

# Mechanics systems on para-Kählerian manifolds of constant $J$ -sectional curvature

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**Abstract.** The goal of this paper is to present Euler-Lagrange and Hamiltonian equations on  $\mathbb{R}_n^{2n}$  which is a model of para-Kählerian manifolds of constant  $J$ -sectional curvature. In conclusion, some differential geometrical and physical results on the related mechanic systems have been given.

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**Key words:** Para-Kählerian manifolds; constant  $J$ -sectional curvature; Lagrangian and Hamiltonian mechanical systems.

## 1 Introduction

It is well-known that Modern Differential Geometry has an important role to obtain different types of Lagrangian and Hamiltonian formalisms of Classical Mechanics. Moreover, it is possible to see a lot of studies in the suitable fields. One may say that Lagrangian and Hamiltonian systems are characterized by a convenient vector field  $X$  defined on the tangent and cotangent bundles which are phase-spaces of velocities and momentum of a given configuration manifold. If  $Q$  is an  $m$ -dimensional configuration manifold and  $L : TQ \rightarrow R$  is a regular Lagrangian function, then there is a unique vector field  $X$  on  $TQ$  such that

$$(1.1) \quad i_{X_L} \omega_L = dE_L,$$

where  $\omega_L$  is the symplectic form and  $E_L$  is energy associated to  $L$ . The so-called Euler-Lagrange vector field  $X$  is a semispray (or *second order differential equation*) on  $Q$  since its integral curves are the solutions of the Euler-Lagrange equations. The triple  $(TQ, \omega_L, L)$  is called *Lagrangian system* on the tangent bundle  $TQ$ . If  $H : T^*Q \rightarrow R$  is a regular Hamiltonian function then there is a unique vector field  $X_H$  on  $T^*Q$  such that

$$(1.2) \quad i_{X_H} \omega = dH$$

where  $\omega$  is the symplectic form and  $H$  stands for Hamiltonian function. The paths of the so-called Hamiltonian vector field  $X_H$  are the solutions of the Hamiltonian

equations. The triple  $(T^*Q, \omega, H)$  is called *Hamiltonian system* on the cotangent bundle  $T^*Q$  endowed with symplectic form  $\omega$ .

From the before some studies given in [5]-[11] and there in; we know that time-dependent or not, constraint, real, complex and paracomplex analogues of the Lagrangian and Hamiltonian systems have detailed been researched. But, we see that is not mentioned about Lagrangian and Hamiltonian mechanics systems on constant  $J$ -sectional curvature. Therefore, in this paper we present the Euler-Lagrange equations and Hamiltonian equations on a model of para-Kählerian manifolds of constant  $J$ -sectional curvature and derive differential geometrical and physical conclusions on related dynamics systems.

In this paper, all the manifolds and geometric objects are  $C^\infty$  and the Einstein summation convention is in use. Also,  $\mathbb{R}$ ,  $\mathcal{F}(M)$ ,  $\chi(M)$  and  $\Lambda^1(M)$  denote the set of real numbers, the set of functions on  $M$ , the set of vector fields on  $M$  and the set of 1-forms on  $M$ , respectively.

## 2 Para-Kählerian manifolds of constant $J$ -sectional curvature

**Definition 1.** [1, 4] Let a manifold  $M$  be endowed with an almost product structure  $J \neq \mp Id$ ; which is a  $(1,1)$ -tensor field such that  $J^2 = Id$ . We say that  $(M, J)$  (resp.  $(M, J, g)$ ) is an almost product (resp. almost Hermitian) manifold, where  $g$  is a semi-Riemannian metric on  $M$  with respect to which  $J$  is skew-symmetric, that is

$$(2.1) \quad g(JX, Y) + g(X, JY) = 0, \forall X, Y \in \chi(M)$$

Then  $(M, J, g)$  is para-Kählerian if  $J$  is parallel with respect to the Levi-Civita connection.

Let  $(M, J, g)$  be a para-Kählerian manifold and let denote the curvature  $(0, 4)$ -tensor field by

$$(2.2) \quad R(X, Y, Z, V) = g(R(X, Y)Z, V); \forall X, Y, Z, V \in \chi(M)$$

where the Riemannian curvature  $(1, 3)$ -tensor field associated to the Levi-Civita connection  $\nabla$  of  $g$  is given by  $R = [\nabla, \nabla] - \nabla[\cdot, \cdot]$ .

Then

$$(2.3) \quad \begin{aligned} R(X, Y, Z, V) &= -R(Y, X, Z, V) = -R(X, Y, V, Z) = R(JX, JY, Z, V) \\ &\text{and } \sum_{\sigma} R(X, Y, Z, V) = 0, \end{aligned}$$

where  $\sigma$  denotes the sum over all cyclic permutations. We know that the following  $(0,4)$ -tensor field is defined by

$$(2.4) \quad R_0(X, Y, Z, V) = \frac{1}{4} \left\{ \begin{aligned} &g(X, Z)g(Y, V) - g(X, V)g(Y, Z) - g(X, JZ)g(Y, JV) \\ &+ g(X, JV)g(Y, JZ) - 2g(X, JY)g(Z, JV) \end{aligned} \right\},$$

where  $\forall X, Y, Z, V \in \chi(M)$ . For any  $p \in M$ , a subspace  $S \subset T_p M$  is called non-degenerate if  $g$  restricted to  $S$  is non-degenerate. If  $\{u, v\}$  is a basis of a plane

$\sigma \subset T_p M$ , then  $\sigma$  is non-degenerate iff  $g(u, u)g(v, v) - [g(u, v)]^2 \neq 0$ . In this case the sectional curvature of  $\sigma = \text{span}\{u, v\}$  is

$$(2.5) \quad k(\sigma) = \frac{R(u, v, u, v)}{g(u, u)g(v, v) - [g(u, v)]^2}$$

From (2.1) it follows that  $X$  and  $JX$  are orthogonal for any  $X \in \chi(M)$ . By a  $J$ -plane we mean a plane which is invariant by  $J$ . For any  $p \in M$ , a vector  $u \in T_p M$  is isotropic provided  $g(u, u) = 0$ . If  $u \in T_p M$  is not isotropic, then the sectional curvature  $H(u)$  of the  $J$ -plane  $\text{span}\{u, Ju\}$  is called the  $J$ -sectional curvature defined by  $u$ . When  $H(u)$  is constant, then  $(M, J, g)$  is called of constant  $J$ -sectional curvature, or a para-Kählerian space form.

**Theorem 1.** *Let  $(M, J, g)$  be a para-Kählerian manifold such that for each  $p \in M$ , there exists  $c_p \in \mathbb{R}$  satisfying  $H(u) = c_p$  for  $u \in T_p M$  such that  $g(u, u)g(Ju, Ju) \neq 0$ . Then the Riemann-Christoffel tensor  $R$  satisfies  $R = cR_0$ , where  $c$  is the function defined by  $p \rightarrow c_p$ . And conversely.*

**Definition 2.** A para-Kählerian manifold  $(M, J, g)$  is said to be of constant paraholomorphic sectional curvature  $c$  if it satisfies the conditions of **Theorem 1**.

**Theorem 2.** *Let  $(M, J, g)$  be a para-Kählerian manifold with  $\dim M > 2$ . Then the following properties are equivalent:*

- 1)  $M$  is a space of constant paraholomorphic sectional curvature  $c = 0$
- 2) The Riemann-Christoffel tensor curvature tensor  $R$  has the expression

$$(2.6) \quad R(X, Y, Z, V) = 0, \forall X, Y, Z, V \in \chi(M).$$

Let  $(x_i, y_i)$  be a real coordinate system on a neighborhood  $U$  of any point  $p$  of  $\mathbb{R}^{2n}$ , and  $\{(\frac{\partial}{\partial x_i})_p, (\frac{\partial}{\partial y_i})_p\}$  and  $\{(dx_i)_p, (dy_i)_p\}$  natural bases over  $\mathbb{R}$  of the tangent space  $T_p(\mathbb{R}^{2n})$  and the cotangent space  $T_p^*(\mathbb{R}^{2n})$  of  $\mathbb{R}^{2n}$ , respectively.

The space  $(\mathbb{R}^{2n}, g, J)$ , is the model of the para-Kählerian space forms of dimension  $2n \geq 2$  and paraholomorphic sectional curvature  $c = 0$ , where  $g$  is the metric

$$(2.7) \quad g = dx_i \otimes dy_i + dy_i \otimes dx_i,$$

and  $J$  the almost product structure

$$(2.8) \quad J = \frac{\partial}{\partial x_i} \otimes dx_i - \frac{\partial}{\partial y_i} \otimes dx_i.$$

Then we have

$$(2.9) \quad J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}, \quad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial y_i}.$$

The dual endomorphism  $J^*$  of the cotangent space  $T_p^*(\mathbb{R}^{2n})$  at any point  $p$  of manifold  $\mathbb{R}^{2n}$  satisfies  $J^{*2} = Id$  and is defined by

$$(2.10) \quad J^*(dx_i) = dx_i, \quad J^*(dy_i) = -dy_i.$$

### 3 Lagrangian mechanical systems

Here, we introduce Euler-Lagrange equations on para-Kählerian manifolds of constant  $J$ -sectional curvature  $(\mathbb{R}_n^{2n}, g, J)$ .

Given by  $J$  almost product structure and by  $(x_i, y_i)$  the coordinates of  $\mathbb{R}_n^{2n}$ . Let semispray be a vector field as follows:

$$(3.1) \quad \xi = X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}, \quad X_i = \dot{x}_i = y_i, \quad Y_i = \dot{y}_i.$$

By *Liouville vector field* on para-Kählerian space form  $(\mathbb{R}_n^{2n}, g, J)$ , we call the vector field determined by  $V = J\xi$  and calculated by

$$(3.2) \quad J\xi = X_i \frac{\partial}{\partial x_i} - Y_i \frac{\partial}{\partial y_i},$$

Denote  $T$  by the kinetic energy and  $P$  by the potential energy of mechanics system on para-Kählerian space of constant  $J$ -sectional curvature,. Then we write by  $L = T - P$  Lagrangian function and by  $E_L = V(L) - L$  the energy function associated  $L$ .

The operator  $i_J$  defined by

$$(3.3) \quad i_J : \wedge^2 \mathbb{R}_n^{2n} \rightarrow \wedge^1 \mathbb{R}_n^{2n}$$

is called the interior product with  $J$ , or sometimes the insertion operator, or contraction by  $J$ . The exterior vertical derivation  $d_J$  is defined by

$$(3.4) \quad d_J = [i_J, d] = i_J d - d i_J,$$

where  $d$  is the usual exterior derivation. For almost product structure  $J$  determined by (2.9), the closed para-Kählerian form is the closed 2-form given by  $\Phi_L = -dd_J L$  such that

$$(3.5) \quad d_J = \frac{\partial}{\partial x_i} dx_i - \frac{\partial}{\partial y_i} dy_i : \mathcal{F}(\mathbb{R}_n^{2n}) \rightarrow \wedge^1 \mathbb{R}_n^{2n}.$$

Thus we get

$$(3.6) \quad \begin{aligned} \Phi_L &= -\frac{\partial^2 L}{\partial x_j \partial x_i} dx_j \wedge dx_i - \frac{\partial^2 L}{\partial y_j \partial x_i} dy_j \wedge dx_i \\ &+ \frac{\partial^2 L}{\partial x_j \partial y_i} dx_j \wedge dy_i + \frac{\partial^2 L}{\partial y_j \partial y_i} dy_j \wedge dy_i. \end{aligned}$$

Then

$$(3.7) \quad \begin{aligned} i_\xi \Phi_L &= -X_i \frac{\partial^2 L}{\partial x_j \partial x_i} \delta_i^j dx_i + X_i \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j - Y_i \frac{\partial^2 L}{\partial y_j \partial x_i} \delta_i^j dx_i + X_i \frac{\partial^2 L}{\partial y_j \partial x_i} dy_j \\ &+ X_i \frac{\partial^2 L}{\partial x_j \partial y_i} \delta_i^j dy_i - Y_i \frac{\partial^2 L}{\partial x_j \partial y_i} dx_j + Y_i \frac{\partial^2 L}{\partial y_j \partial y_i} \delta_i^j dy_i - Y_i \frac{\partial^2 L}{\partial y_j \partial y_i} dy_j. \end{aligned}$$

Because of the closed para-Kählerian form  $\Phi_L$  on para-Kählerian space form  $(\mathbb{R}_n^{2n}, g, J)$  is para-symplectic structure, one may obtain

$$(3.8) \quad E_L = X_i \frac{\partial L}{\partial x_i} - Y_i \frac{\partial L}{\partial y_i} - L,$$

and thus

$$(3.9) \quad \begin{aligned} dE_L = & X_i \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j - Y_i \frac{\partial^2 L}{\partial x_j \partial y_i} dx_j - \frac{\partial L}{\partial x_j} dx_j \\ & + X_i \frac{\partial^2 L}{\partial y_j \partial x_i} dy_j - Y_i \frac{\partial^2 L}{\partial y_j \partial y_i} dy_j - \frac{\partial L}{\partial y_j} dy_j \end{aligned}$$

Taking care of  $i_\xi \Phi_L = dE_L$ , we have

$$(3.10) \quad \begin{aligned} -X_i \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j - Y_i \frac{\partial^2 L}{\partial y_j \partial x_i} dx_j + \frac{\partial L}{\partial x_j} dx_j \\ + X_i \frac{\partial^2 L}{\partial x_j \partial y_i} dy_j + Y_i \frac{\partial^2 L}{\partial y_j \partial y_i} dy_j + \frac{\partial L}{\partial y_j} dy_j = 0. \end{aligned}$$

If the curve  $\alpha$  on  $\mathbb{R}_n^{2n}$  be integral curve of  $\xi$ , which satisfies

$$(3.11) \quad \begin{aligned} -[X_i \frac{\partial^2 L}{\partial x_j \partial x_i} + Y_i \frac{\partial^2 L}{\partial y_j \partial x_i}] dx_j + \frac{\partial L}{\partial x_j} dx_j \\ + [X_i \frac{\partial^2 L}{\partial x_j \partial y_i} + Y_i \frac{\partial^2 L}{\partial y_j \partial y_i}] dy_j + \frac{\partial L}{\partial y_j} dy_j = 0. \end{aligned}$$

follow the equations

$$(3.12) \quad \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial x_j} \right) - \frac{\partial L}{\partial x_j} = 0, \quad \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial y_j} \right) + \frac{\partial L}{\partial y_j} = 0$$

so-called *Euler-Lagrange equations* whose solutions are the paths of the semispray  $\xi$  on para-Kählerian space form  $(\mathbb{R}_n^{2n}, g, J)$ . Finally one may say that the triple  $(\mathbb{R}_n^{2n}, \Phi_L, \xi)$  is *mechanical system* on para-Kählerian manifolds of constant J-sectional curvature  $(\mathbb{R}_n^{2n}, g, J)$ . Therefore we say

**Proposition 1.** *Let  $J$  almost product structure on para-Kählerian space of constant  $J$ -sectional curvature  $(\mathbb{R}_n^{2n}, g, J)$ . Also let  $(f_1, f_2)$  be natural bases of  $\mathbb{R}_n^{2n}$ . Then it follows*

$$\begin{aligned} J(f_1) - f_1 = 0 & \iff \dot{f}_{1,L} - f_{1,L} = 0, \\ J(f_2) + f_2 = 0 & \iff \dot{f}_{2,L} + f_{2,L} = 0, \end{aligned}$$

where  $f_{1,L} = \frac{\partial L}{\partial x_i}$ ,  $f_{2,L} = \frac{\partial L}{\partial y_i}$ ,  $\dot{f}_{1,L} = \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial x_i} \right)$ ,  $\dot{f}_{2,L} = \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial y_i} \right)$ .

## 4 Hamiltonian mechanical systems

We further present the Hamiltonian equations on para-Kählerian manifolds of constant  $J$ -sectional curvature  $(\mathbb{R}_n^{2n}, g, J)$ .

Let  $J^*$  be an almost product structure defined by (2.10) and  $\lambda$  Liouville form determined by  $J^*(\omega) = \frac{1}{2}y_i dx_i - \frac{1}{2}x_i dy_i$  such that  $\omega = \frac{1}{2}y_i dx_i + \frac{1}{2}x_i dy_i$  1-form on  $\mathbb{R}_n^{2n}$ . If  $\Phi = -d\lambda$  is closed para-Kählerian form, then it is also a para-symplectic structure on  $\mathbb{R}_n^{2n}$ .

Let  $(\mathbb{R}_n^{2n}, g, J)$  be para-Kählerian manifolds of constant J-sectional curvature with closed para-Kählerian form  $\Phi$ . Suppose that Hamiltonian vector field  $Z_H$  associated to Hamiltonian energy  $H$  is given by

$$(4.1) \quad Z_H = X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}.$$

For the closed para-Kählerian form  $\Phi$  on  $\mathbb{R}_n^{2n}$ , we have

$$(4.2) \quad \Phi = -d\lambda = -d\left(\frac{1}{2}y_i dx_i - \frac{1}{2}x_i dy_i\right) = dx_i \wedge dy_i.$$

Then it follows

$$(4.3) \quad i_{Z_H} \Phi = \Phi(Z_H) = -Y_i dx_i + X_i dy_i.$$

Otherwise, one may calculate the differential of Hamiltonian energy as follows:

$$(4.4) \quad dH = \frac{\partial H}{\partial x_i} dx_i + \frac{\partial H}{\partial y_i} dy_i.$$

From (4.3) and (4.4) with respect to  $i_{Z_H} \Phi = dH$ , we find para-Hamiltonian vector field on para-Kählerian space of constant  $J$ -sectional curvature to be

$$(4.5) \quad Z_H = \frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y_i}.$$

Suppose that the curve

$$(4.6) \quad \alpha : I \subset \mathbb{R} \rightarrow \mathbb{R}_n^{2n}$$

be an integral curve of Hamiltonian vector field  $Z_H$ , i.e.,

$$(4.7) \quad Z_H(\alpha(t)) = \dot{\alpha}, \quad t \in I.$$

In the local coordinates we have

$$(4.8) \quad \alpha(t) = (x_i(t), y_i(t)),$$

and

$$(4.9) \quad \dot{\alpha}(t) = \frac{dx_i}{dt} \frac{\partial}{\partial x_i} + \frac{dy_i}{dt} \frac{\partial}{\partial y_i}.$$

Now, by means of (4.7), from (4.5) and (4.9), we deduce the equations so-called *para-Hamiltonian equations*

$$(4.10) \quad \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}.$$

In the end, we may say to be *para-mechanical system*  $(\mathbb{R}_n^{2n}, \Phi, Z_H)$  triple on para-Kählerian manifolds of constant  $J$ -sectional curvature  $(\mathbb{R}_n^{2n}, g, J)$ .

## 5 Conclusions

We obtain that the Lagrangian and the Hamiltonian formalisms in generalized Classical Mechanics and field theory can be intrinsically characterized on  $(\mathbb{R}_n^{2n}, g, J)$  being a model of para-Kählerian space of constant  $J$ -sectional curvature. So, the paths of semisprays  $\xi$  on  $\mathbb{R}_n^{2n}$  are the solutions of the Euler-Lagrange equations given by (3.12) on the mechanical system  $(\mathbb{R}_n^{2n}, \Phi_L, \xi)$ . Also, the solutions of the Hamiltonian equations determined by (4.10) on the mechanical system  $(\mathbb{R}_n^{2n}, \Phi, Z_H)$  are the paths of vector field  $Z_H$  on  $\mathbb{R}_n^{2n}$ .

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