Position vectors of timelike biharmonic Legendre curves in the Lorentzian Heisenberg group $Heis^3$

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Abstract. In this paper, we study timelike biharmonic Legendre curves in the Lorentzian Heisenberg group $Heis^3$. We characterize the biharmonic curves in terms of their curvature and torsion. We prove that all of timelike biharmonic curves are helices. Moreover, we obtain the position vectors of timelike biharmonic Legendre curves in the Lorentzian Heisenberg group $Heis^3$. Also, by using the position vector, we give some characterizations for timelike biharmonic Legendre curves.

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1 Introduction

Harmonic maps $f : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds are the critical points of the energy

$$E(f) = \frac{1}{2}\int_M ||df||^2 v_g,$$

and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f) = \text{trace } \nabla df. \tag{1.1}$$

As suggested by Eells and Sampson in [4], we can define the bienergy of a map f by

$$E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 v_g,$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [7, 8], showing that the Euler–Lagrange equation associated to E_2 is

$$\tau_2(f) = -\mathcal{J}^f(\tau(f)) = -\Delta\tau(f) - \text{trace } R^N(df, \tau(f))\, df = 0,$$

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where $\mathcal{J}^f$ is the Jacobi operator of f . The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In the last decade there have been a growing interest in the theory of biharmonic functions, which can be divided into two main research directions. On one side, the differential geometric aspect has driven attention to the construction of examples and classification results. The other side is the analytic aspect from the point of view of PDE: biharmonic functions are solutions of a fourth order strongly elliptic semilinear PDE.

In [1] the authors completely classified the biharmonic submanifolds of the three-dimensional sphere, while in [2] there were given new methods to construct biharmonic submanifolds of codimension greater than one in the n-dimensional sphere. The biharmonic submanifolds into a space of nonconstant sectional curvature were also investigated. The proper biharmonic curves on Riemannian surfaces were studied in [3]. Inoguchi classified the biharmonic Legendre curves and the Hopf cylinders in three-dimensional Sasakian space forms in [6]. Then, Sasahara gave in [17] the explicit representation of the proper biharmonic Legendre surfaces in five-dimensional Sasakian space forms.

The second variation formula for biharmonic maps in spheres was deduced [14] and the stability of certain classes of biharmonic maps in spheres was discussed in [11]. Also, in [18] there were given some sufficient conditions for the instability of Legendre proper biharmonic submanifolds in Sasakian space forms and the author proved the instability of Legendre curves and surfaces in Sasakian space forms.

In this paper, we prove that the timelike biharmonic curves of Lorentzian Heisenberg group $Heis^3$ are helices. We obtain the position vectors of timelike biharmonic Legendre curves in the Lorentzian Heisenberg group $Heis^3$. Also, by using the position vectors of the curves, we give some characterizations of timelike biharmonic Legendre curves.

2 The Lorentzian Heisenberg group $Heis^3$

The Heisenberg group $Heis^3$ is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = (x + \bar{x}, y + \bar{y}, z + \bar{z} - \bar{x}y + x\bar{y}).$$

The identity of the group is $(0, 0, 0)$ and the inverse of (x, y, z) is given by $(-x, -y, -z)$. The left-invariant Lorentz metric on $Heis^3$ is

$$g = -dx^2 + dy^2 + (xdy + dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{ e_1 = \frac{\partial}{\partial z},\ e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial x} \right\}. \tag{2.1}$$

The characterizing properties of this algebra are the following commutation relations:

$$[e_2, e_3] = 2e_1, \quad [e_3, e_1] = 0, \quad [e_2, e_1] = 0,$$

with $g(e_1, e_1) = g(e_2, e_2) = 1$, $g(e_3, e_3) = -1$.

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & e_3 & e_2 \\ e_3 & 0 & e_1 \\ e_2 & -e_1 & 0 \end{pmatrix}, \tag{2.2}$$

where the (i, j)-element in the table above equals $\nabla_{e_i} e_j$ for our basis $\{e_1, e_2, e_3\}$.

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

The Riemannian curvature tensor is given by

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

Moreover, we put

$$R_{ijk} = R(e_i, e_j)e_k, \quad R_{ijkl} = R(e_i, e_j, e_k, e_l),$$

where the indices i, j, k and l take the values $1, 2$ and 3.

$$R_{121} = -e_2, \quad R_{131} = -e_3, \quad R_{232} = 3e_3$$

and

$$R_{1212} = -1, \quad R_{1313} = 1, \quad R_{2323} = -3. \tag{2.3}$$

3 Timelike biharmonic curves in the Lorentzian Heisenberg group $Heis^3$

Let $\gamma : I \longrightarrow Heis^3$ be a non geodesic timelike curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. Let $\{T, N, B\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $Heis^3$ along γ defined as follows:

T is the unit vector field γ' tangent to γ, N is the unit vector field in the direction of $\nabla_T T$ (normal to γ), and B is chosen so that $\{T, N, B\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{cases} \nabla_T T = \kappa_1 N \\ \nabla_T N = \kappa_1 T + \kappa_2 B \\ \nabla_T B = -\kappa_2 N, \end{cases} \tag{3.1}$$

where $\kappa_1 = |\kappa_2(\gamma)| = |\nabla_T T|$ is the curvature of γ and κ_2 is its torsion. With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ we can write

$$\begin{cases} T = T_1 e_1 + T_2 e_2 + T_3 e_3, \\ N = N_1 e_1 + N_2 e_2 + N_3 e_3, \\ B = T \times N = B_1 e_1 + B_2 e_2 + B_3 e_3. \end{cases}$$

Theorem 3.1. *$\gamma : I \longrightarrow Heis^3$ is a timelike biharmonic curve if and only if*

$$\begin{aligned} \kappa_1 &= \text{constant} \neq 0, \\ \kappa_1^2 - \kappa_2^2 &= 1 - 4B_1^2, \\ \kappa_2' &= 2N_1B_1. \end{aligned} \tag{3.2}$$

Proof. Using (3.1), we have

$$\begin{aligned} \tau_2(\gamma) &= \nabla_T^3 T - \kappa_1 R(T,N)T \\ &= (3\kappa_1'\kappa_1)T + (\kappa_1'' + \kappa_1^3 - \kappa_1\kappa_2^2)N + (2\kappa_2\kappa_1' + \kappa_1\kappa_2')B - \kappa_1 R(T,N)T. \end{aligned}$$

By (1.1), we see that γ is a biharmonic curve if and only if

$$\begin{aligned} \kappa_1\kappa_1' &= 0, \\ \kappa_1'' + \kappa_1^3 - \kappa_1\kappa_2^2 &= \kappa_1 R(T,N,T,N), \\ 2\kappa_2\kappa_1' + \kappa_2'\kappa_1 &= \kappa_1 R(T,N,T,B). \end{aligned}$$

Since $\kappa_1 \neq 0$ by the assumption that is non-geodesic

$$\begin{aligned} \kappa_1 &= \text{constant} \neq 0, \\ \kappa_1^2 - \kappa_2^2 &= R(T,N,T,N), \\ \kappa_2' &= R(T,N,T,B). \end{aligned} \tag{3.3}$$

A direct computation using (2.3), yields

$$\begin{aligned} R(T,N,T,N) &= 1 - 4B_1^2, \\ R(T,N,T,B) &= 2N_1B_1. \end{aligned}$$

These, together with (3.3), complete the proof of the theorem. □

Theorem 3.2. *Let $\gamma : I \longrightarrow Heis^3$ be a timelike curve with constant curvature. If $\kappa_2' \neq 0$, then γ is not biharmonic .*

Proof. We can use (2.2) to compute the covariant derivatives of the vector fields T, N and B as:

$$\begin{aligned} \nabla_T T &= T_1'e_1 + (T_2' + 2T_1T_3)e_2 + (T_3' + 2T_1T_2)e_3, \\ \nabla_T N &= (N_1' + T_2N_3 - T_3N_2)e_1 + (N_2' + T_1N_3 - T_3N_1)e_2 \\ &\quad + (N_3' + T_2N_1 - T_1N_2)e_3, \\ \nabla_T B &= (B_1' + T_2B_3 - T_3B_2)e_1 + (B_2' + T_1B_3 - T_3B_1)e_2 \\ &\quad + (B_3' + T_2B_1 - T_1B_2)e_3. \end{aligned}$$

It follows that the first components of these vectors are given by

$$\begin{cases} \langle \nabla_T T, e_1 \rangle = T_1', \\ \langle \nabla_T N, e_1 \rangle = N_1' + T_2N_3 - T_3N_2, \\ \langle \nabla_T B, e_1 \rangle = B_1' + T_2B_3 - T_3B_2. \end{cases} \tag{3.4}$$

On the other hand, using Frenet formulas (3.1), we have

$$\begin{cases} \langle \nabla_T T, e_1 \rangle = \kappa_1 N_1, \\ \langle \nabla_T N, e_1 \rangle = \kappa_1 T_1 + \kappa_2 B_1, \\ \langle \nabla_T B, e_1 \rangle = -\kappa_2 N_1. \end{cases} \tag{3.5}$$

These, together with (3.4) and (3.5), give

$$\begin{aligned} T_1' &= \kappa_1 N_1, \\ N_1' + T_2 N_3 - T_3 N_2 &= \kappa_1 T_1 + \kappa_2 B_1, \\ B_1' + T_2 B_3 - T_3 B_2 &= -\kappa_2 N_1. \end{aligned} \tag{3.6}$$

Assume that γ is biharmonic. Then using $\kappa_2' = 2N_1 B_1 \neq 0$ and (3.2), we obtain $-2\kappa_2'\kappa_2 = 8B_1 B_1'$, and $\kappa_2 N_1 B_1 = 2B_1 B_1'$. Then

$$\kappa_2 = \frac{2B_1'}{N_1}. \tag{3.7}$$

If we use $T_2 B_3 - T_3 B_2 = N_1$ and (3.6), we get $B_1' = (1 - \kappa_2) N_1$. Substituting B_1' in equation (3.7), we obtain $\kappa_2 = \frac{2}{3} = \text{constant}$.. Therefore also κ_2 is constant and we have a contradiction that is $\kappa_2' = N_1 B_1 \neq 0$. □

Corollary 3.3. $\gamma : I \longrightarrow Heis^3$ *is timelike biharmonic if and only if*

$$\begin{aligned} \kappa_1 &= \text{constant} \neq 0, \\ \kappa_2 &= \text{constant}, \\ N_1 B_1 &= 0, \\ \kappa_1^2 - \kappa_2^2 &= 1 - 4B_1^2. \end{aligned}$$

Corollary 3.4. *If* $N_1 \neq 0$, then γ *is not timelike biharmonic.*

Proof. We use the third equation of (3.6) we obtain $(1 - \kappa_2) N_1 = 0$. Using $N_1 \neq 0$, we have

$$\kappa_2 = 1. \tag{3.8}$$

Assume now that γ is biharmonic. If we substitue (3.8) in (3.2), we obtain

$$\kappa_1^2 = -4B_1^2. \tag{3.9}$$

By multipliying both sides of (3.9) with N_1, we obtain

$$\kappa_1^2 N_1 = -4B_1 (B_1 N_1).$$

Using (3.2) and $N_1 \neq 0$ we have $\kappa_1 = 0$. These, together with Theorem 3.1 complete the proof of the corollary. □

Corollary 3.5. *If* $N_1 = 0$*, then*

$$T(s) = \sinh \mu_0 e_1 + \cosh \mu_0 \sinh \Omega(s) e_2 + \cosh \mu_0 \cosh \Omega(s) e_3, \text{ where } \mu_0 \in \mathbb{R}. \tag{3.10}$$

Proof. Since γ is s parametrized by arc length, we can write

$$T(s) = \sinh\mu(s)e_1 + \cosh\mu(s)\sinh\Omega(s)e_2 + \cosh\mu(s)\cosh\Omega(s)e_3. \tag{3.11}$$

From (3.6), we obtain $T_1' = \kappa_1 N_1$. Since $N_1 = 0$, we have $T_1' = 0$. Then T_1 is constant. Using (3.11), we get $T_1 = \sinh\mu_0 =$ constant. We obtain (3.10) and the corollary is proved. □

4 Position vectors of a timelike biharmonic Legendre curve in $Heis^3$

A curve $\gamma : I \longrightarrow Heis^3$ is said to be Legendre if it is an integral curve of the contact distribution $\mathcal{D}$ =Kerω (equivalently, $\omega(\gamma') = 0$).

Lemma 4.1. *Let $\gamma : I \longrightarrow Heis^3$ be a timelike Legendre curve. Then,*

$$z'(s) + x(s)y'(s) = 0. \tag{4.1}$$

Proof. Using the orthonormal left-invariant frame (3.1), we have

$$\gamma'(s) = x'(s)\partial_x + y'(s)\partial_y + z'(s)\partial_z = x'(s)e_3 + y'(s)e_2 + \omega(\gamma'(s))\partial_z.$$

Then, $\gamma(s)$ is a timelike Legendre curve iff

$$\gamma'(s) = x'(s)e_3 + y'(s)e_2, \quad \omega(\gamma'(s)) = z'(s) + x(s)y'(s).$$

We obtain (4.1) and the Lemma is proved.

Lemma 4.2. *If $\gamma(s)$ is a timelike Legendre curve, then*

$$x'(s)e_3 + y'(s)e_2 = x'(s)\frac{\partial}{\partial x} + y'(s)\frac{\partial}{\partial y} - x(s)y'(s)\frac{\partial}{\partial z}.$$

Theorem 4.3. *Let $\gamma : I \longrightarrow Heis^3$ be a timelike biharmonic Legendre curve. Then the position vector of the curve $\gamma = \gamma(s)$ is:*

$$\gamma(s) = \left(\frac{1}{\zeta}\sinh(\zeta s + \sigma) + c_1, \frac{1}{\zeta}\cosh(\zeta s + \sigma) + c_2, \frac{s}{2\zeta} - \frac{1}{4\zeta}\sinh 2(\zeta s + \sigma\right) + c_3),$$

where $\zeta = \dfrac{\kappa_1 - \sinh 2\mu_0}{\cosh\mu_0}$ *and* $c_1, c_2, c_3, \sigma \in \mathbb{R}$.

Proof. If $\gamma : I \longrightarrow Heis^3$ is a timelike biharmonic Legendre curve, then we can write its position vector as follows:

$$\gamma(s) = x(s)\partial_x + y(s)\partial_y + z(s)\partial_z. \tag{4.2}$$

Differentiating the above equation with respect to s and by using the corresponding orthonormal the left-invariant frame (3.1), we find

$$\gamma'(s) = x'(s)e_3 + y'(s)e_2 + \omega(\gamma'(s))\partial_z.$$

The covariant derivative of the vector field T is:

$$\nabla_T T = T_1' e_1 + (T_2' + 2T_1T_3)e_2 + (T_3' + 2T_1T_2)e_3.$$

From (3.10), we have

$$\begin{aligned}\nabla_T T &= (\Omega' \cosh\mu_0 \cosh\Omega(s) + 2\sinh\mu_0\cosh\mu_0\cosh\Omega(s))e_2 + \\ &\quad (\Omega' \cosh\mu_0 \sinh\Omega(s) + 2\sinh\mu_0\cosh\mu_0\cosh\Omega(s))e_3.\end{aligned}$$

Since $|\nabla_T T| = \kappa_1$, we obtain

$$\Omega(s) = \left(\frac{\kappa_1 - \sinh 2\mu_0}{\cosh\mu_0}\right)s + \sigma, \text{ where } \sigma \in \mathbb{R}.$$

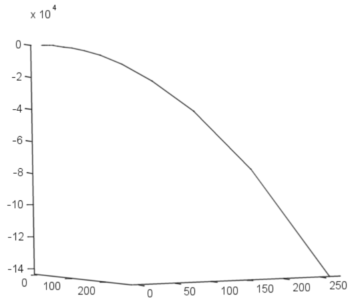
We choose $\zeta = \kappa_1 - \sinh 2\mu_0/\cosh\mu_0$. Using (3.10) and Lemma 4.1, we get

$$\begin{aligned} x'(s) &= \cosh(\zeta s + \sigma), \\ y'(s) &= \sinh(\zeta s + \sigma), \\ z'(s) + x(s)y'(s) &= \sinh\mu_0 = 0. \end{aligned} \tag{4.3}$$

If the system (4.3) is integrated, we obtain

$$\begin{aligned} x(s) &= \frac{1}{\zeta}\sinh(\zeta s + \sigma) + c_1, \\ y(s) &= \frac{1}{\zeta}\cosh(\zeta s + \sigma) + c_2, \\ z(s) &= \frac{s}{2\zeta} - \frac{1}{4\zeta}\sinh 2(\zeta s + \sigma) + c_3. \end{aligned}$$

Finally, the plot of the curve $\gamma(s)$ at $\sigma = c_1 = c_2 = c_3 = 0$ and $\zeta = 1$ has the form:



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