Warped product semi-slant submanifolds of trans-Sasakian manifolds

Siraj Uddin and Khalid Ali Khan

Abstract. In the present paper, we study warped product semi-slant submanifolds of trans-Sasakian manifolds. We have obtained results on the existence of warped product semi-slant submanifolds of trans-Sasakian manifolds in term of the canonical structure F.

M.S.C. 2000: 53C40, 53C42, 53B25.
Key words: Warped product; doubly warped product; slant submanifold; semi-slant submanifold; trans-Sasakian manifold; canonical structure.

1 Introduction

The notion of semi-slant submanifolds of almost Hermitian manifolds was introduced by N. Papaghuic [14]. In fact, semi-slant submanifolds in almost Hermitian manifolds are defined on the line of CR-submanifolds. In the setting of almost contact metric manifolds, semi-slant submanifolds are defined and investigated by J.L. Cabrerizo et. al [4].

In [13] J.A. Oubina studied a new class of almost contact metric structure, called *trans-Sasakian* which is, in some sense, an analogue of a locally conformal Kaehler structure on an almost Hermitian manifold.

On the other hand, in Gray-Hervella classification of almost Hermitian manifolds [7], there appears a class $\mathcal{W}_4$, of Hermitian manifolds which are closely related to a locally conformal Kaehler manifolds. An almost contact metric structure on a manifold $\bar{M}$ is called *trans-Sasakian* structure [13] if the product manifold $\bar{M} \times \mathbb{R}$ belongs to class $\mathcal{W}_4$. In [6], the authors introduced two subclasses of trans-Sasakian structure, the $\mathcal{C}_5$ and $\mathcal{C}_6$-structures, which contains the Kenmotsu and Sasakian structures, respectively. The class $\mathcal{C}_6 \oplus \mathcal{C}_5$ [11] coincides with the class of trans-Sasakian of type (α, β). We note that the trans-Sasakian structures of type $(0,0), (0,\beta)$ and $(\alpha, 0)$ are cosymplectic [3], $\beta-$Kenmotsu and $\alpha-$Sasakian [8], respectively.

Warped product manifolds were introduced by R.L. Bishop and B. O'Neill in [2]. The problem of existence or non-existence of warped product manifolds plays some important role in differential geometry and physics. The study of warped product semi-slant submanifolds of Kaehler manifolds was introduced by B. Sahin [15]. Later, K.A. Khan et.al studied warped product semi-slant submanifolds in cosymplectic

Differential Geometry - Dynamical Systems, Vol.12, 2010, pp. 260-270.

manifolds and showed that there exist no proper warped product semi-slant submanifolds in the forms $N_T \times {}_f N_\theta$ and reversing the two factors in cosymplectic manifolds [9].

Recently, M. Atçeken proved that the warped product submanifolds of the types $M = N_\theta \times {}_f N_T$ and $M = N_\theta \times {}_f N_\perp$ of a Kenmotsu manifold $\bar{M}$ do not exist where the manifolds N_θ and N_T (resp. $N_\perp$) are proper slant and $\phi-$invariant (resp. anti-invariant) submanifolds of a Kenmotsu manifold $\bar{M}$, respectively [1]. In this paper, we have obtained some results for the existence of warped product semi-slant submanifolds of trans-Sasakian manifolds.

2 Preliminaries

Let $\bar{M}$ be a $(2m+1)-$dimensional manifold with almost contact structure (ϕ, ξ, η) is defined on $\bar{M}$ by a $(1,1)$ tensor field ϕ, a vector field ξ and the dual $1-$form η of ξ, satisfying the following properties [3]

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1. \tag{2.1}$$

There always exists a Riemannian metric g on an almost contact manifold $\bar{M}$ satisfying the following compatibility condition

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

An almost contact structure (ϕ, ξ, η) is said to be *normal* if the almost complex structure J on the product manifold $\bar{M} \times \mathbb{R}$ given by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \ \eta(X)\frac{d}{dt}),$$

has no torsion i.e., J is integrable where f is a $C^\infty-$function on $\bar{M} \times \mathbb{R}$, in other words the tensor $[\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically on $\bar{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ [17].

An almost contact metric manifold $\bar{M}$ is called *trans-Sasakian* [13] if

$$(\bar{\nabla}_X \phi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}. \tag{2.3}$$

for all $X, Y \in T\bar{M}$, where $\alpha,\ \beta$ are smooth function on $\bar{M}$ and $\bar{\nabla}$ is the Levi-Civita connection of g and in this case we say that the *trans-Sasakian structure* is of type (α, β).

If $\alpha = 0$ then $\bar{M}$ is $\beta-$*Kenmotsu* manifold and if $\beta = 0$ then $\bar{M}$ is $\alpha-$*Sasakian* manifold. Moreover, if $\alpha = 1$ and $\beta = 0$ then $\bar{M}$ is a *Sasakian* manifold and if $\alpha = 0$ and $\beta = 1$ then $\bar{M}$ is a *Kenmotsu* manifold. From (2.3), it follows that

$$\bar{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{2.4}$$

Let M be submanifold of an almost contact metric manifold $\bar{M}$ with induced metric g and if ∇ and $\nabla^\perp$ are the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M, respectively then Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.5}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.6}$$

for each $X,\ Y \in TM$ and $N \in T^{\perp}M$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into $\bar{M}$. They are related as [17]

$$g(h(X,Y),N) = g(A_N X, Y), \tag{2.7}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as induced on M.

For any $X \in TM$, we write

$$\phi X = PX + FX, \tag{2.8}$$

where PX is the tangential component and FX is the normal component of ϕX.

Similarly, for any $N \in T^{\perp}M$, we write

$$\phi N = BN + CN, \tag{2.9}$$

where BN is the tangential component and CN is the normal component of ϕN. The covariant derivatives of the tensor fields P and F are defined as

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y, \tag{2.10}$$

$$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y \tag{2.11}$$

for all $X,\ Y \in TM$. The canonical structures P and F on a submanifold M are said to be *parallel* if $\nabla P = 0$ and $\bar{\nabla} F = 0$, respectively.

We shall always consider ξ to be tangent to M. The submanifold M is said to be *invariant* if F is identically zero, that is, $\phi X \in TM$ for any $X \in TM$. On the other hand M is said to be *anti-invariant* if P is identically zero, that is, $\phi X \in T^{\perp}M$, for any $X \in TM$.

For each non zero vector X tangent to M at x, such that X is not proportional to ξ, we denote by $\theta(X)$, the angle between ϕX and PX.

M is said to be *slant* [5] if the angle $\theta(X)$ is constant for all $X \in TM - \{\xi\}$ and $x \in M$. The angle θ is called *slant angle* or *Wirtinger angle*. Obviously if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is an anti-invariant submanifold. If the slant angle of M is different from 0 and $\pi/2$ then it is called *proper slant*.

A characterization of slant submanifolds is given by the following

Theorem 2.1 [5]. *Let M be a submanifold of an almost contact metric manifold $\bar{M}$, such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{2.12}$$

Furthermore, if θ is slant angle, then $\lambda = \cos^2\theta$.

The following relations are straightforward consequences of equation (2.12):

$$g(PX, PY) = \cos^2\theta[g(X,Y) - \eta(X)\eta(Y)] \tag{2.13}$$

$$g(FX, FY) = \sin^2\theta[g(X, Y) - \eta(X)\eta(Y)] \tag{2.14}$$

for any X, Y tangent to M.

The study of semi-slant submanifolds of almost Hermitian manifolds was introduced by N. Papaghuic [14], which were latter extended to almost contact metric manifold by J.L. Cabrerizo et.al [4]. We say M is a *semi-slant submanifold* of $\bar{M}$ if there exist an orthogonal direct decomposition of TM as

$$TM = D_1 \oplus D_2 \oplus \{\xi\}$$

where D_1 is an invariant distribution i.e., $\phi(D_1) = D_1$ and D_2 is slant with slant angle $\theta \neq 0$. The orthogonal complement of $F\mathcal{D}_2$ in the normal bundle $T^{\perp}M$, is an invariant subbundle of $T^{\perp}M$ and is denoted by μ. Thus, we have

$$T^{\perp}M = F\mathcal{D}_2 \oplus \mu.$$

Similarly, we say that M is *anti-slant submanifold* of $\bar{M}$ if D_1 is an anti-invariant distribution of M i.e., $\phi D_1 \subseteq T^{\perp}M$ and D_2 is slant with slant angle $\theta \neq 0$.

3 Warped and doubly warped product submanifolds

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and f, a positive differentiable function on N_1. The warped product of N_1 and N_2 is the Riemannian manifold $N_1 \times {}_f N_2 = (N_1 \times N_2, g)$, where

$$g = g_1 + f^2 g_2. \tag{3.1}$$

A warped product manifold $N_1 \times {}_f N_2$ is said to be *trivial* if the warping function f is constant.We recall the following general formula on a warped product [2].

$$\nabla_X V = \nabla_V X = (X \ln f)V, \tag{3.2}$$

where X is tangent to N_1 and V is tangent to N_2.

Let $M = N_1 \times {}_f N_2$ be a warped product manifold, this means that N_1 is totally geodesic and N_2 is totally umbilical submanifold of M, respectively.

Doubly warped product manifolds were introduced as a generalization of warped product manifolds by B. Ünal [16]. A *doubly warped product manifold* of N_1 and N_2, denoted as ${}_{f_2}N_1 \times {}_{f_1}N_2$ is endowed with a metric g defined as

$$g = f_2^2 g_1 + f_1^2 g_2 \tag{3.3}$$

where f_1 and f_2 are positive differentiable functions on N_1 and N_2 respectively.

In this case formula (3.2) is generalized as

$$\nabla_X V = (X \ln f_1)V + (V \ln f_2)X \tag{3.4}$$

for each $X \in TN_1$ and $V \in TN_2$ [12].

If neither f_1 nor f_2 is constant we have a non trivial doubly warped product $M = {}_{f_2}N_1 \times {}_{f_1}N_2$. Obviously in this case both N_1 and N_2 are totally umbilical submanifolds of M.

Theorem 3.1. *There do not exist proper doubly warped product submanifolds $M = {}_{f_2}N_1 \times {}_{f_1}N_2$ of a trans-Sasakian manifold $\bar{M}$ where N_1 and N_2 are any Riemannian submanifolds of $\bar{M}$.*

Proof. For any $X \in TN_1$ and $Z \in TN_2$ then by (3.4) we have

$$\nabla_Z X = \nabla_X Z = (X \ln f_1)Z + (Z \ln f_2)X.$$

If $\xi \in TN_1$ then above equation gives

$$\nabla_Z \xi = (\xi \ln f_1)Z + (Z \ln f_2)\xi. \tag{3.5}$$

On the other hand, using (2.4) and the fact that ξ is tangent to N_1 we have

$$\bar{\nabla}_Z \xi = -\alpha\phi Z + \beta Z.$$

By (2.5) and (2.8), we get

$$\nabla_Z \xi + h(Z, \xi) = -\alpha PZ + \beta Z - \alpha FZ.$$

Using (3.5) and then comparing the tangential and normal components we obtain

$$(\xi \ln f_1)Z + (Z \ln f_2)\xi = -\alpha PZ + \beta Z \tag{3.6}$$

and

$$h(Z, \xi) = -\alpha FZ.$$

Taking product with Z in (3.6) and using the fact that ξ, Z and PZ are orthogonal vector fields then

$$\xi \ln f_1 = \beta, \quad Z \ln f_2 = 0.$$

This shows that f_2 is constant.

Similarly, if the structure vector field ξ is tangent to N_2 and for any $X \in TN_1$ we obtain

$$\xi \ln f_2 = \beta, \quad X \ln f_1 = 0,$$

showing that f_1 is constant. This completes the proof. □

The following corollary is an immediate consequence of the above theorem:

Corollary 3.1. *There do not exist warped product submanifolds $M = N_1 \times {}_fN_2$ of a trans-Sasakian manifold $\bar{M}$ such that $\xi \in TN_2$, where N_1 and N_2 are any Riemannian submanifolds of $\bar{M}$.*

Thus, the only remaining case to study of warped product submanifolds $N_1 \times {}_fN_2$ of trans-Sasakian manifolds is that the structure vector field ξ tangential to the first factor i.e., $\xi \in TN_1$. In this case, first we obtain some useful formulae for later use.

Lemma 3.1. *Let $M = N_1 \times {}_fN_2$ be warped product submanifolds of a trans-Sasakian manifold $\bar{M}$ such that ξ is tangent to N_1, where N_1 and N_2 are any Riemannian submanifolds of $\bar{M}$. Then*

(i) $\xi \ln f = \beta$,

(ii) $A_{FZ}X = -Bh(X, Z)$,

(iii) $g(h(X,Y),FZ) = g(h(X,Z),FY)$,

(iv) $g(h(X,Z),FW) = g(h(X,W),FZ)$

for any $X,\ Y \in TN_1$ *and* $Z,\ W \in TN_2$ *where* β *is a smooth function on* $\bar{M}$.

Proof. First part follows by Theorem 3.1. Now, for any $X \in TN_1$ and $Z \in TN_2$ we have

$$(\bar{\nabla}_X \phi)Z = \bar{\nabla}_X \phi Z - \phi \bar{\nabla}_X Z.$$

Using (2.3) and the fact that $\xi \in TN_1$, left hand side of the above equation is zero by orthogonality of two distributions, then

$$\bar{\nabla}_X \phi Z = \phi \bar{\nabla}_X Z.$$

By (2.5), (2.6), (2.8) and (2.9) we obtain

$$\nabla_X PZ + h(X,PZ) - A_{FZ}X + \nabla^{\perp}_X FZ = P\nabla_X Z + F\nabla_X Z + Bh(X,Z) + Ch(X,Z).$$

Equating the tangential and normal components and using (3.2), we get

$$A_{FZ}X = -Bh(X,Z) \tag{3.7}$$

and

$$\nabla^{\perp}_X FZ = (X \ln f)FZ + Ch(X,Z) - h(X,PZ). \tag{3.8}$$

Equation (3.7) follows part (ii). Parts (iii) and (iv) follow by taking the product in (ii) with $Y \in TN_1$ and $W \in TN_2$, respectively. □

In the following section we shall investigate warped product semi-slant submanifolds of trans-Sasakian manifolds.

4 Warped product semi-slant submanifolds

We have seen that the warped products of the type $N_1 \times {}_f N_2$ of trans-Sasakian manifolds do not exist if the structure vector field ξ is tangent to N_2. Thus, in this section we study warped product semi-slant submanifolds $N_1 \times {}_f N_2$ of trans-Sasakian manifolds only when $\xi \in TN_1$. If the manifolds N_θ and N_T (resp. $N_\perp$) are slant and invariant (resp. anti-invariant) submanifolds of a trans-Sasakian manifold $\bar{M}$, then their warped product semi-slant submanifolds may given by one of the following forms:

(i) $N_T \times {}_f N_\theta$, (ii) $N_\perp \times {}_f N_\theta$,

(iii) $N_\theta \times {}_f N_T$, (iv) $N_\theta \times {}_f N_\perp$.

For the warped products of the type (i), we have

Theorem 4.1. *Let* $M = N_T \times {}_f N_\theta$ *be warped product semi-slant submanifolds of a trans-Sasakian manifold* $\bar{M}$ *such that* ξ *is tangent to* N_T. *Then* $(\bar{\nabla}_X F)Z$ *lies in the invariant normal subbundle for all* $X \in TN_T$ *and* $Z \in TN_\theta$ *where* N_T *and* N_θ *are invariant and proper slant submanifolds of* $\bar{M}$.

Proof. For any $X \in TN_T$ and $Z \in TN_\theta$ we have

$$g(\phi\bar{\nabla}_X Z, \phi Z) = g(\bar{\nabla}_X Z, Z).$$

Then from (3.2), we obtain

$$g(\phi\bar{\nabla}_X Z, \phi Z) = (X \ln f)\|Z\|^2. \tag{4.1}$$

On the other hand we have

$$(\bar{\nabla}_X \phi)Z = \bar{\nabla}_X \phi Z - \phi\bar{\nabla}_X Z,$$

for any $X \in TN_T$ and $Z \in TN_\theta$. Using (2.3) and the fact that ξ is in TN_T, left hand side of the above equation is zero, then

$$\phi\bar{\nabla}_X Z = \bar{\nabla}_X \phi Z.$$

Taking the product with ϕZ and then using (2.8), we get

$$g(\phi\bar{\nabla}_X Z, \phi Z) = g(\bar{\nabla}_X (PZ + FZ), PZ + FZ).$$

Then on applying (2.5) and (2.6) we obtain

$$\begin{aligned} g(\phi\bar{\nabla}_X Z, \phi Z) &= g(\nabla_X PZ, PZ) + g(h(X, PZ), FZ) \\ &-g(A_{FZ}X, PZ) + g(\nabla^{\perp}_X FZ, FZ). \end{aligned}$$

Thus from (2.7) and (3.2) we have

$$g(\phi\bar{\nabla}_X Z, \phi Z) = (X \ln f)g(PZ, PZ) + g((\bar{\nabla}_X F)Z, FZ) + (X \ln f)g(FZ, FZ).$$

On using (2.13) and (2.14), we get

$$g(\phi\bar{\nabla}_X Z, \phi Z) = (X \ln f)\cos^2\theta\|Z\|^2 + g((\bar{\nabla}_X F)Z, FZ) + (X \ln f)\sin^2\theta\|Z\|^2,$$

or

$$g(\phi\bar{\nabla}_X Z, \phi Z) = (X \ln f)\|Z\|^2 + g((\bar{\nabla}_X F)Z, FZ). \tag{4.2}$$

By equations (4.1) and (4.2), it follows that

$$g((\bar{\nabla}_X F)Z, FZ) = 0. \tag{4.3}$$

for any $X \in TN_T$ and $Z \in TN_\theta$. Since N_θ is a proper slant submanifold of $\bar{M}$, then (4.3) implies $(\bar{\nabla}_X F)Z \in \mu$. The proof is complete. □

For the warped product of the type $N_\perp \times {}_f N_\theta$ of a trans-Sasakian manifold $\bar{M}$ such that $\xi \in TN_\perp$, we have the following theorem

Theorem 4.2. *Let $M = N_\perp \times {}_f N_\theta$ be warped product semi-slant submanifolds of a trans-Sasakian manifold $\bar{M}$ such that ξ is tangent to $N_\perp$. Then*

$$Z \ln f = \beta\eta(Z) \tag{4.4}$$

for any $Z \in TN_\perp$ where β is a smooth function on $\bar{M}$ and $N_\perp$ and N_θ are anti-invariant and proper slant submanifolds of $\bar{M}$, respectively.

Proof. For any $X \in TN_\theta$ and $Z \in TN_\perp$ we have

$$(\bar{\nabla}_X \phi)Z = \bar{\nabla}_X \phi Z - \phi \bar{\nabla}_X Z.$$

On using (2.5), (2.6), (2.8), (2.9) and the structure equation of trans-Sasakian and the fact that $\xi \in TN_\perp$, we obtain

$$\begin{aligned} -\eta(Z)\alpha X - \beta\eta(Z)PX - \beta\eta(Z)FX = -A_{FZ}X + \nabla_X^\perp FZ - P\nabla_X Z \\ -F\nabla_X Z - Bh(X,Z) - Ch(X,Z). \end{aligned}$$

Equating the tangential components and using (3.2), we get

$$\eta(Z)\alpha X + \beta\eta(Z)PX = A_{FZ}X + (Z\ln f)PX + Bh(X,Z). \tag{4.5}$$

Taking the product with PX in (4.5) and using the fact that X and PX are mutually orthogonal vector fields, then

$$\beta\eta(Z)g(PX,PX) = g(A_{FZ}X,PX) + (Z\ln f)g(PX,PX) + g(Bh(X,Z),PX).$$

Thus from (2.7) and (2.13) we get

$$\{\beta\eta(Z) - (Z\ln f)\}\cos^2\theta\|X\|^2 = g(h(X,PX),FZ) - g(h(X,Z),FPX). \tag{4.6}$$

As N_θ is proper slant, interchanging X by PX in (4.6) and taking account of equation (2.12), we deduce that

$$\{\beta\eta(Z) - (Z\ln f)\}\cos^2\theta\|X\|^2 = -g(h(X,PX),FZ) + g(h(PX,Z),FX). \tag{4.7}$$

On adding (4.6) and (4.7), we obtain

$$2\{\beta\eta(Z) - (Z\ln f)\}\cos^2\theta\|X\|^2 = g(h(PX,Z),FX) - g(h(X,Z),FPX). \tag{4.8}$$

Thus by Lemma 3.1 (*iv*), the right hand side of the above equation is zero then we have

$$\{\beta\eta(Z) - (Z\ln f)\}\cos^2\theta\|X\|^2 = 0.$$

Since N_θ is proper slant and X is non-null, then $Z\ln f = \beta\eta(Z)$. Hence the theorem is proved. □

Now, the case (iii) is dealt with the following

Proposition 4.1. *Let $\bar{M}$ be a $(2n+1)-$dimensional trans-Sasakian manifold and $M = N_\theta \times_f N_T$ be warped product semi-slant submanifolds of $\bar{M}$ such that ξ is tangent to N_θ. Then*

$$g((\bar{\nabla}_X F)Z, FZ) = \sec^2\theta g((\bar{\nabla}_X F)PZ, FPZ) \tag{4.9}$$

for any $X \in TN_T$ and $Z \in TN_\theta$ where N_T and N_θ are invariant and proper slant submanifolds of $\bar{M}$, respectively.

Proof. Assume $M = N_\theta \times {}_f N_T$ be warped product semi-slant submanifolds of a trans-Sasakian manifold $\bar{M}$ such that ξ is tangent to N_θ. Then for any $X \in TN_T$ and $Z \in TN_\theta$ we have

$$(\bar{\nabla}_X \phi)Z = \bar{\nabla}_X \phi Z - \phi \bar{\nabla}_X Z.$$

Since $\xi \in TN_\theta$ then from (2.3), (2.5), (2.6), (2.8) and (2.9), we get

$$\begin{aligned} -\eta(Z)\alpha X - \beta\eta(Z)\phi X &= \nabla_X PZ + h(X, PZ) - A_{FZ}X + \nabla^\perp_X FZ \\ &-P\nabla_X Z - F\nabla_X Z - Bh(X, Z) - Ch(X, Z). \end{aligned}$$

On comparing the tangential and normal parts we have

$$-\eta(Z)\alpha X - \beta\eta(Z)\phi X = \nabla_X PZ - A_{FZ}X - P\nabla_X Z - Bh(X, Z) \tag{4.10}$$

and

$$(\bar{\nabla}_X F)Z = Ch(X, Z) - h(X, PZ). \tag{4.11}$$

Taking the product with FZ in (4.11) we have

$$\begin{aligned} g((\bar{\nabla}_X F)Z, FZ) &= g(Ch(X, Z), FZ) - g(h(X, PZ), FZ) = \\ &= g(\phi h(X, Z), \phi Z) - g(Bh(X, Z), PZ) - g(h(X, PZ), \phi Z). \end{aligned}$$

That is,

$$g((\bar{\nabla}_X F)Z, FZ) = -g(Bh(X, Z), PZ) + g(Bh(X, PZ), Z). \tag{4.12}$$

As $\theta \neq \frac{\pi}{2}$, then substituting Z by PZ in (4.12) and using (2.12) we obtain

$$g((\bar{\nabla}_X F)PZ, FPZ) = \cos^2\theta\{-g(Bh(X, Z), PZ) + g(Bh(X, PZ), Z)\}.$$

Using (4.12), we get

$$g((\bar{\nabla}_X F)PZ, FPZ) = \cos^2\theta g((\bar{\nabla}_X F)Z, FZ).$$

This proves our assertion. □

The case (iv) is dealt in the following Theorem.

Theorem 4.3. *Let $M = N_T \times {}_f N_\perp$ be warped product submanifold of a trans-Sasakian manifold $\bar{M}$ such that N_T an invariant submanifold tangent to ξ and $N_\perp$ is an anti-invariant submanifold of $\bar{M}$. Then $(\bar{\nabla}_X F)Z$ lies in the invariant normal subbundle for each $X \in TN_T$ and $Z \in TN_\perp$.*

Proof. As $M = N_T \times {}_f N_\perp$ be warped product submanifold with ξ is tangent to N_T, then for any $X \in TN_T$ and $Z \in TN_\perp$ we have

$$(\bar{\nabla}_X \phi)Z = \bar{\nabla}_X \phi Z - \phi \bar{\nabla}_X Z.$$

On using (2.3) and the fact that ξ tangential to N_T the left hand side of the above equation is zero. Thus, we have

$$\bar{\nabla}_X \phi Z = \phi \bar{\nabla}_X Z.$$

Then from (2.5) and (2.6) we obtain

$$-A_{FZ}X + \nabla^{\perp}_{X}FZ = \phi(\nabla_X Z + h(X, Z)).$$

Which on using (2.8) and (2.9) yields

$$-A_{FZ}X + \nabla^{\perp}_{X}FZ = P\nabla_X Z + F\nabla_X Z + Bh(X, Z) + Ch(X, Z).$$

From the normal components of the above equation and formula (3.2) gives

$$\nabla^{\perp}_{X}FZ = (X \ln f)FZ + Ch(X, Z). \tag{4.13}$$

Taking the product in (4.13) with FW_1 for any $W_1 \in TN_{\perp}$, we get

$$g(\nabla^{\perp}_{X}FZ, FW_1) = (X \ln f)g(FZ, FW_1) + g(Ch(X, Z), FW_1)$$

or,

$$g(\nabla^{\perp}_{X}FZ, FW_1) = (X \ln f)g(\phi Z, \phi W_1) + g(\phi h(X, Z), \phi W_1).$$

Then from (2.2) we have

$$g(\nabla^{\perp}_{X}FZ, FW_1) = (X \ln f)g(Z, W_1). \tag{4.14}$$

On the other hand for any $X \in TN_T$ and $Z \in TN_{\perp}$ we have

$$(\bar{\nabla}_X F)Z = \nabla^{\perp}_{X}FZ - F\nabla_X Z.$$

Taking the product with FW_1, for any $W_1 \in TN_{\perp}$ and using (3.2), we get

$$g((\bar{\nabla}_X F)Z, FW_1) = g(\nabla^{\perp}_{X}FZ, FW_1) - (X \ln f)g(Z, W_1). \tag{4.15}$$

Equations (4.14) and (4.15), it follows that

$$g((\bar{\nabla}_X F)Z, FW_1) = 0, \tag{4.16}$$

for any $X \in TN$ and Z, $W_1 \in TN_{\perp}$. Now, if $W_2 \in TN_T$ then using the formula (2.11), we get

$$g((\bar{\nabla}_X F)Z, \phi W_2) = g(\nabla^{\perp}_{X}FZ, \phi W_2) - g(F\nabla_X Z, \phi W_2).$$

As N_T is an invariant submanifold then $\phi W_2 \in TN_T$ for any $W_2 \in TN_T$, thus using the fact that the product of tangential component with normal is zero, we obtain that

$$g((\bar{\nabla}_X F)Z, \phi W_2) = 0, \tag{4.17}$$

for any $X, W_2 \in TN_T$ and $Z \in TN_{\perp}$. Thus from equations (4.16) and (4.17) it follows that $(\bar{\nabla}_X F)Z \in \mu$. This proves the theorem completely. □

Acknowledgement. The authors are thankful to the referee for his valuable suggestions and comments.

References

[1] M. Atçeken, *Warped product semi-slant submanifolds in Kenmotsu manifolds*, Turk J. Math. 33 (2009), 1-8.

[2] R.L. Bishop and B. O'Neill, *Manifolds of Negative curvature,* Trans. Amer. Math. Soc. 145 (1969), 1-49.

[3] D.E. Blair, *Contact manifolds in Riemannian geometry,* Lecture Notes in Mathematics, Vol. 509. Springer-Verlag, New York, 1976.

[4] J.L. Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez, *Semi-slant submanifolds of a Sasakian manifold*, Geom. Dedicata 78 (1999), 183-199.

[5] J.L. Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez, *Slant submanifolds in Sasakian manifolds*, Glasgow Math. J. 42 (2000), 125-138.

[6] D. Chinea and C. Gonzalez, *A classification of almost contact metric manifolds*, Ann. Mat. Pura Appl. 156 (1990), 15-36.

[7] A. Gray and L.M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. 123 (1980), 35-58.

[8] D. Janssens and L. Vanhecke, *Almost contact structure and curvature tensors*, Kodai Math. J. 4 (1981), 1-27.

[9] K.A. Khan, V.A. Khan and Sirajuddin, *Warped product submanifolds of cosymplectic manifolds*, Balkan J. Geom. Appl. 13 (2008), 55-65.

[10] A. Lotta, *Slant submanifolds in contact geometry*, Bull. Math. Soc. Roumanie 39 (1996), 183-198.

[11] J.C. Marrero, *The local structure of trans-Sasakian manifolds*, Ann. Mat. Pura Appl. 162 (1992), 77-86.

[12] M.I. Munteanu, *A note on doubly warped contact CR-submanifolds in trans-Sasakian manifolds*, Acta Math. Hungar. 16 (2007), 121-126.

[13] J. Oubina, *New classes of almost contact metric structures*, Publ. Math. Debrecen 32 (1985), 187-193.

[14] N. Papaghiuc, *Semi-slant submanifolds of a Kaehlerian manifold*, An. Stiint. Al. I. Cuza. Univ. Iasi. 40 (1994), 55-61.

[15] B. Sahin, *Nonexistence of warped product semi-slant submanifolds of Kaehler manifold*, Geom. Dedicata 117 (2006), 195-202.

[16] B. Ünal, *Doubly warped products*, Diff. Geom. Appl. 15 (2001), 253-263.

[17] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics, World Scientific Publishing Co., Singapore, 1984.

Authors' addresses:

Siraj Uddin
Institute of Mathematical Sciences, Faculty of Science,
University of Malaya, 50603 Kuala Lumpur, Malaysia.
E-mail: siraj.ch@gmail.com

Khalid Ali Khan
School of Engineering & Logistics, Faculty of Technology,
Charles Darwin University, NT-0909, Australia.
E-mail: khalid.mathematics@gmail.com