Landsberg spaces satisfying semi-T'-condition

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Abstract. The T-tensor in the theory of Finsler spaces was introduced as indicatrized tensor of the v-covariant derivative of $(h)hv$-torsion tensor by Matsumoto[5].The vanishing of T-tensor is called T-condition. A Finsler space with T-condition has been studied by many authors ([2], [7],[10]). Recently semi-T'-condition of Finsler spaces has been introduced as a generalisation of T-condition in the previous paper [11] and a S-4 like Finsler space with semi-T'-condition has been studied. The purpose of the present paper is to study special Finsler spaces with semi-T'-condition and find the condition for a Landsberg space with semi-T'-condition to vanish h-curvature tensor R_{ijkh}.Finally the condition for a Finsler space with semi-T'-condition to be locally Minkowski is obtained.The notations and terminologies are referred to the monograph [5].

M.S.C. 2000: 53B40, 53C60.
Key words: Finsler space; Landsberg space; T-condition.

1 Introduction

Let M^n be an $n(\geq 3)$ dimensional Finsler space endowed with a fundamental function L = L(x,y),where $x = (x^i)$ is a point and $y = (y^i)$ is a supporting element of M^n.The metric tensor g_{ij} and $(h)hv$-torsion tensor C_{ijk} of M^n are given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j},\ C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

The T- tensor T_{ijkh} of M^n is defined as

$$T_{ijkh} = LC_{ijk}|_h + C_{ijk}l_h + C_{jkh}l_i + C_{khi}l_j + C_{hij}l_k, \tag{1.1}$$

where $l_i = L^{-1}g_{ir}y^r$ and the symbol | means the v-covariant derivative with respect to Cartan connection $C\Gamma$ of M^n.Transvection of (1.1) by the reciprocal metric tensor g^{kh} of g_{kh} gives

$$T_{ij} = LC_i|_j + C_il_j + C_jl_i, \tag{1.2}$$

where $T_{ij}(= g^{kh}T_{ijkh})$ and $C_i(= g^{jk}C_{ijk})$ are called T'-tensor [2] and the torsion vector of M^nrespectively. If the T'-tensor T_{ij} of M^n vanishes, then M^n is called

Differential Geometry - Dynamical Systems, Vol.12, 2010, pp. 246-251.

Finsler space with T'-condition. For instance,a C^v-reducible Finsler space [8] satisfy T-condition as well as T'-condition. The T'-tensor of C-reducible Finsler space [6] can be written as

$$T_{ij} = LC_i|_j + C_i l_j + C_j l_i = \alpha h_{ij}, \tag{1.3}$$

where $\alpha = L(n-1)^{-1}C^i|_i$ and $h_{ij}(= g_{ij} - l_i l_j)$ is angular metric tensor.

Definition ([11]). If the T'-tensor T_{ij} of a Finsler space M^n is written in the form (1.3) then M^n is called a Finsler space with semi-T'-condition.

For instance, a C-reducible Finsler space satisfy semi-T'-condition. For later use, one Ricci identity with respect to torsion vector C_i and one Bianchi identity is written below ([5]):

$$C_i|_j|_r - C_i|_r|_j = -C_t S^t_{ijr}, \tag{1.4}$$

$$R^h_{jr}C^r_{ki} - R^h_{kr}C^r_{ji} - P^h_{jr}P^r_{ki} - P^h_{kr}P^r_{ji} + P^h_{ki|j} - P^h_{ji|k} + R^h_{jk}|_i - R^h_{ijk} = 0, \tag{1.5}$$

where R^h_{ijk}, S^h_{ijk}, R^r_{jk}, P^r_{jk} are the h-curvature tensor, v-curvature tensor, the $(v)h$-torsion tensor, and (v)hv- torsion tensor of M^n respectively and the symbol $|$ represents the h-covariant derivative with respect to $C\Gamma$.

2 Finsler spaces with semi-T'-condition

Let M^n be a Finsler space with semi-T'-condition. Differentiating equation (1.3) v-covariantly with respect to y^r, equation (1.4) can be written as

$$-L^2 C_t S^t_{ilr} = A_r h_{il} - A_l h_{ir}, \tag{2.1}$$

where

$$A_r = (L\alpha)|_r + C_r. \tag{2.2}$$

In view of $l_r R^r_{jk} = 0$, transvection of (2.1) by R^r_{jk} gives

$$L^2 C_t S^t_{ilr} R^r_{jk} - R_{ijk} A_l + h_{il} A_r R^r_{jk} = 0. \tag{2.3}$$

Now we prove the following lemma,

Lemma 2.1. *The $(v)h$-torsion tensor R_{ijk}; of a Finsler space M^n of dimension $n(\geq 3)$ satisfying*

$$-R_{ijk} X_l + h_{il} X_r R^r_{jk} = 0, \tag{2.4}$$

for some vector X_r ; vanishes identically provided that $X^2 \neq 0$, where $X^2 = g_{ij}X^i X^j$.

Proof. Transvection of (2.4) by g^{il} gives $(n-2)X_r R^r_{jk} = 0$. Since $n \geq 3$ we obtain $X_r R^r_{jk} = 0$.Substituting this value again in (2.4) we have $X_l R_{ijk} = 0$, which gives $R_{ijk} = 0$ provided that $X^2 \neq 0$. □

Now we consider the Finsler spaces whose v-curvature tensor S^r_{ijk} are of special forms. First we consider a Finsler space whose v-curvature tensor S^r_{ijk} vanishes. Thus in view of equation (2.3) and lemma (2.1) we have,

Theorem 2.2. *If the v-curvature tensor S^r_{ijk},of an $n(\geq 3)$ dimensional Finsler space M^n with semi-T'-condition, vanishes then its $(v)h$-torsion tensor R^i_{jk} vanishes identically provided that $A^2 \neq 0$, where $A^2 = g^{rs}A_rA_s$ and A_r is given by (2.2).*

Since the v-curvature tensor of a C-2 like Finsler space vanishes [5], the following result is evident,

Corollary 2.3. *The (v)h- torsion tensor R^i_{jk} of an $n(\geq 3)$dimensional C-2 like Finsler space M^n with semi-T'-condition vanishes identically provided that $A^2 \neq 0$.*

Further if we assume the Finsler space M^n is S-3 like [5], for which the v-curvature tensor S_{ijkl} is represented as

$$S_{ijkl} = S(h_{ik}h_{jl} - h_{il}h_{jk}). \tag{2.5}$$

Substituting this in equation (2.3) we have

$$R_{ijk}B_l - h_{il}B_rR^r_{jk} = 0,$$

where

$$B_r = (L\alpha)|_r + C_r(1 + L^2S). \tag{2.6}$$

Applying lemma (2.1) we have

Theorem 2.4. *The $(v)h$-torsion tensor R_{ijk},of an $n(\geq 4)$ dimensional S-3 like Finsler space M^n with semi-T'-condition, vanishes identically provided that $B^2 \neq 0$, where $B^2 = g^{rs}B_rB_s$ and B_r given by (2.6).*

Since the v-curvature tensor of a three dimensional Finsler space can also be written in the form (2.5), the following corollary is evident

Corollary 2.5. *The (v)h- torsion tensor R_{ijk} of a three dimensional Finsler space M^3 with semi-T'-condition vanishes identically provided that $B^2 \neq 0$.*

The S-4 like Finsler spaces has been studied in detail ([5], [7],[11]). The v-curvature tensor S_{ijkl} of a S-4 like Finsler space can be written as

$$S_{ijkl} = h_{il}M_{jk} + h_{jk}M_{il} - h_{ik}M_{jl} - h_{jl}M_{ik} \tag{2.7}$$

where M_{ij} is symmetric tensor satisfying $M_{ij}y^j = 0$ ([4]). It has been seen ([11, Theorem 2.1]) that the tensor M_{ij}, of an $n(\geq 5)$dimensional S-4 like Finsler space M^n with semi-T'-condition, can be written as

$$M_{ij} = \lambda h_{ij} + \mu C_iC_j, \tag{2.8}$$

where

$$\lambda = L^{-2}C^{-2}(n-3)^{-1}[(ML^2 - 1)C^2 - L(\alpha|_kC^k)],$$
$$\mu = L^{-2}C^{-2}(n-3)^{-1}[(n-1) - 2ML^2 + (n-1)C^{-2} - (L\alpha)|_kC^k] \text{ and } M = g^{ij}M_{ij}.$$

Thus in view of equations (2.7) and (2.8) the tensor $C_tS^t_{ilr}R^r_{jk}$ for a S-4 like Finsler space M^n with semi-T'-condition can be written as

(2.9) $$C_t S^t_{ilr} R^r_{jk} = (2\lambda + \mu C^2)[R_{ijk} C_l - h_{il} C_r R^r_{jk}].$$

Therefore the equation (2.3) can be written in the following form

(2.10) $$R_{ijk} D_l - h_{il} D_r R^r_{jk} = 0,$$

where

(2.11) $$D_r = A_r - L^2(2\lambda + \mu C^2) C_r.$$

Applying lemma (2.1) we have

Theorem 2.6. *The $(v)h$-torsion tensor R_{ijk},of an $n(\geq 5)$ dimensional S-4 like Finsler space M^n with semi-T'-condition, vanishes identically provided that $D^2 \neq 0$, where $D^2 = g^{rs} D_r D_s$ and D_r given by (2.11).*

Since the v-curvature tensor of a four dimensional Finsler space can also be written in the form (2.7), the following corollary is evident

Corollary 2.7. *The (v)h- torsion tensor R_{ijk} of a four dimensional Finsler space M^4 with semi-T'-condition vanishes identically provided that $D^2 \neq 0$.*

3 Landsberg spaces

Definition (D1), [5]. An n-dimensional Finsler space M^n is called a Landsberg space if the (v)hv-torsion tensor P^r_{jk} of M^n vanishes identically.
Definition (D2), [5]. If the h-covariant derivative of $(h)hv$-torsion tensor C_{ijk} with respect to Cartan connection $C\Gamma$ of an n-dimensional Finsler space M^n vanishes then M^n is called a Berwald space.
Definition (D3), [5]. If the h-curvature tensor R^h_{ijk} of an n-dimensional Berwald space M^n vanishes then M^n is called a locally Minkowski space.
In terms of the Cartan connection $C\Gamma$, we have:

Landsberg space $\Leftrightarrow C_{ijk|h} y^h = 0$

Berwald space $\Leftrightarrow C_{ijk|h} = 0$

Locally Minkowski space $\Leftrightarrow$ Berwald space and $R^h_{ijk} = 0$.

Let M^n be an $n(\geq 3)$-dimensional Landsberg space then equation (1.5) can be written as

$$R^h_{jr} C^r_{ki} - R^h_{kr} C^r_{ji} + R^h_{jk}|_i - R^h_{ijk} = 0$$

If the $(v)h$-torsion tensor R_{ijk} of M^n vanishes then $R^h_{ijk} = 0$. Thus in view of above, Theorems 2.2, 2.4 and 2.6 can be written as,

Theorem 3.1. **(i)** *If the v-curvature tensor S^r_{ijk}, of an $n(\geq 3)$-dimensional Landsberg space M^n with semi-T'-condition, vanishes then its h-curvature tensor R^r_{ijk} also vanishes identically provided that $A^2 \neq 0$, where $A^2 = g^{rs} A_r A_s$ and A_r is given by (2.2).* **(ii)** *If the v-curvature tensor S^r_{ijk},of an $n(\geq 3)$-dimensional Berwald space M^n with semi-T'-condition, vanishes then the space is locally Minkowski provided that $A^2 \neq 0$.*

Corollary 3.2. **(i)** *The h-curvature tensor* R^r_{ijk}*,of an* $n(\geq 3)$*-dimensional C-2 like Landsberg space* M^n *with semi-*T'*-condition vanishes provided that* $A^2 \neq 0$*.* **(ii)** *An* $n-$*dimensional* $(n \geq 3)$ *C-2 like Berwald space* M^n *with semi-*T'*-condition is locally Minkowski provided that* $A^2 \neq 0$*.*

Theorem 3.3. **(i)** *The h-curvature tensor* R^r_{ijk}*, of an* $n(\geq 4)$*-dimensional S-3 like Landsberg space* M^n *with semi-*T'*-condition vanishes provided that* $B^2 \neq 0$*.* **(ii)** *An* $n(\geq 3)$*-dimensional S-3 like Berwald space* M^n *with semi-*T'*-condition is locally Minkowski provided that* $B^2 \neq 0$*.*

Corollary 3.4. **(i)** *The h-curvature tensor* R^r_{ijk} *of a three dimensional Landsberg space* M^3 *with semi-*T'*-condition vanishes provided that* $B^2 \neq 0$*.* **(ii)** *An three dimensional Berwald space* M^3 *with semi-*T'*-condition is locally Minkowski provided that* $B^2 \neq 0$*.*

Theorem 3.5. **(i)** *The h-curvature tensor* R^r_{ijk}*,of an* $n(\geq 5)$ *dimensional S-4 like Landsberg space* M^n *with semi-*T'*-condition vanishes provided that* $D^2 \neq 0$*.* **(ii)** *An* $n(\geq 5)$*dimensional S-4 like Berwald space*M^n *with semi-*T'*-condition is locally Minkowski provided that* $D^2 \neq 0$*.*

Corollary 3.6. **(i)** *The h-curvature tensor* R^r_{ijk} *of a four dimensional Landsberg space* M^4 *with semi-*T'*-condition vanishes provided that* $D^2 \neq 0$*.* **(ii)** *A four dimensional Berwald space* M^4 *with semi-*T'*-condition is locally Minkowski provided that* $D^2 \neq 0$*.*

4 Semi-C-reducible Finsler spaces

In this section semi-C-reducible Finsler spaces are considered. The $(h)hv$-torsion C_{ijk} of a semi-C-reducible Finsler space is written as

$$C_{ijk} = p[h_{ij}C_k + h_{jk}C_i + h_{ki}C_j] + q[C_iC_jC_k], \tag{4.1}$$

where p,q are functions satisfying $p(n+1) + qC^2 = 0$ and $C^2 = g_{ij}C^iC^j$ ([9],[5],[7]). The v-curvature tensor S_{ijkl} of a semi-C-reducible Finsler space can be written as

$$\begin{aligned} S_{ijkl} &= p^2C^2(h_{il}h_{jk} - h_{ik}h_{jl}) \\ &\quad +(p^2 + pqC^2)[h_{il}C_jC_k + h_{jk}C_iC_l - h_{ik}C_jC_l - h_{jl}C_iC_k]. \end{aligned} \tag{4.2}$$

In view of (4.2), equation (2.3) can be written as

$$R_{ijk}E_l - h_{il}E_rR^r_{jk} = 0, \tag{4.3}$$

where

$$E_r = L^2C^2p^2(n-1)C_r + A_r. \tag{4.4}$$

Applying Lemma 2.1, we have

Theorem 4.1. *Let* M^n *be an* $n(\geq 3)$*dimensional semi-*C*-reducible Finsler space with semi-*T'*-condition, then the* $(v)h$*-torsion tensor* R^i_{jk} *of* M^n *vanishes provided that* $E^2 \neq 0$*, where* $E^2 = g^{rs}E_rE_s$ *and* E_r *is given by (4.4).*

Theorem 4.2. **(i)** *Let M^n be an $n(\geq 3)$dimensional semi-C-reducible Landsberg space with semi-T'-condition, then its h-curvature tensor R^r_{ijk} also vanishes provided that $E^2 \neq 0$, where $E^2 = g^{rs}E_rE_s$ and E_r is given by (4.4).* **(ii)** *Let M^n be an $n(\geq 3)$dimensional semi-C-reducible Berwald space with semi-T'-condition, then M^n is locally Minkowski provided that $E^2 \neq 0$, where $E^2 = g^{rs}E_rE_s$ and E_r is given by (4.4).*

Further, since a C-reducible Landsberg space is Berwald ([5]) and a C-reducible Finsler space satisfies the semi-T'-condition, we have

Theorem 4.3. *Let M^n be an $n(\geq 3)$dimensional C-reducible Landsberg space with semi-T'-condition, then M^n is locally Minkowski provided that $E^2 \neq 0$.*

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