Invariant submanifolds of trans-Sasakian manifolds

Aysel Turgut Vanli and Ramazan Sari

Abstract. In this paper, invariant submanifolds of a trans-Sasakian manifold are studied. Necessary and sufficient conditions are given on a submanifold of a trans-Sasakian manifold to be invariant submanifold.In this case, we investigate further properties of invariant submanifolds of a trans-Sasakian manifold. An addition, some theorems are given related to an invariant submanifold of a trans-Sasakian manifold.

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1 Introduction

In the Gray-Hervella classication of almost Hermitian manifolds [5], there appears a class, W_4,of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [7] if the product manifold $M \times \mathbb{R}$ belongs to the class W_4. The class $C_5 \oplus C_6$ [8] coincides with the class of trans-Sasakian structures of (α, β). In recent works many authors, (for example, in [1],[9], D.G. Prakasha, C.S. Bagevadi and P. Venkatesha) study trans-Sasakian manifolds. In [6], Karadag and Atceken studied invariant submanifold of a Sasakian manifold. Necessary and sufficient conditions are given on a submanifold of a Sasakian manifold to be an invariant submanifold in [6].

In this paper, we give necessary and succient conditions for a submanifold of a trans-Sasakian manifold to be an invariant submanifold and we consider the invariant case.

2 Preliminaries

Let M be a $(2n+1)$-dimensional almost contact manifold with an almost contact metric structure (Φ, ξ, η, g), where Φ is a (1,1) tensor field, ξ is a vector field; η is 1-form and g is a compatible Riemannian metric such that

$$\Phi^2 = -I + \eta \otimes \xi. \ \ \eta(\xi) = 1, \quad \Phi(\xi) = 0, \quad \eta \circ \Phi = 0 \tag{2.1}$$

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

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$$g(\Phi X, Y) = -g(X, \Phi Y),\ g(X,\xi) = \eta(X) \tag{2.3}$$

for all $X, Y \in \Gamma(TM)$.

An almost contact metric structure (Φ, ξ, η, g) on M is called a trans-Sasakian structure of type (α, β) if

$$(\nabla_X \Phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\Phi X, Y)\xi - \eta(Y)\Phi X) \tag{2.4}$$

for some smooth functions α and β on M. From the formula (2.4) it follows that

$$\nabla_X \xi = -\alpha \Phi X + \beta(X - \eta(X)\xi) = -\alpha\Phi X - \beta\Phi^2 X. \tag{2.5}$$

Moreover, from [2]

$$(\nabla_X \eta)(Y) = -\alpha g(\Phi X, Y) + \beta(g(X,Y) - \eta(X)\eta(Y)),$$

(2.6)

$$(\nabla_X \Omega)(Y,Z) = \alpha(g(X,Z)\eta(Y) - g(X,Y)\eta(Z)) - \beta(g(X,\Phi Z)\eta(Y) - g(X,\Phi Y)\eta(Z))$$

where Ω is the fundamental 2-form of the structure given by $\Omega(X,Y) = g(X, \Phi Y)$.Hence

$$(\nabla_X \Phi)(X,\xi) = -\alpha,\ (\nabla_X \eta)(X) = \beta$$

for X orthogonal to ξ, and $g(X,X) = 1$.

If trans-Sasakian structure of type (1,0) is *Sasakian*, trans-Sasakian structure of type (0,1) is *Kenmotsu*, trans-Sasakian structures of type (α,0) are *α-Sasakian*, trans-Sasakian structures of type (0,β) are *β-Kenmotsu* and trans-Sasakian structures of type (0,0) are *cosymplectic*. Thus the structural equations for $\alpha-$Sasakian, Sasakian, $\beta-$Kenmotsu, Kenmotsu and cosymplectic manifolds are given by

$$(\nabla_X \Phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X)$$
$$(\nabla_X \Phi)Y = g(X,Y)\xi - \eta(Y)X$$
$$(\nabla_X \Phi)Y = \beta(g(\Phi X,Y)\xi - \eta(Y)\Phi X)$$
$$(\nabla_X \Phi)Y = g(\Phi X,Y)\xi - \eta(Y)\Phi X$$
$$(\nabla_X \Phi)Y = 0$$

respectively.

3 Submanifolds of an almost contact metric manifold

Let M be an (m+1) dimensional immersed submanifold of an almost contact metric manifold $(\bar{M}, \Phi, \bar{\eta}, \xi, \bar{g})$, where $\bar{M}$ is (2n+1)-dimensional.

Let $i : M \longmapsto \bar{M}$ be an immersion; we denote by B the differential of i. The induced Riemannian metric g of M is given by $g = i^* \bar{g}$.

$TM = T_xM \oplus T_xM^{\perp}$ where T_xM the tangent space of M at $x \in M$, $T_xM^{\perp}$ the normal space of M in $\bar{M}$, respectively. Moreover, we denote by $\{N_1, N_2, ..., N_t\}$, $t = 2n - m$, an orthonormal basis of the normal space $T_xM^{\perp}$ .Then

$$\Phi BX = B\varphi X + \sum_{l=1}^{t} v_l(X) N_l \tag{3.1}$$

for any $X \in T_xM$, where φ are induced $(1,1)$ tensor and v_l are induced 1-forms on M. Similarly, ΦN_l

$$\Phi N_l = BU_l + \sum_{l=1}^{t} \lambda_{ls} N_s \tag{3.2}$$

where, U_l are vector fields on M and λ_{ls} are functions on M. Furthermore, the vector field ξ can be expressed as follows:

$$\xi = BV + \sum_{l=1}^{t} \alpha_l N_l \tag{3.3}$$

where, V is a vector field on M, α_l are functions on M. Thus

$$\begin{aligned} g(\varphi X, Y) &= \bar{g}(B\varphi X, BY) = \bar{g}(\Phi BX - \sum_{l=1}^{t} v_l(X) N_l, BY) \\ &= \bar{g}(\Phi BX, BY) - \sum_{l=1}^{t} v_l(X) \bar{g}(N_l, BY) \\ &= -\bar{g}(BX, \Phi BY) = -\bar{g}(BX, B\varphi Y) = -g(X, \varphi Y). \end{aligned}$$

Hence we get

$$g(\varphi X, Y) = -g(X, \varphi Y) \tag{3.4}$$

for any $X, Y \in \Gamma(TM)$. Moreover, from (2.3), $\bar{g}(\Phi BX, N_l) = -\bar{g}(BX, \Phi N_l)$ and $\bar{g}(\Phi N_l, N_s) = -\bar{g}(N_l, \Phi N_s)$ we get the equations $v_s(X) = -g(X, U_s)$, $\lambda_{ls} = -\lambda_{sl}$. So λ_{ls} is skew-symmetric.

The following Lemmas will be needed later. This Lemmas provided that for an immersed submanifold of a Sasakian manifold [6]. But this Lemmas true for an immersed submanifold of any almost contact metric manifold.

Lemma 3.1 *Let M be an immersed submanifold of an almost contact metric manifold $\bar{M}$. Then we have*

$$\Phi^2 = -I + \eta \otimes V - \sum_{l=1}^{t} v_l \otimes U_l \tag{3.5}$$

$$v_p(\varphi X) + \sum_{l=1}^{t} v_l(X) \lambda_{lp} - \alpha_p \eta(X) = 0 \tag{3.6}$$

and

$$\varphi U_p - \sum_{l=1}^{t} \lambda_{lp} U_l - \alpha_p V = 0,$$

where η is an induced 1-form on M and $\eta(X) = g(X, V)$.

Lemma 3.2 *Let M be an immersed submanifold of an almost contact metric manifold $\bar{M}$. Then the following equations:*

$$\varphi V + \sum_{l=1}^{t} \alpha_l U_l = 0, \tag{3.7}$$

$$v_k(V) + \sum_{l=1}^{t} \alpha_l \lambda_{lk} = 0 \tag{3.8}$$

and

$$\eta(V) = 1 - \sum_{l=1}^{t} \alpha_l^2.$$

4 Invariant submanifolds of a trans-Sasakian manifold

Let M be an immersed submanifold of a trans-Sasakian manifold $\bar{M}$. If $\Phi(B(T_xM)) \subset T_xM$, for any point $x \in M$, then M is called an invariant submanifold of $\bar{M}$. In this case, we have

$$\Phi BX = B\varphi X \tag{4.1}$$

$$\Phi N_l = \sum_{l=1}^{t} \lambda_{ls} N_s \tag{4.2}$$

$$\xi = BV + \sum_{l=1}^{t} \alpha_l N_s. \tag{4.3}$$

Let ∇ be the Levi-Civita connection of M with respect to the induced metric g. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{4.4}$$

$$\bar{\nabla}_X N = \nabla^{\perp}_X N - A_N X \tag{4.5}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM)^{\perp}$. $\nabla^{\perp}$ is the connection in the normal bundle, h is the second fundamental from of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form h and the shape operator A related by

$$g(h(X, Y), N) = g(A_N X, Y). \tag{4.6}$$

The curvature transformations of M and $\bar{M}$ will be denote by

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad X, Y \in \Gamma(TM)$$

and

$$\bar{R}(X,Y) = \bar{\nabla}_X\bar{\nabla}_Y - \overline{\nabla}_Y\bar{\nabla}_X - \bar{\nabla}_{[X,Y]} \quad X,Y \in \Gamma(T\bar{M})$$

respectively.

Using (4.4) and (4.5) of the Gauss and the Weingarten formulas, we obtain for any vector fields X, Y and Z tangent to M

$$\bar{R}(X,Y)Z = R(X,Y)Z - A_{h(Y,Z)}(X) + A_{h(X,Z)}(Y) + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z).$$

Thus, if W is tangent to M, then we get the Gauss equation

$$\bar{g}(\bar{R}(X,Y)Z,W) = g(R(X,Y)Z,W) + \bar{g}(h(Y,W),h(X,Z)) - \bar{g}(h(X,W),h(Y,Z)).$$

Lemma 4.1 *Let M be an invariant submanifold of a trans-Sasakian manifold $\bar{M}$. Then we have*

$$\varphi^2 = -I + \bar{\eta}\otimes V, \quad \alpha_l\bar{\eta} = 0 \quad l,k = 1,2,...,t \tag{4.7}$$

$$\varphi V = 0, \quad \sum_{l=1}^{t}\alpha_l\lambda_{lk} = 0.$$

Proof. It is clear from the Lemma 4.2 in [6].

Theorem 4.2 *Let M be an invariant submanifold of a trans-Sasakian manifold $\bar{M}$. The ξ is tangent to M if and only if then the induced structure (φ, V, η, g) on M is a trans-Sasakian structure.*

Proof. ξ is tangent to M. V$\neq$0; that is, $\alpha_l = 0$. Then from (2.3) we have

$$\xi = BV \tag{4.8}$$

from (3.1) we have

$$\bar{g}(\Phi X,Y) = \bar{g}(B\varphi X,Y) + \sum_{l=1}^{t} v_l(X)\bar{g}(N_l,Y) = g(\varphi X,Y). \tag{4.9}$$

Then, from (2.4) we get

$$(\bar{\nabla}_X\Phi)Y = \alpha(\bar{g}(X,Y)\xi - \bar{\eta}(Y)X) + \beta(\bar{g}(\Phi X,Y)\xi - \bar{\eta}(Y)\Phi X).$$

By using (4.8) and (2.3), we obtain

$$\begin{aligned}(\bar{\nabla}_X\Phi)Y &= \alpha(\bar{g}(X,Y)BV - \bar{g}(Y,\xi)X) + \beta(\bar{g}(\Phi X,Y)BV - \bar{g}(Y,\xi)\Phi X)\\ &= \alpha(\bar{g}(X,Y)BV - \bar{g}(Y,BV)X) + \beta(\bar{g}(\Phi X,Y)BV - \bar{g}(Y,BV)\Phi X)\end{aligned}$$

From (4.9), we get

$$\begin{aligned}(\bar{\nabla}_X\Phi)Y &= \alpha(g(X,Y)V - g(Y,V)X) + \beta(g(\varphi X,Y)V - g(Y,V)\varphi X)\\ &= \alpha(g(X,Y)V - \eta(Y)X) + \beta(g(\varphi X,Y)V - \eta(Y)\varphi X)\\ &= (\nabla_X\varphi)Y.\end{aligned}$$

Hence by using (2.5) and (4.6) , we have

$$\bar{\nabla}_X \xi = \bar{\nabla}_X BV$$

$$-\alpha \Phi X - \beta \Phi^2 X = \nabla_X V$$

$$-\alpha \Phi BX - \beta \Phi^2 BX = \nabla_{BX} V$$

Hence by using (2.1) and (3.1) it follows that

$$\begin{aligned} \nabla_{BX} V &= -\alpha(B\varphi X + \sum_{l=1}^{t} v_l(X) N_l) \\ &\quad +\beta(B\varphi^2 X + \sum_{l=1}^{t} v_l(\varphi X) N_l + \sum_{i=1}^{t} v_i(X) BU_i + \sum_{i=1}^{t} v_i(X) \sum_{l=1}^{t} \lambda_{ls} N_s) \end{aligned}$$

Thus, we have

$$\nabla_{BX} V = -\alpha(B\varphi X) - \beta(B\varphi^2 X)$$

from which

$$\nabla_X V = -\alpha(\varphi X) - \beta(\varphi^2 X) \qquad \text{or} \qquad \nabla_X V = -\alpha(\varphi X) + \beta(X - \eta(X)V).$$

Then M is a trans-Sasakian manifold with trans-Sasakian structure (φ, V, η, g).

Vice versa, from (2.1) and (3.5) we get $\sum_{l=1}^{t} v_l(X) U_l = 0$ therefore, we have $v_l(X) = 0$. By using (3.6) we get $v_p(\varphi X) = \alpha_p \eta(X)$ which implies that $\alpha_p = 0$. Then $\xi = BV$; that is, $\xi \in T_x M$. □

Theorem 4.3 *Let M be an immersed submanifold of a trans-Sasakian manifold $\bar{M}$. Then M is an invariant submanifold of a trans-Sasakian manifold $\bar{M}$ if and only if the induced structure (φ, V, η, g) on M is a trans-Sasakian structure.*

Proof. Let M be an invariant submanifold of the trans-Sasakian manifold $\bar{M}$. Then, from (3.5) and (4.7) we get

$$\sum_{l=1}^{t} v_l(X) U_l = 0 \Rightarrow v_l(X) = 0.$$

By using (3.6) it follows that

$$v_p(\varphi X) = \alpha_p \eta(X), \ -g(\varphi X, U_p) = \alpha_p g(X, V).$$

Thus we have $g(X, \varphi U_p) = \alpha_p g(X, V)$; that is, $g(\varphi U_p - \alpha_p V, X) = 0$. Since g is non-degenere, we have $\varphi U_p = \alpha_p V$. Thus we get,

$$\alpha_p = 0$$

Then, from (3.3) it follows that $\xi = BV$; that is, $\xi \in T_x M$.

Vice versa, the Reeb vector field ξ is tangent to M from Theorem 4.2 We have $\alpha_l = 0$, $l = 1, 2, , , t$ by using (4.3), Hence, we get $v_k = 0$, $k = 1, 2, , , t$ from (2.8).

Thus, we have $\Phi BX = B\varphi X$ using (3.1). Hence M is invariant submanifold of $\bar{M}$. (We note as well that we identify X and BX, since X is a vector field in M). □

Theorem 4.4 *Let M be an invariant submanifold of a trans-Sasakian manifold $\bar{M}$, and let $\overline{\Omega}$ (resp. Ω) be fundamental 2-form of $\bar{M}$ (resp. M) then we have*
(i) $\quad d\overline{\Omega}(X,Y,Z) = d\Omega(X,Y,Z)$
(ii) $\quad (\nabla_X\Omega)(Y,N_l) = 0$, *for all* $N_l \in \Gamma(TM)^{\perp}$.

Proof. (i) Since $(\bar{\nabla}_X\Phi)$ is tangent to M and by direct computations using (4.9).We have

$$\begin{aligned} 3d\overline{\Omega}(X,Y,Z) &= \bar{g}((\bar{\nabla}_X\Phi)Y,Z) + \bar{g}((\bar{\nabla}_Y\Phi)Z,X) + \bar{g}((\overline{\nabla}_Z\Phi)X,Y) \\ &= g((\nabla_X\varphi)Y,Z) + g((\nabla_Y\varphi)Z,X) + g((\nabla_Z\varphi)X,Y) \\ &= 3d\Omega(X,Y,Z). \end{aligned}$$

(ii) By using (2.6) it follows that

$$\begin{aligned} (\nabla_X\Omega)(Y,N_l) &= \alpha(g(X,N_l)\eta(Y) - g(X,Y)\eta(N_l)) \\ &\quad -\beta(g(X,\Phi N_l)\eta(Y) - g(X,\Phi Y)\eta(N_l)) \\ &= 0 \end{aligned}$$

The following Lemma 4,5, Theorem 4,6 and Corollary 4,1 provided in [3]. But we proved different a way these theorems.

Lemma 4.5 *Let M be an invariant submanifold of a trans-Sasakian manifold $\bar{M}$. Then we have,*

$$h(X,\xi) = 0 \tag{4.10}$$

for any $X \in TM$.

Proof. For the trans-Sasakian manifolds using (4.4), we get

$$\bar{\nabla}_X\xi = -\alpha\Phi X + \beta(X - \eta(X)\xi).$$

On the other hand, we get

$$\nabla_X\xi + h(X,\xi) = -\alpha\Phi X + \beta(X - \eta(X)\xi)$$

from the Gauss formula then the equation is implied that $h(X,\xi) = 0$. □

Theorem 4.6 *Let M be an invariant submanifold of the trans-Sasakian manifold $\bar{M}$. Then we have,*

$$h(\Phi X,Y) = \Phi(h(X,Y)) = h(X,\Phi Y) \tag{4.11}$$

for all $X,Y \in \Gamma(TM)$.

Proof. For all X,Y$\in \Gamma$(TM),

$$\begin{aligned} (\bar{\nabla}_X\Phi)Y &= \bar{\nabla}_X\Phi Y - \Phi(\bar{\nabla}_X Y) \\ &= \nabla_X\Phi Y + h(X,\Phi Y) - \Phi(\nabla_X Y) - \Phi(h(X,Y)) \\ &= (\nabla_X\Phi)Y + h(X,\Phi Y) - \Phi(h(X,Y)). \end{aligned}$$

Then we have,

$$\alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\Phi X,Y)\xi - \eta(Y)\Phi X) = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\Phi X,Y)\xi - \eta(Y)\Phi X) + h(X,\Phi Y) - \Phi(h(X,Y)).$$

Thus, we get

$$h(X,\Phi Y) = \Phi(h(X,Y)).$$

On the other hand, it follows that

$$(\bar{\nabla}_Y \Phi)X = (\nabla_Y \Phi)X + h(Y,\Phi X) - \Phi(h(Y,X)).$$

Hence we get

$$h(Y,\Phi X) = \Phi(h(Y,X)).$$

From Theorem 4.6 we have $h(\varphi X,Y) = \Phi(h(X,Y)) = h(X,\varphi Y)$. □

Corollary 4.1 *Let M be an invariant submanifold of the trans-Sasakian manifold $\bar{M}$. Then we have*

$$h(\Phi X,\Phi Y) = -h(X,Y). \tag{4.12}$$

Proof. From (4.11) and (2.1) we get

$$\begin{aligned} h(\Phi X,\Phi Y) &= \Phi(h(X,\Phi Y)) = \Phi^2(h(X,Y)) \\ &= -h(X,Y) + \eta(h(X,Y))\xi = -h(X,Y). \end{aligned}$$

Lemma 4.7 *Let M be invariant submanifold of the trans-Sasakian manifold $\bar{M}$. Then,*

$$(\nabla_X h)(Y,\xi) = -h(Y,\nabla_X \xi) \tag{4.13}$$

for any $X,Y \in \Gamma(TM)$.

Proof. By using (4.10) we get

$$(\nabla_X h)(Y,\xi) = \nabla_X h(Y,\xi) - h(\nabla_X Y,\xi) - h(Y,\nabla_X \xi).$$

Then, we have

$$(\nabla_X h)(Y,\xi) = -h(Y,\nabla_X \xi).$$

Corollary 4.2 *Let M be an invariant submanifold of the trans-Sasakian manifold $\bar{M}$. Then,*

$$(\nabla_X h)(Y,\xi) = h(Y,\alpha\Phi X) + h(Y,\beta\Phi^2 X) \tag{4.14}$$

for any $X,Y \in \Gamma(TM)$.

Proof**.** *By using* (4.13)*, we get*

$$\begin{aligned} (\nabla_X h)(Y,\xi) &= -h(Y,\nabla_X \xi) = -h(Y,-\alpha\Phi X - \beta\Phi^2 X) \\ &= h(Y,\alpha\Phi X) + h(Y,\beta\Phi^2 X). \end{aligned}$$

Corollary 4.3 *Let M be an invariant submanifold of the trans-Sasakian manifold $\bar{M}$. Then,*

$$(\nabla_X h)(Y,\xi) = \alpha h(Y,\Phi X) - \beta h(Y,X) \tag{4.15}$$

for any $X, Y \in \Gamma(TM)$.

Proof. By using (4.14), we get

$$\begin{aligned}(\nabla_X h)(Y,\xi) &= h(Y,\alpha\Phi X) + h(Y,\beta\Phi^2 X)\\ &= \alpha h(Y,\Phi X) + \beta h(Y,-X+\eta(X)\xi)\\ &= \alpha h(Y,\Phi X) - \beta h(Y,X).\end{aligned}$$

Theorem 4.8 *Let M be an invariant submanifold of a trans-Sasakian manifold $\bar{M}$. Then h is parallel if and only if M is totally geodesic.*

Proof. Suppose that h is parallel. For each $X, Y \in \Gamma(TM)$ and using (4.13) we get

$$(\nabla_X h)(Y,\xi) = 0 \Rightarrow h(Y,\nabla_X \xi) = 0$$

or

$$h(Y,-\alpha\Phi X - \beta\Phi^2 X) = 0$$

Hence

$$-\alpha h(Y,\Phi X) - \beta h(Y,\Phi^2 X) = 0.$$

Since M is an invariant submanifold of $\bar{M}$, we have,

$$\Phi(h(X,Y)) = 0.$$

From (4.11) it follows that

$$\Phi(h(X,Y)) = h(Y,\Phi X) = 0.$$

Then we get

$$\beta h(Y,\Phi^2 X) = 0.$$

Hence it follows that

$$h(Y,-X+\eta(X)\xi) = 0.$$

So

$$h(Y,X) = 0.$$

Vice versa, let M is totally geodesic. Then $h = 0$. For all $X, Y, Z \in TM$,

$$(\nabla_X h)(Y,Z) = \nabla_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z) = 0.$$

Thus we have $\nabla h = 0$. □

Theorem 4.9 *Let M be an invariant submanifold of a trans-Sasakian manifold $\bar{M}$. Then following conditions are equivalent:*

(*i*) *M is totally geodesic;*
(*ii*) $(\nabla_X \nabla_Y h)(\xi,\xi) = 0$;
(*iii*) $\bar{R}(X,\xi)h = 0$;
(*iv*) $\bar{R}(X,Y)h = 0$;

where X and Y are arbitrary vector fields on M. [3]

Corollary 4.4 *Let M be an invariant submanifold of a trans-Sasakian manifold $\bar{M}$. Then following conditions are equivalent:*
(i) M *is totally geodesic;*
(ii) h *is parallel;*
(iii) $(\nabla_X \nabla_Y h)(\xi, \xi) = 0;$
(iv) $\bar{R}(X, \xi)h = 0;$
(v) $\bar{R}(X, Y)h = 0;$
where X and Y are arbitrary vector fields on M.

Let M be an invariant submanifold of a trans-Sasakian manifold $\bar{M}$. The second fundemantal form h of M is said to be $\eta - parallel$ if $(\nabla_X h)(\Phi Y, \Phi Z) = 0$ for all vector fields X, Y and Z tangent to M.

Theorem 4.10 *Let M be an invariant submanifold of a trans-Sasakian manifold $\bar{M}$. Then we have, the second fundemental form h is η-parallel if and only if*

$$(\nabla_X h)(Y, Z) = \alpha(\eta(Y)\Phi h(X, Z) + \eta(Z)\Phi h(X, Y)) - \beta(\eta(Y)h(X, Z) + \eta(Z)h(X, Y)).$$

Proof. First of all, we have

$$(\nabla_X h)(\Phi Y, \Phi Z) = \nabla_X h(\Phi Y, \Phi Z) - h(\nabla_X \Phi Y, \Phi Z) - h(\Phi Y, \nabla_X \Phi Z).$$

Then

$$\nabla_X h(\Phi Y, \Phi Z) = h(\nabla_X \Phi Y, \Phi Z) + h(\Phi Y, \nabla_X \Phi Z).$$

Using (4.12)

$$\begin{aligned} -\nabla_X h(Y, Z) &= h((\nabla_X \Phi)Y + \Phi(\nabla_X Y), \Phi Z) + h(\Phi Y, (\nabla_X \Phi)Z + \Phi(\nabla_X Z)) \\ &= h((\nabla_X \Phi)Y, \Phi Z) - h(\nabla_X Y, Z) + h(\Phi Y, (\nabla_X \Phi)Z) - h(Y, \nabla_X Z). \end{aligned}$$

Thus, by using (2.4) we have

$$-(\nabla_X h)(Y, Z) = -\alpha\eta(Y)h(X, \Phi Z) - \beta\eta(Y)h(\Phi X, \Phi Z) - \alpha\eta(Z)h(\Phi Y, X) - \beta\eta(Z)h(\Phi Y, \Phi X).$$

From (3.11) and (4.12) we have

$$(\nabla_X h)(Y, Z) = \alpha(\eta(Y)\Phi h(X, Z) + \eta(Z)\Phi h(X, Y)) - \beta(\eta(Y)h(X, Z) + \eta(Z)h(X, Y)).$$

Theorem 4.11 *Let M be an invariant submanifold of a trans-Sasakian manifold $\bar{M}$. Then*

$$\Phi(A_N X) = A_{\Phi N} X = -A_N \Phi X$$

for all $X \in \Gamma(TM)$, $N_l \in \Gamma(TM)^{\perp}$.

Proof. By using (2.3) and (4.6) for all $X \in \Gamma(TM)$, $N_l \in \Gamma(TM)^{\perp}$ we get

$$\begin{aligned} g(\Phi(A_N X), Y) &= -g(A_N X, \Phi Y) = -\, g(h(X, \Phi Y), N) \\ &= -\, g(h(\Phi X, Y), N) = -\, g(A_N \Phi X, Y). \end{aligned}$$

Then, we have $\Phi(A_N X) = -A_N \Phi X$. Now using (4.11), we get

$$\begin{aligned} g(A_{\Phi N}X, Y) &= g(h(X,Y), \Phi N) = -g(\Phi(h(X,Y)), N) = -g(h(X, \Phi Y), N) \\ &= -g(A_N X, \Phi Y) = g(\Phi(A_N X), Y). \end{aligned}$$

Then we have $A_{\Phi N} X = \Phi(A_N X)$. □

Lemma 4.12 *In a (2n+1)-dimensional trans-Sasakian manifold, we have*

$$\begin{aligned} R(X,Y)\xi &= (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\Phi X - \eta(X)\Phi Y) \\ &\quad +(Y\alpha)\Phi X - (X\alpha)\Phi Y + (Y\beta)\Phi^2 X - (X\beta)\Phi^2 Y \end{aligned}$$

where R is the curvature tensor [4].

Theorem 4.13 *Let M be an invariant submanifold of a trans-Sasakian manifold $\bar{M}$. Then $\bar{R}(X,Y)\xi$ is tangent to M for any $X, Y \in \Gamma(TM)$.*

Proof. For each $N_l \in \Gamma(TM)^{\perp}$ we have

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)\xi, N_l) &= \bar{g}((\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) \\ &\quad +2\alpha\beta(\eta(Y)\Phi X - \eta(X)\Phi Y) \\ &\quad +(Y\alpha)\Phi X - (X\alpha)\Phi Y + (Y\beta)\Phi^2 X - (X\beta)\Phi^2 Y, N_l) \\ &= \bar{g}((\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y), N_l) \\ &\quad +\bar{g}(2\alpha\beta(\eta(Y)\Phi X - \eta(X)\Phi Y), N_l) \\ &\quad +\bar{g}((Y\alpha)\Phi X, N_l) - \bar{g}((X\alpha)\Phi Y, N_l) \\ &\quad +\bar{g}((Y\beta)\Phi^2 X, N_l) - \bar{g}((X\beta)\Phi^2 Y, N_l) \\ &= (\alpha^2 - \beta^2)(\eta(Y)\bar{g}(X, N_l) - \eta(X)\bar{g}(Y, N_l)) \\ &\quad +2\alpha\beta(\eta(Y)\bar{g}(\Phi X, N_l) - \eta(X)\bar{g}(\Phi Y, N_l)) \\ &\quad +(Y\alpha)\bar{g}(\Phi X, N_l) - (X\alpha)\bar{g}(\Phi Y, N_l) + (Y\beta)\bar{g}(\Phi^2 X, N_l) \\ &\quad -(X\beta)\bar{g}(\Phi^2 Y, N_l) \\ &= (Y\beta)\bar{g}(-X + \eta(X)\xi, N_l) - (X\beta)\bar{g}(-Y + \eta(Y)\xi, N_l) \\ &= (Y\beta)(-\bar{g}(X, N_l) + \eta(X)\bar{g}(\xi, N_l)) \\ &\quad -(X\beta)(-\bar{g}(Y, N_l) + \eta(Y)\bar{g}(\xi, N_l)) \\ &= (Y\beta)(\eta(X)g(V, N_l)) - (X\beta)(\eta(Y)g(V, N_l)) = 0. \end{aligned}$$

Proposition 4.14 *Let M be an invariant submanifold of a trans-Sasakian manifold $\bar{M}$. Then we have,*

$$g(R(X,\varphi X)\varphi X, X) = \bar{g}(\bar{R}(X,\varphi X)\varphi X, X) - 2\bar{g}(h(X,X), h(X,X)).$$

Proof. First of all, we have

$$\begin{aligned} \bar{g}(\bar{R}(X,\varphi X)\varphi X, X) &= g(R(X,\varphi X)\varphi X, X) - \bar{g}(h(X,X), h(\varphi X, \varphi X)) \\ &\quad +\bar{g}(h(\varphi X, X), h(X, \varphi X)) \end{aligned}$$

From (4.11), we get

$$\begin{aligned} g(R(X,\varphi X)\varphi X, X) &= \bar{g}(\bar{R}(X,\varphi X)\varphi X, X) + \bar{g}(h(X,X), h(\varphi X, \varphi X)) \\ &\quad -\bar{g}(\Phi h(X,X), \Phi h(X,X)). \end{aligned}$$

By using (4.12) we have

$$g(R(X, \varphi X)\varphi X, X) = \bar{g}(\bar{R}(X, \varphi X)\varphi X, X) - 2\bar{g}(h(X, X), h(X, X)).$$

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Authors' addresses:

Aysel Turgut Vanlı and Ramazan Sarı
Department of Mathematics, Faculty of Arts and Sciences,
Gazi University, 06500 Ankara, Turkey.
Email: avanli@gazi.edu.tr and ramazansr@gmail.com