{e_i} e_i)(x), \quad \forall x \in M$$

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or

$$\tau(\Phi)(x) = \sum_{i=1}^{m} ({}^N\tilde{\nabla}_{\Phi_* e_i} \Phi_* e_i - \Phi_* \nabla_{e_i} e_i)(x), \quad \forall x \in M$$

and $\left.\dfrac{d}{dt}\right|_{t=0} E(\Phi_t) = -\displaystyle\int_M h(V, \tau(\Phi)) v_g.$

Proof. We can write

$$\left.\frac{d}{dt}\right|_{t=0} E(\Phi_t) = \frac{1}{2}\int_M \sum_{i=1}^{m} \frac{d}{dt} h(\Phi_{t_*} e_i, \Phi_{t_*} e_i) v_g = \frac{1}{2}\int_M \sum_{i=1}^{m} \frac{d}{dt} h(F_* e_i, F_* e_i)(t,x) v_g =$$

$$= \frac{1}{2}\int_M \sum_{i=1}^{m} \left(\frac{\partial}{\partial t}\right)_{(t,x)} h(F_* e_i, F_* e_i) v_g = \int_M \sum_{i=1}^{m} h\left(\tilde{\nabla}_{\frac{\partial}{\partial t}} F_* e_i, F_* e_i\right) v_g =$$

$$= \int_M \sum_{i=1}^{m} h\left(\tilde{\nabla}_{e_i} F_*\left(\frac{\partial}{\partial t}\right), F_* e_i\right) = \int_M \sum_{i=1}^{m} \left(e_i h\left(F_* \frac{\partial}{\partial t}, F_* e_i\right)\right) - h\left(F_* \frac{\partial}{\partial t}, \tilde{\nabla}_{e_i} F_* e_i\right) v_g.$$

We have

$$\sum_{i=1}^{m} \left(e_i h\left(F_* \frac{\partial}{\partial t}, F_* e_i\right)\right) = \sum_{i=1}^{m} e_i g(X_t, e_i) = \sum_{i=1}^{m} (g(\nabla_{e_i} X_t, e_i) + g(X_t, \nabla_{e_i} e_i)) =$$

$$= \mathrm{div} X_t + \sum_{i=1}^{m} g(X_t, \nabla_{e_i} e_i) = \mathrm{div} X_t + \sum_{i=1}^{m} h\left(F_* \frac{\partial}{\partial t}, F_* \nabla_{e_i} e_i\right),$$

so $\left.\dfrac{d}{dt}\right|_{t=0} E(\Phi_t) = -\displaystyle\int_M h(\nabla, \tau(\Phi)) v_g$ and $\tau(\Phi) = \displaystyle\sum_{i=1}^{m} (\tilde{\nabla}_{e_i} \Phi_* e_i - \Phi_* \nabla_{e_i} e_i)$. Locally,

$$\tau(\Phi)^\gamma(x) = \sum_{i,j=1}^{m} g^{ij} \left(\frac{\partial^z \Phi^\gamma}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial \Phi^\gamma}{\partial x^k} + \sum_{\alpha,\beta=1}^{n} {}^N\Gamma_{\alpha\beta}^\gamma \frac{\partial \Phi^\alpha}{\partial x^i} \frac{\partial \Phi^\beta}{\partial x^j}\right),$$

$$\tau(\Phi)^\gamma = \Delta \Phi^\gamma + \sum_{i,j=1}^{m} g^{ij} \left(\sum_{\alpha,\beta=1}^{n} {}^N\Gamma_{\alpha\beta}^\gamma \frac{\partial \Phi^\alpha}{\partial x^i} \frac{\partial \Phi^\beta}{\partial x^j}\right).$$

The Euler-Lagrange equation is a non-linear partial differential equation.

Examples.

1. $\Phi : (M, g) \to (N, h)$, $\Phi(x) = q$, $\forall x \in M$ is a constant harmonic map ($\tau(\Phi) = 0$).
2. $\Phi : (M, g) \to (\mathbf{R}^n, g_0)$, $\Phi = (\Phi^1, \ldots, \Phi^n) \in C^\infty(M, R^k)$ is a harmonic linear map if and only if Φ^i is a harmonic function, $\forall i = \overline{1, n}$.
3. $\Phi : (S^1, g) \to (N, h)$, where $e_1 = \dfrac{\partial}{\partial x}$, $x \in \mathbf{R}$, $\nabla_{e_1} e_1 = 0$. The function Φ is a harmonic map if and only if $\nabla_\phi \phi' = 0$ (i.e. the equation of geodesics for $\Phi'_1 = \Phi^* e_1$).
4. $\Phi : (M, g) \to (N, h)$, an isometric harmonic immersion is equivalent to minimality.

5. $\Phi : (M, g) \to (N, h)$, a harmonic Riemannian submersion is equivalent to the minimality of the submanifolds $\Phi^{-1}(\Phi(x))$ in M.

6. $\Phi : (M, g) \to (N, h)$ is a holomorphic harmonic map between two Kahler manifolds.

7. $\pi : (\mathbf{C}^n, g_0) \to (\mathbf{C}^n|\Lambda, g_\Lambda)$ is the projection between two Kähler manifolds with zero sectional curvature ($\mathbf{C}^n|\Lambda$ is the complex thorus, $\Lambda = \mathbf{C}^n$).

8 $\pi : (S^{n+1}, g_S^{n+1}) \to (P^n(\mathbf{C}), h)$ is the Hopf map, i.e. a Riemannian harmonic submersion between the sphere $S^{m+1} = SU(n+1)/SU(n)$ (with sectional curvature equal to 1) and the projective complex space $\mathbf{P}^n(C) = SU(n+1)/S(U(1) \times U(n))$ (with sectional curvature in [1,4]).

2 The second variational formula

Let us consider a new continuous variation with respect of two parameters s, t,

$$F : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times M \to N,\ F(s,t,x) = \Phi_{s,t}(x),\ F(0,0,x) = \Phi(x),\ \forall x \in M.$$

The variational vector fields are as follows:

$$V(x) = \tfrac{d}{ds}\big|_{s=0} \Phi_s(x) = F_* \left(\tfrac{\partial}{\partial s}\right)\big|_{(s,t)=(0,0)},$$
$$W(x) = \tfrac{d}{dt}\big|_{t=0} \Phi_t(x) = F_* \left(\tfrac{\partial}{\partial t}\right)\big|_{(s,t)=(0,0)} \in \Gamma(\Phi_*^{-1} TN).$$

The integral of energy is

$$E(\Phi_{s,t}) = \frac{1}{2} \int_M \sum_{i=1}^m h(\phi_{s,t_*} e_i, \phi_{s,t_*} e_i) v_g$$

and the Hessian of energy is Hess $(E)_\Phi(V, W) = \dfrac{\partial^2}{\partial s \partial t}\Big|_{(s,t)=(0,0)} E(\Phi_{s,t})$.

Theorem 2.1. *(the second variational formula). If Φ is a harmonic map between M and N, then the Hessian of the energy is Hess $(E)_\Phi(V, W) = \int_M h(J_\Phi(V), W) v_g$, where the elliptic differential Jacobi operator has $J_\phi(V)$ of the form*

$$J_\Phi(V) = -\sum_{i=1}^m (\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} - \tilde{\nabla}_{\nabla_{e_i} e_i}) V - \sum_{i=1}^m R(V, \Phi_* e_i) \Phi_* e_i, J_\Phi = \bar{\Delta}_\Phi - R_\Phi,$$

the classical Laplacian $\bar{\Delta}_\phi$ being defined by

$$\int_M h(\bar{\Delta}_\Phi V, W) v_g = \int_M h(\tilde{\nabla} V, \tilde{\nabla} W) v_g = \int_M h(V, \bar{\Delta}_\Phi W) v_g,\ \forall V, W \in \Gamma(\phi_*^{-1} TN).$$

Proof. Indeed, by straightforward calculation, we obtain:

$$\frac{\partial}{\partial t} E(\Phi_{s,t}) = -\int_M h\left(F_* \frac{\partial}{\partial t}, \sum_{i=1}^m \left(\tilde{\nabla}_{e_i} F_* e_i - F_* \nabla_{e_i} e_i\right)\right) v_g.$$

We differentiate with respect to s:

$$\frac{\partial^2}{\partial s\partial t}E(\Phi_{s,t}) = -\int_M h\left(\tilde{\nabla}_{\frac{\partial}{\partial t}} F_*\frac{\partial}{\partial t}, \sum_{i=1}^m \left(\tilde{\nabla}_{e_i}F_*e_i - F_*\nabla_{e_i}e_i\right)\right)v_g -$$

$$- \int_M h\left(F_*\frac{\partial}{\partial t}, \sum_{i=1}^m \tilde{\nabla}_{\frac{\partial}{\partial s}}\left(\tilde{\nabla}_{e_i}F_*e_i - F_*\nabla_{e_i}e_i\right)\right)v_g.$$

We find

$$\tilde{\nabla}_{\frac{\partial}{\partial s}}\tilde{\nabla}_{e_i}F_*e_i = \tilde{\nabla}_{e_i}\tilde{\nabla}_{\frac{\partial}{\partial s}}F_*e_i + {}^N R\left(F_*\frac{\partial}{\partial s}, F_*e_i\right)F_*e_i + \tilde{\nabla}_{\left[\frac{\partial}{\partial s}, e_i\right]}F_*e_i$$

and

$$\tilde{\nabla}_{\frac{\partial}{\partial s}}F_*\nabla_{e_i}e_i = \tilde{\nabla}_{\nabla_{e_i}e_i}F_*\frac{\partial}{\partial s}, \quad \text{because} \quad \left[\frac{\partial}{\partial s}, e_i\right] = 0,\ \forall i = \overline{1,m}.$$

So, we have $\text{Hess}(E)_\Phi(V,W) = \int_M h(J_\Phi V, W)$, where the Jacobi's operator is $J_\Phi(V) = \bar{\Delta}_\Phi V - R_\Phi V$. Moreover,

$$h(\bar{\Delta}_\Phi V, W) = -\sum_{i=1}^m (e_i h(\tilde{\nabla}_{e_i}V, W) - h(\tilde{\nabla}_{e_i}V, \tilde{\nabla}_{e_i}W)) + \sum_{i=1}^m h(\tilde{\nabla}_{\nabla_{e_i}e_i}V, W) =$$

$$= -\text{div}X + \sum_{i=1}^m h(\tilde{\nabla}_{e_i}V, \tilde{\nabla}_{e_i}W), g(X,Y) = h(\tilde{\nabla}_Y V, W).$$

By integration, we obtain the final conclusion. □

3 The instability theorem

For a harmonic map $\Phi : (M,g) \to (N,h)$, we define the following notions:

- index(Φ) = $\sup\{\dim F | F \subset \Gamma(\Phi_*^{-1}TN)$ a subspace with Hess $(E)_\Phi$ non-positive definite $\}$,
- null $(\Phi) = \dim\{V \in \Gamma(\Phi_*^{-1}TN) | \text{Hess}\,(E)_\Phi(V,W) = 0,\ \forall W \in \Gamma(\Phi_*^{-1}TN)\}$,
- $V_\lambda(\Phi) = \{V \in \Gamma(\Phi_*^{-1}TN) | J_\Phi V = \lambda V\}$, $\dim V_\lambda(\Phi)$ = the multiplicity of λ,
- $\text{Spec}(J_\Phi) = \{\lambda | J_\Phi V = \lambda V,\ V \in \Gamma(\Phi_*^{-1}TN)\}$ (with increasing elements).

We say that Φ is *low - stable* if and only if the index vanishes $(\Phi) = 0$ (with positive eigenvalues for the Hessian) and if it isn't, Φ is called *instable.* We can write the index $(\Phi) = = \sum_{\lambda<0} \dim V_\lambda(\phi)$, null $(\Phi) = \dim V_0(\Phi) = \dim(\ker J_\Phi)$.

Theorem 3.1. *1. If (N,h) has a non-positive sectional curvature, then Φ is low-stable.*

2. We have $T_\Phi Har(M,N) \subset Ker J_\Phi$ (in general, the equality is false) and $dim\, T_\Phi Har(M,N) \leq null(\Phi)$ (as a vector space).

3. (Xin, 1980). For (S^m, g_S^m) (the sectional curvature is equal with 1), $m \geq 3$, (N,h) a compact Riemannian manifold, any non-constant harmonic map $\Phi : S^m \to N$ is unstable.

4. By generalization, for a compact Riemannian manifold (M,g), $\pi_1(M) = \pi_2(M) = 0$ and (N,h), (M',g') two arbitrary compact Riemannian manifolds, the harmonic maps $\Phi : M \to N$ and $\Psi : M \to M'$ are unstable (index $(\Phi) > 0$, index $(\Psi) > 0$).

Proof. **1**. The condition ${}^N k(u,v) \leq 0$ is equivalent to $h({}^N R(u,v)v,u) \leq 0$, which is equivalent to $h(R_\Phi V, V) \leq 0$, $\forall V \in (\Phi_*^{-1}TN)$. So,

$$\int h(\bar{\Delta}_\Phi V, V)v_g = \int h(\tilde{\nabla} V, \tilde{\nabla} V)v_g \geq 0$$

and $\int h(J_\Phi V, V)v_g \geq 0$, $\forall V \in (\phi_*^{-1}TN)$.

2. If $(\Phi_s)_s \subset \mathrm{Har}(M,N)$ is a 1-parameter family of harmonic maps, $\Phi_0 = \Phi$, $\Phi_{s,t} \in C^\infty(M,N)$ a variation of Φ_s, $\Phi_{s,0} = \Phi_s$, then $\left.\frac{\partial}{\partial t}\right|_{t=0} E(\Phi_{s,t}) = 0$ (ϕ_s is harmonic, $\forall s$) and

$$\int h(J_\Phi V, W)v_g = \left.\frac{\partial^z}{\partial s \partial t}\right|_{(s,t)=(0,0)} E(\Phi_{s,t}) = 0, \quad \forall W \in \Gamma(\Phi_*^{-1}TN).$$

So, $V \in \mathrm{Ker}\,(J_\phi)$ and $\dim T_\Phi \mathrm{Har}(M,N) \leq \mathrm{null}(\Phi)$.

3. **Step 1**. We decompose the tangent vector space in $x \in M$, in a direct sum $T_x\mathbf{R}^{m+1} = T_xS^m \oplus T_xS^{m\perp}$ s.t., for any vector field $V = \sum_{i=1}^{m+1} a^i \frac{\partial}{\partial x_i} = V^T + V^\perp$, $V^T = \sum_{i=1}^{m+1}(a_i - x_i(a,x))\frac{\partial}{\partial x_i}$, $V^\perp = (a,x)\sum_{i=1}^{m+1} x_i \frac{\partial}{\partial x_i} \overset{not}{=} W$. We obtain $\nabla_X W = = -(a,x)X$, $\forall X \in T_xS^m$.

Step 2. We consider an orthonormal local frame on (S^m, g_S^m), $(e_i)_{i=1,m}$, and we prove $\bar{\Delta}W = W = \mathrm{grad}S^m(a,x)$. Indeed,

$$\begin{aligned}\bar{\Delta}W &= -\textstyle\sum_{i=1}^m (\tilde{\nabla}_{ei}\tilde{\nabla}_{ei}W - \tilde{\nabla}_{\nabla_{e_i}e_i}W = \\ &= -\textstyle\sum_{i=1}^m (\tilde{\nabla}_{ei}(-(a,x)e_i) - (a,x)\tilde{\nabla}_{ei}e_i) = \sum_{i=1}^m (e_i(a,x))e_i = W.\end{aligned}$$

Step 3. We have $\nabla_Y e_i = 0$, $\forall Y \in \chi(S^m)$ and

$$\bar{\Delta}\Phi_* W = -\sum_{i=1}^m (\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i} - \tilde{\nabla}_{\nabla_{e_i}e_i})\Phi_* W = -\sum_{i=1}^m \tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}\Phi_* W.$$

But $\Phi_* W = d\Phi(W)$, $\nabla_X W = -(a,x)W$, so

$$\tilde{\nabla}_{e_i}\Phi_* W = (\tilde{\nabla}_{e_i} d\Phi)(W) + d\Phi(\nabla_{e_i} W) = (\tilde{\nabla}_{e_i} d\Phi)(W) - d\Phi((a,x)e_i).$$

By analogy,

$$\bar{\Delta}\Phi_* W = -\sum_{i=1}^m (\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i} d\Phi)(W) - \sum_{i=1}^m (\tilde{\nabla}_{e_i} d\Phi)(\nabla_{e_i} W) + \sum_{i=1}^m \tilde{\nabla}_{e_i}(d\Phi((a,x)e_i))$$

and because $d\Phi(a,x)e_i = (a,x)\Phi_* e_i$, Φ is harmonic, $\nabla_{e_i} e_i = 0$,

$$\Phi_* W = \sum_{i=1}^{m} \tilde{\nabla}_{e_i}(d\Phi((a,x)e_i)),$$

we obtain the relation: $(\bar{\Delta} d\Phi)(W) = \sum_{i=1}^{m} {}^N R(\Phi_* W, \Phi_* e_i)\phi_* e_i - \Phi_* \rho(W)$, where $\rho(W) = \sum_{i=1}^{m} R(W, e_i)e_i = (m-1)W$. So,

$$\bar{\Delta}\Phi_* W = \sum_{i=1}^{m} {}^N R(\Phi_* W, \Phi_* e_i)\Phi_* e_i + (2-m)\Phi_* W.$$

Step 4.

$$0 \le \int_M h(J_\Phi(\Phi_* W), \Phi_* W)v_g = \int_M h(\bar{\Delta}\Phi_* W - \\ - \sum_{i=1}^{m} {}^N R(\Phi_* W, \Phi_* e_i)\phi_* e_i, \Phi_* W)v_g = (2-m)\int_M h(\Phi_* W, \Phi_* W)v_g.$$

For $m \ge 3$ and index $(\Phi) = 0$, we have $\int_M h(\Phi_* W, \Phi_* W)v_g = 0$, i.e. $\Phi_* W = 0$, $\forall W \in T_x S^m$, so $\Phi_* = 0$ and Φ is constant. □

For related problems, see [1]-[10].

References

[1] M. P. do Carmo, *Riemannian Geometry*, Boston, 1992.

[2] S. Ianuş, *Differential Geometry with Applications in Relativity Theory* (in Romanian), Romanian Academy Publishers, 1983.

[3] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, vol. I, II*, Interscience Publishers, 1963.

[4] A. Mishchenko and A. Fomenko, *A Course of Differential Geometry and Topology*, Mir Publishers, Moscow, 1988.

[5] L. Nicolescu, *Lie Groups*, University Publishers, Bucharest, 1994.

[6] T. Rapcsak, *Smooth Nonlinear Optimization in R^n*, Kluwer Acad. Publishers, 1997.

[7] C. Udrişte, *Field Lines*, Technical House Eds., Bucharest, 1988.

[8] C. Udrişte, *Convex Functions and Optimization Methods on Riemannian Manifolds*, Kluwer Acad. Publishers, 1994.

[9] H. Urakawa, *Calculus of Variations and Harmonic Maps*, American Mathematical Society, Providence, 1990.

[10] J. C. Wood, *Harmonic morphisms and Hermitian structures on Einstein 4-manifolds*, Int. J. Math. 3. (1992), 415439.

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