Non-existence of contact totally umbilical proper slant submanifolds of a cosymplectic manifold

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Abstract. In this paper, we prove that there do not exist contact totally umbilical proper slant submanifolds of cosymplectic manifolds.

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1 Introduction

The notion of a slant submanifold of an almost Hermitian manifold was introduced by Chen ([4], [5]). On the other hand, A. Lotta [6] has defined and studied slant submanifolds of an almost contact metric manifold. He has also studied the intrinsic geometry of 3-dimensional non-anti-invariant slant submanifolds of K-Contact manifolds [7]. Later, L. Cabrerizo et al [3] investigated slant submanifolds of a Sasakian manifold and obtained many interesting results. Afterwards, we have also studied slant submanifolds of Cosymplectic manifolds, Kenmotsu manifolds and trans-Sasakian manifolds ([12], [10], [8], [9], [11]).

2 Preliminaries

An odd-dimensional Riemannian manifold $\bar{M}$ is said to be an almost contact metric manifold if there exist structure tensors $\{\varphi, \xi, \eta, g\}$, where φ is a (1,1) tensor field, ξ a vector field, η a 1-form and g is the Riemannian metric on $\bar{M}$ satisfying

$$\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\varphi X) = 0 \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{2.2}$$

for any $X, Y \in T\bar{M}$, where $T\bar{M}$ denotes the Lie algebra of vector fields on $\bar{M}$. An almost contact metric manifold is called a cosymplectic manifold if [1]

$$(\overline{\nabla}_X \varphi)Y = 0 \qquad and \qquad \overline{\nabla}_X \xi = 0 \tag{2.3}$$

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where $\overline{\nabla}$ denotes the Levi-Civita connection on $\bar{M}$.

Let M be an m-dimensional Riemannian manifold with induced metric g isometrically immersed in an n-dimensional Cosymplectic manifold $\bar{M}$. We denote by TM the Lie algebra of vector fields on M and by $TM^{\perp}$ the set of all vector fields normal to M. For any $X \in TM$ and $N \in TM^{\perp}$, we write

$$\varphi X = PX + FX, \quad \varphi N = tN + fN \tag{2.4}$$

where PX (resp. FX) denotes the tangential (resp. normal) component of φX, and tN (resp. fN) denotes the tangential (resp. normal) component of φN. In view of (2.4), we can have

$$T_x\overline{M} = T_xM \oplus F(T_xM) \oplus \mu_x$$

where μ_x is orthogonal complement to $F(T_xM)$ in $T_xM^{\perp}$.

In what follows, we suppose that the structure vector field ξ is tangent to M. Hence if we denote by D the orthogonal distribution to ξ in TM, we can consider the orthogonal direct decomposition $TM = D \oplus \xi$.

For each non zero X tangent to M at x such that it is not proportional to ξ_x, we denote by $\theta(X)$ the Wirtinger angle of X, that is, the angle between φX and T_xM.

The submanifold M is called slant if the Wirtinger angle $\theta(X)$ is a constant, which is independent of the choice of $x \in M$ and $X \in T_xM - \{\xi_x\}$ ([6]). The Wirtinger angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle θ equal to 0 and $\frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion.

Let ∇ be the Riemannian connection on M. Then the Gauss and Weingarten formulae are

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.5}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.6}$$

for $X, Y \in TM$ and $N \in TM^{\perp}$; h and A_N are the second fundamental forms related by

$$g(A_N X, Y) = g(h(X,Y), N) \tag{2.7}$$

where $\nabla^{\perp}$ is the connection in the normal bundle $TM^{\perp}$.

Similar to the concept of contact totally umbilical submanifold of a Sasakian manifold introduced in the book of Yano and Kon (cf. [13, page 374]), we define:

Definition 2.1. If the second fundamental form h of a submanifold M, tangent to the structure vector field ξ, of a cosymplectic manifold, is of the form

$$h(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]\alpha \tag{2.8}$$

for any $X, Y \in TM$, where α is a vector field normal to M, then M is called contact totally umbilical. Further if $\alpha = 0$, then M is called totally geodesic.

The mean curvature vector H is defined by $H = (\frac{1}{m})\, trace\; h$. We say that M is minimal if H vanishes identically.

We mention the following results for later use:

Theorem A. [3] *Let M be a submanifold of an almost contact metric manifold $\bar{M}$ such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = -\lambda(I - \eta \otimes \xi) \tag{2.9}$$

Furthermore, if θ is the slant angle of M, then $\lambda = cos^2\theta$.

Corollary B. [3] *Let M be a submanifold of an almost contact metric manifold $\bar{M}$ with slant angle θ. Then for any $X, Y \in TM$, we have*

$$g(PX, PY) = cos^2\theta(g(X,Y) - \eta(X)\eta(Y)) \tag{2.10}$$

$$g(FX, FY) = sin^2\theta(g(X,Y) - \eta(X)\eta(Y)) \tag{2.11}$$

Let M be an m-dimensional proper slant submanifold of an n-dimensional cosymplectic manifold $\bar{M}$. Then $F(T_xM)$ is a subspace of $T_xM^{\perp}$. For $x \in M$, there exists an invariant subspace μ_x of $T_x\overline{M}$ such that in view of (2.4), we can have

$$T_x\overline{M} = T_xM \oplus F(T_xM) \oplus \mu_x.$$

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We have:

Theorem 3.1. Every contact totally umbilical proper slant submanifold M of a Cosymplectic manifold $\overline{M}$ such that structure vector field ξ is tangent to M is totally geodesic if $\nabla^{\perp}_X\alpha \in \mu$.

Proof. As M is contact totally umbilical slant submanifold of $\overline{M}$, from (2.8) and (2.10), we have

$$h(PX, PY) = cos^2\theta[g(X,Y) - \eta(X)\eta(Y)]\alpha \tag{3.1}$$

for any $X, Y \in TM$.
Further, from (2.4)$\sim$(2.6), we find

$$(\overline{\nabla}_{PX}\varphi)X + \varphi\overline{\nabla}_{PX}X = h(PX,PX) + \nabla_{PX}PX - A_{FX}PX + \nabla^{\perp}_{PX}FX$$

Using (2.3) and considering the normal parts in the above equation, we obtain

$$F\nabla_{PX}X + fh(PX,X) = h(PX,PX) + \nabla^{\perp}_{PX}FX$$

or

$$F\nabla_{PX}X = h(PX,PX) + \nabla^{\perp}_{PX}FX$$

Using (3.1) in the above equation, we get

$$cos^2\theta[g(X,X) - \eta(X)\eta(X]\alpha = F\nabla_{PX}X - \nabla^{\perp}_{PX}FX \tag{3.2}$$

Taking the inner product with FX for $X \in TM$ in (3.2), we obtain

$$cos^2\theta[g(X,X) - \eta(X)\eta(X)]g(\alpha, FX) = g(F\nabla_{PX}X, FX) - g(\nabla^{\perp}_{PX}FX, FX)$$

Using (2.11), we get

$$\begin{aligned}(3.3)\ \ cos^2\theta[g(X,X) - \eta(X)\eta(X)]g(\alpha, FX) = sin^2\theta[g(\nabla_{PX}X, X) - \eta(\nabla_{PX}X)\eta(X)] \\ -g(\nabla^{\perp}_{PX}FX, FX).\end{aligned}$$

Taking the covariant derivative of (2.11) with respect to PX and using Gauss and Weingarten formulae, we find

$$sin^2\theta[g(\nabla_{PX}X, X) - \eta(\nabla_{PX}X)\eta(X)] = g(\nabla^{\perp}_{PX}FX, FX) \tag{3.4}$$

Therefore, (3.3) and (3.4) imply that

$$cos^2\theta[g(X,X) - \eta(X)\eta(X)]g(\alpha, FX) = 0$$

Since M is proper slant and $g(X,X) - \eta(X)\eta(X) \neq 0$ for all $X \in TM$, we have $g(\alpha, FX) = 0$ which implies that $\alpha \in TM^{\perp}$ is orthogonal to FX, hence

$$\alpha \in \Gamma(\mu) \tag{3.5}$$

Similarly, for $X, Y \in TM$ and using (2.3)$\sim$ (2.6) and (2.8), we obtain

$$\nabla_X PY + g(X, PY)\alpha - A_{FY}X + \nabla^{\perp}_X FY = P\nabla_X Y + [g(X,Y) - \eta(X)\eta(Y)]\varphi\alpha + F\nabla_X Y \tag{3.6}$$

Taking inner product in (3.6) with $\varphi\alpha$ and using the fact that μ is invariant with respect to φ, we find

$$g(\bar{\nabla}_X FY, \varphi\alpha) = [g(X,Y) - \eta(X)\eta(Y)]g(\alpha, \alpha) \tag{3.7}$$

From (2.3), for $X \in TM$, we have

$$\bar{\nabla}_X \varphi\alpha = \varphi\bar{\nabla}_X\alpha$$

or

$$-A_{\varphi\alpha}X + \nabla^{\perp}_X\varphi\alpha = -PA_\alpha X - FA_\alpha X + \varphi\nabla^{\perp}_X\alpha$$

Taking inner product in the above equation with FY, we find

$$g(\nabla^{\perp}_X\varphi\alpha, FY) = -g(FA_\alpha X, FY) + g(\varphi\nabla^{\perp}_X\alpha, FY) \tag{3.8}$$

Since $\varphi\nabla^{\perp}_X\alpha \in \mu$ and using (2.11), we obtain

$$g(\nabla^{\perp}_X\varphi\alpha, FY) = -sin^2\theta[g(A_\alpha X, Y) - \eta(A_\alpha X)\eta(Y)]$$

Using (2.3) and (2.7), we get

$$g(\nabla^{\perp}_{X}\varphi\alpha, FY) = -sin^{2}\theta g(A_{\alpha}X, Y) \tag{3.9}$$

Since FY and $\varphi\alpha$ are orthogonal, taking covariant derivative of $g(FY, \varphi\alpha) = 0$, we have

$$g(\bar{\nabla}_{X}FY, \varphi\alpha) = -g(\nabla^{\perp}_{X}\varphi\alpha, FY)$$

implying that

$$g(\bar{\nabla}_{X}FY, \varphi\alpha) = sin^{2}\theta g(A_{\alpha}X, Y) \tag{3.10}$$

Using (2.7) and (2.8) in the above equation, we get

$$g(\bar{\nabla}_{X}FY, \varphi\alpha) = sin^{2}\theta[g(X, Y) - \eta(X)\eta(Y)]g(\alpha, \alpha) \tag{3.11}$$

Then, from (3.7) and (3.11), we find

$$cos^{2}\theta[g(X, Y) - \eta(X)\eta(Y)]g(\alpha, \alpha) = 0 \tag{3.12}$$

Since M is proper slant and $g(X, Y) - \eta(X)\eta(Y) \neq 0$ in general for all $X, Y \in TM$, we conclude that $\alpha = 0$ which shows that M is totally geodesic, thus proving the theorem. □

Remark. It is easy to see that an invariant submanifold of a Cosymplectic manifold $\overline{M}$ with structure vector field tangent to M is minimal. As $\theta = 0$ for invariant submanifold M of Cosymplectic manifold $\overline{M}$, from (3.12), it follows that $\alpha = 0$ which implies that M is minimal. Thus, from Theorem 3.1, we can say that every contact totally umbilical invariant submanifold of a Cosymplectic manifold is totally geodesic. We can aslo see that if M is $(m + 1)$-dimensional proper slant submanifold of $(2m + 1)$-dimensional Cosymplectic manifold $\overline{M}$, then μ={0} which shows that $F(T_pM) = T_pM^{\perp}$. Thus, from (3.5), we get that $\alpha = 0$ which implies that the proof of Theorem 3.1 is valid in this case as well.

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