On a semi-symmetric metric connection in a Lorentzian para-Sasakian manifold

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Abstract. In this paper we study some properties of curvature tensor, conformal curvature tensor, quasi-conformal curvature tensor, projective curvature tensor with respect to semi-symmetric metric connection in a Lorentzian para-Sasakian (briefly LP-Sasakian) manifold. We investigate the conditions for a Lorentzian para-Sasakian manifold to be conformally flat and quasi-conformally flat with respect to a semi-symmetric metric connection.

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1 Introduction

In 1924, Friedmann and Schouten [7] introduced the idea of semi-symmetric linear connection on a differentiable manifold. A linear connection $\widetilde{\nabla}$ in an n-dimensional differentiable manifold M is said to be semi-symmetric connection if its torsion $\widetilde{T}$ is of the form

$$\widetilde{T}(X,Y) = u(X)Y - u(Y)X, \tag{1.1}$$

where u is a 1-form. The connection $\widetilde{\nabla}$ is a metric connection if there is a Riemannian metric g in M such that $\widetilde{\nabla}g = 0$, otherwise it is non-metric. In 1930, H. A. Hayden [8] defined a semi-symmetric metric connection on a Riemannian manifold and this was further developed by K. Yano [23]. In [1], Agashe and Chafle introduced a semi-symmetric non-metric connection on a Riemannian manifold and this was further studied by U. C. De and D. Kamilya [4], J. Sengupta, U. C. De and T. Q. Binh [16], S. C. Biswas and U. C. De[2], B. B. Chaturvedi and P. N. Pandey [3], and others. In [15], Sharfuddin and Hussian defined a semi-symmetric metric connection in an almost contact manifold by identifying the 1-form u in (1.1) with the contact form η, that is, by setting

$$\widetilde{T}(X,Y) = \eta(X)Y - \eta(Y)X.$$

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U. C. De and J. Sengupta [5] investigated the curvature tensor of an almost contact metric manifold that admit a type of semi-symmetric metric connection and studied the properties of curvature tensor, conformal curvature tensor and projective curvature tensor. M. M. Tripathi [18] studied the semi-symmetric metric connection in a Kenmotsu manifolds. In [19], the semi-symmetric non-metric connection in a Kenmotsu manifold was studied by M. M. Tripathi and N. Nakkar. Also in [20], M. M. Tripathi proved the existence of a new connection and he showed that in particular cases, this connection reduces to semi-symmetric connections; even some of them are not introduced so far. On the other hand, there is a class of almost paracontact metric manifolds, namely Lorentzian para-Sasakian manifolds. In 1989, K. Matsumoto [10], introduced the notion of Lorentzian para-Sasakian manifold. Then I. Mihai and R. Rosca [12] introduced the same notion independently and they obtained several results on this manifolds. Lorentzian para-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [11], U. C. De and et al.,[13], A. A. Shaikh and S. Biswas [14], M. M. Tripathi and U. C. De [21].

In this paper, we study the semi-symmetric metric connection in a Lorentzian para-Sasakian manifold. Section 2 is devoted to preliminaries. In section 3, we find the expression for curvature tensor (resp. Ricci tensor) with respect to the semi-symmetric metric connection and investigate relations between curvature tensor (resp. Ricci tensor) with respect to the semi-symmetric metric connection and curvature tensor (resp. Ricci tensor) with respect to Levi-Civita connection. In section 4 projective curvature tensor of the semi-symmetric metric connection is studied. In section 5 it is shown that the conformal curvature tensor of the semi-symmetric metric connection is equal to the conformal curvature tensor of the Lorentzian para-Sasakian manifold. In section 6, the quasi-conformal curvature tensor of the semi-symmetric metric connection is studied and the conditions for a Lorentzian para-Sasakian manifold to be quasi-conformally flat with respect to a semi-symmetric metric connection are investigated. In the last section deformation algebra of the Levi-Civita connection and the semi-symmetric metric connection of a Lorentzian para-Sasakian manifold is established.

2 Preliminaries

A differentiable manifold of dimension n is called Lorentzian para-Sasakian (briefly, LP-Sasakian)[10, 12], if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2(X) = X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X,\xi) = \eta(X), \tag{2.4}$$

$$\nabla_X \xi = \phi X, \tag{2.5}$$

$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.6}$$

where ∇ denotes the covariant differentiation with respect to the Lorentzian metric g.

It can be easily seen that in an LP-Sasakian manifold the following relations hold:

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.7}$$

$$rank\ \phi = n - 1. \tag{2.8}$$

If we put

$$\Phi(X, Y) = g(X, \phi Y) \tag{2.9}$$

for any vector fields X and Y, then the tensor field $\Phi(X, Y)$ is a symmetric $(0, 2)$ tensor field [10]. Also since the 1-form η is closed in an LP-Sasakian manifold, we have [10, 13]

$$(\nabla_X \eta)(Y) = \Phi(X, Y), \quad \Phi(X, \xi) = 0 \tag{2.10}$$

for all X , $Y \in TM$.

Also in an LP-Sasakian manifold, the following relations hold [11, 13]:

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.11}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.12}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.13}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{2.14}$$

$$S(X, \xi) = (n - 1)\eta(X), \tag{2.15}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y) \tag{2.16}$$

for any vector fields X, Y, Z where R and S are the Riemannian curvature and the Ricci tensors of M, respectively.

Let M be an n-dimensional LP-Sasakian manifold and ∇ be the Levi-Civita connection on M. A linear connection $\widetilde{\nabla}$ on M is said to be semi-symmetric if the torsion tensor

$$\widetilde{T}(X, Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y]$$

satisfies

$$\widetilde{T}(X, Y) = \eta(Y)X - \eta(X)Y \tag{2.17}$$

for all X , $Y \in TM$. A semi-symmetric connection $\widetilde{\nabla}$ is called semi-symmetric metric connection, if it further satisfies

$$\widetilde{\nabla} g = 0. \tag{2.18}$$

A semi-symmetric metric connection $\widetilde{\nabla}$ in a LP-Sasakian manifold can be defined by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi, \tag{2.19}$$

where ∇ is the Levi-Civita connection on M (see [23, 15]).

3 Curvature tensor

Let M be an n-dimensional LP-Sasakian manifold. The curvature tensor $\widetilde{R}$ of M with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ is defined by

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z. \tag{3.1}$$

From (2.19) and (3.1) we have,

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y \\ &\quad -g(Y,Z)\beta(X) + g(X,Z)\beta(Y),\end{aligned} \tag{3.2}$$

where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

is the curvature tensor of M with respect to Levi-Civita connection ∇, α is a tensor field of type $(0,2)$ defined by

$$\alpha(X,Y) = (\widetilde{\nabla}_X\eta)(Y) + \frac{1}{2}g(X,Y) \tag{3.3}$$

and

$$\beta(X) = \widetilde{\nabla}_X\xi + \frac{1}{2}X. \tag{3.4}$$

Lemma 3.1. *Let M be an LP-Sasakian manifold with the semi-symmetric metric connection $\widetilde{\nabla}$. Then*

$$\alpha(X,Y) = g(\beta(X),Y) \tag{3.5}$$

for all X , $Y \in TM$.

Proof. By using the definition of β, (2.1) and (2.4), we have

$$\begin{aligned} g(\beta(X),Y) &= g(\widetilde{\nabla}_X\xi + \frac{1}{2}X, Y) \\ &= g(\nabla_X\xi + \eta(\xi)X - g(X,\xi)\xi + \frac{1}{2}X, Y) \\ &= g(\nabla_X\xi, Y) - \eta(X)\eta(Y) - \frac{1}{2}g(X,Y). \end{aligned}$$

On the other hand from (2.1), (2.2), (2.5) and (2.7), we obtain

$$\begin{aligned} (\widetilde{\nabla}_X\eta)(Y) &= \widetilde{\nabla}_X\eta(Y) - \eta(\widetilde{\nabla}_X Y) \\ &= X\eta(Y) - \eta(\nabla_X Y + \eta(Y)X - g(X,Y)\xi) \\ &= g(\nabla_X\xi, Y) - \eta(X)\eta(Y) - g(X,Y). \end{aligned}$$

This completes the proof. □

From Lemma (3.1) we have the following corollary

Corollary 3.2. *In an LP-Sasakian manifold, the tensor field β of type (1,1) is self-adjoint.*

Let K and $\widetilde{K}$ be the curvature tensors of type $(0,4)$ given by

$$K(X,Y,Z,U)=g(R(X,Y)Z,U)$$

and

$$\widetilde{K}(X,Y,Z,U)=g(\widetilde{R}(X,Y)Z,U).$$

Theorem 3.3. *In an LP-Sasakian manifold with semi-symmetric metric connection $\widetilde{\nabla}$ we have*

$$\widetilde{R}(X,Y)Z+\widetilde{R}(Y,Z)X+\widetilde{R}(Z,X)Y = 0, \tag{3.6}$$

$$\widetilde{K}(X,Y,Z,U)+\widetilde{K}(Y,X,Z,U) = 0, \tag{3.7}$$

$$\widetilde{K}(X,Y,Z,U)+\widetilde{K}(X,Y,U,Z) = 0, \tag{3.8}$$

$$\widetilde{K}(X,Y,Z,U)-\widetilde{K}(Z,U,X,Y,) = 0. \tag{3.9}$$

Proof. Using (3.2) and the first Bianchi identity

$$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0$$

with respect to Levi-Civita connection ∇, we obtain

$$\begin{aligned}\widetilde{R}(X,Y)Z+\widetilde{R}(Y,Z)X+\widetilde{R}(Z,X)Y &= (\alpha(Z,Y)-\alpha(Y,Z))X\\ &\quad +(\alpha(X,Z)-\alpha(Z,X))Y\\ &\quad +(\alpha(Y,X)-\alpha(X,Y))Z.\end{aligned}$$

Since α is symmetric, then we get (3.6).

From (3.2) we have

$$\begin{aligned}\widetilde{K}(X,Y,Z,U) &= K(X,Y,Z,U)-\alpha(Y,Z)g(X,U)+\alpha(X,Z)g(Y,U)\\ &\quad -g(Y,Z)g(\beta(X),U)+g(X,Z)g(\beta(Y),U).\end{aligned} \tag{3.10}$$

Since $K(X,Y,Z,U)=-K(Y,X,Z,U)$, we obtain (3.7). By using (3.10), (3.5) and the equation $K(X,Y,Z,U)=-K(X,Y,U,Z)$ we have

$$\begin{aligned}\widetilde{K}(X,Y,Z,U) &= K(X,Y,U,Z)+\alpha(Y,U)g(X,Z)-\alpha(X,U)g(Y,Z)\\ &\quad +g(Y,U)g(\beta(X),Z)-g(X,U)g(\beta(Y),Z)\end{aligned}$$

which implies (3.8). Again from (3.10), (3.5) and the equation $K(X,Y,Z,U)=K(Z,U,X,Y)$ we obtain (3.9). □

Let M be an n-dimensional LP-Sasakian manifold. Then the Ricci tensor $\widetilde{S}$ of the manifold M with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ is defined by

$$\widetilde{S}(X,Y)=\sum_{i=1}^{n}\varepsilon_i g(\widetilde{R}(e_i,X)Y,e_i)$$

and the scalar curvature of the manifold M with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ is given by

$$\widetilde{r} = \sum_{i=1}^{n} \varepsilon_i \widetilde{S}(e_i, e_i)$$

where X , $Y \in TM$, $\{e_1, e_2, ..., e_n\}$ is an orthonormal frame and $\varepsilon_i = g(e_i, e_i)$.

Theorem 3.4. *In an n-dimensional LP-Sasakian manifold the Ricci tensor $\widetilde{S}$ and scalar curvature $\widetilde{r}$ of semi-symmetric metric connection $\widetilde{\nabla}$ are given by*

$$\widetilde{S}(X, Y) = S(X, Y) - (n-2)\alpha(X, Y) - trace(\alpha)g(X, Y) \tag{3.11}$$

and

$$\widetilde{r} = r - 2(n-1)trace(\alpha) \tag{3.12}$$

where S and r denote the Ricci tensor and scalar curvature of Levi-Civita connection ∇, respectively. Consequently, $\widetilde{S}$ is symmetric.

Proof. By using (3.2), (3.5) and (3.10), we have

$$\begin{aligned} \widetilde{S}(X, Y) &= \sum_{i=1}^{n} \varepsilon_i \{g(R(e_i, X)Y, e_i) - \varepsilon_i \alpha(X, Y) + \alpha(e_i, Y)g(X, e_i) \\ &\quad - g(X, Y)g(\beta(e_i), e_i) + g(e_i, Y)g(\beta(X), e_i)\} \\ &= \sum_{i=1}^{n} \varepsilon_i \{g(R(e_i, X)Y, e_i) - \varepsilon_i \alpha(X, Y) + \alpha(g(X, e_i)e_i, Y) \\ &\quad g(X, Y)\alpha(e_i, e_i) + g(\beta(X), g(e_i, Y)e_i)\}. \end{aligned} \tag{3.13}$$

Since the Ricci tensor of Levi-Civita connection ∇ is given by

$$S(X, Y) = \sum_{i=1}^{n} \varepsilon_i g(R(e_i, X)Y, e_i)$$

then (3.13) implies (3.11). (3.12) follows from (3.11). Also from (3.11), it is obvious that $\widetilde{S}$ is symmetric. □

Lemma 3.5. *Let M be an n-dimensional LP-Sasakian manifold with the semi-symmetric metric connection $\widetilde{\nabla}$. Then*

$$\begin{aligned} g(\widetilde{R}(X, Y)Z, \xi) &= \eta(\widetilde{R}(X, Y)Z) \\ &= (\widetilde{\nabla}_X \eta)(Z)\eta(Y) - (\widetilde{\nabla}_Y \eta)(Z)\eta(X) \end{aligned} \tag{3.14}$$

$$\widetilde{R}(\xi, X)\xi = -\widetilde{\nabla}_X \xi, \tag{3.15}$$

$$\widetilde{R}(X, Y)\xi = \eta(X)\widetilde{\nabla}_Y \xi - \eta(Y)\widetilde{\nabla}_X \xi, \tag{3.16}$$

$$\widetilde{R}(\xi, X)Y = \eta(Y)\widetilde{\nabla}_X \xi - g(Y, \widetilde{\nabla}_X \xi)\xi, \tag{3.17}$$

$$\widetilde{S}(X, \xi) = (\frac{n}{2} - trace(\alpha))\eta(X), \tag{3.18}$$

$$\begin{aligned} \widetilde{S}(\phi X, \phi Y) &= S(X, Y) + (\frac{n}{2} - trace(\alpha))\eta(X)\eta(Y) \\ &\quad -(n-2)\alpha(X, Y) - trace(\alpha)g(X, Y) \end{aligned} \tag{3.19}$$

for all $X, Y, Z, \xi \in TM$.

Proof. From (2.11) and (3.2)-(3.5), we have

$$\begin{aligned} g(\widetilde{R}(X,Y)Z,\xi) &= \eta(\widetilde{R}(X,Y)Z) \\ &= g(R(X,Y)Z,\xi) - \alpha(Y,Z)\eta(X) + \alpha(X,Z)\eta(Y) \\ &\quad -g(Y,Z)\eta(\beta(X)) + g(X,Z)\eta(\beta(Y)) \\ &= g(Y,Z)\eta(X) - g(X,Z)\eta(Y) - \alpha(Y,Z)\eta(X) + \alpha(X,Z)\eta(Y) \\ &\quad -\frac{1}{2}g(Y,Z)\eta(X) + \frac{1}{2}g(X,Z)\eta(Y) \\ &= (\frac{1}{2}g(Y,Z) - \alpha(Y,Z))\eta(X) - (\frac{1}{2}g(X,Z) - \alpha(X,Z))\eta(Y) \\ &= (\widetilde{\nabla}_X\eta)(Z)\eta(Y) - (\widetilde{\nabla}_Y\eta)(Z)\eta(X). \end{aligned}$$

By using (2.12) and (3.2) , one can easily see that

$$\widetilde{R}(X,Y)\xi = X - \nabla_X\xi + \eta(X)\xi$$

which implies (3.15). If we use (3.2)-(3.4), (2.13) and (2.14), we can obtain (3.16) and (3.17), respectively. By taking $Y=\xi$ in (3.11) and then using (2.15), (2.4) and (3.3), we get (3.18). Since

$$\alpha(\phi X,\phi Y) = \alpha(X,Y) + \frac{1}{2}\eta(X)\eta(Y),$$

then by using (2.3), (2.16) and (3.11) we obtain (3.19). □

4 Projective curvature tensor

Let M be an n-dimensional LP-Sasakian manifold. The projective curvature tensor $\widetilde{P}$ of type $(1,3)$ of M with respect to semi-symmetric metric connection is defined by

$$\widetilde{P}(X,Y)Z = \widetilde{R}(X,Y)Z - \frac{1}{n-1}\{\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y\}. \tag{4.1}$$

By using (3.2) and (3.11), we get from (4.1),

$$\begin{aligned} \widetilde{P}(X,Y)Z &= P(X,Y)Z - \frac{1}{n-1}\{\alpha(Y,Z)X - \alpha(X,Z)Y\} \\ &\quad -\frac{trace(\alpha)}{n-1}\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad -g(Y,Z)\beta(X) + g(X,Z)\beta(Y) \end{aligned} \tag{4.2}$$

where

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}$$

is the projective curvature tensor of M with respect to Levi-Civita connection ∇. Since

$$P(X,Y)Z = -P(Y,X)Z,$$

from (4.2) we have

$$\widetilde{P}(X,Y)Z + \widetilde{P}(Y,X)Z = 0. \tag{4.3}$$

Furthermore, by using (4.2) we obtain

$$\widetilde{P}(X,Y)Z + \widetilde{P}(Y,Z)X + \widetilde{P}(Z,X)Y = 0. \tag{4.4}$$

5 Conformal curvature tensor

Let M be an n-dimensional LP-Sasakian manifold. The conformal curvature tensor of M with respect to semi-symmetric metric connection $\widetilde{\nabla}$ is defined by

$$\begin{aligned}\widetilde{C}(X,Y,Z,U) &= \widetilde{K}(X,Y,Z,U)\\ &-\frac{1}{n-2}\{g(Y,Z)\widetilde{S}(X,U) - g(X,Z)\widetilde{S}(Y,U)\\ &+\widetilde{S}(Y,Z)g(X,U) - \widetilde{S}(X,Z)g(Y,U)\}\\ &+\frac{\widetilde{r}}{(n-1)(n-2)}\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\}.\end{aligned} \tag{5.1}$$

Proposition 5.1. *The conformal curvature tensors of an LP-Sasakian manifold with respect to the semi-symmetric metric connection and Levi-Civita connection are equal.*

Proof. Let C and $\widetilde{C}$ denote the conformal curvature tensors with respect to Levi-Civita connection ∇ and semi-symmetric metric connection $\widetilde{\nabla}$, respectively. Then by using (3.2), (3.11) and (3.12) in (5.1), we get

$$\widetilde{C}(X,Y,Z,U) = C(X,Y,Z,U). \tag{5.2}$$

This proves the proposition. □

Theorem 5.2. *If the Ricci tensor $\widetilde{S}$ of the semi-symmetric metric connection $\widetilde{\nabla}$ in an LP-Sasakian manifold vanishes, then the curvature tensor with respect to $\widetilde{\nabla}$ is equal to the conformal curvature tensor of the manifold with respect to Levi-Civita connection ∇.*

Proof. Since $\widetilde{S} = 0$, (5.1) gives

$$\widetilde{C}(X,Y,Z,U) = \widetilde{K}(X,Y,Z,U). \tag{5.3}$$

From (5.2) and (5.3), we obtain

$$\widetilde{K}(X,Y,Z,U) = C(X,Y,Z,U). \tag{5.4}$$

This completes the proof. □

A manifold is said to be conformally flat if the conformal curvature tensor vanishes.

Corollary 5.3. *If the curvature tensor $\widetilde{K}$ of the semi-symmetric metric connection $\widetilde{\nabla}$ in an LP-Sasakian manifold vanishes, then the manifold is conformally flat.*

Proof. If $\widetilde{K} = 0$, then from (5.4) we have $C(X, Y, Z, U) = 0$ which gives the assertion of the corollary. □

From proposition (5.1) and corollary (5.1), we have

Theorem 5.4. *An LP-Sasakian manifold with vanishing Ricci tensor $\widetilde{S}$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ is conformally flat if and only if the curvature tensor $\widetilde{K}$ with respect to $\widetilde{\nabla}$ vanishes.*

In [17], the authors proved that a conformally flat Lorentzian para-Sasakian manifold is locally isometric to a unit Lorentzian sphere. Thus we can state

Theorem 5.5. *Let M be a conformally flat LP-Sasakian manifold with respect to the semi-symmetric metric connection $\widetilde{\nabla}$. Then it is locally isometric to a unit Lorentzian sphere.*

6 Quasi-conformal curvature tensor

The notion of the quasi-conformal curvature tensor was introduced by Yano and Sawaki [24]. They define a quasi-conformal curvature tensor by

$$\begin{aligned} W(X,Y)Z &= aR(X,Y)Z + b\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY\} - \frac{r}{n}[\frac{a}{n-1} + 2b]\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{6.1}$$

where a and b are constants such that $ab \neq 0$, R is the Riemannian curvature tensor, S is the Ricci tensor, Q is the Ricci operator defined by $g(QX, Y) = S(X, Y)$ and r is the scalar curvature of the manifold. If $a = 1$ and $b = -\frac{1}{n-2}$, then (6.1) takes the form

$$\begin{aligned} W(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \\ &= C(X,Y)Z, \end{aligned}$$

where C is conformal curvature tensor [6]. Thus the conformal curvature tensor C is a particular case of the quasi-conformal curvature tensor W. An n- dimensional $(n > 3)$ manifold is called quasi-conformally flat if the quasi-conformal curvature tensor W vanishes.

A quasi-conformal curvature tensor $\widetilde{W}$ with respect to a semi-symmetric metric connection in an n-dimensional Lorentzian para-Sasakian manifold is defined by

$$\begin{aligned} \widetilde{W}(X,Y)Z &= a\widetilde{R}(X,Y)Z + b\{\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y + g(Y,Z)\widetilde{Q}X \\ &\quad -g(X,Z)\widetilde{Q}Y\} - \frac{\widetilde{r}}{n}[\frac{a}{n-1} + 2b]\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{6.2}$$

where a and b arbitrary constants such that a and b are not zero simultaneously, $\widetilde{Q}$ is the Ricci operator with respect to the semi-symmetric metric connection i.e.,

$g(\widetilde{Q}X,Y)=\widetilde{S}(X,Y)$ and $\widetilde{r}$ is the scalar curvature of the manifold with respect to the semi-symmetric metric connection. Let us denote

$$\widetilde{W}(X,Y,Z,U)=g(\widetilde{W}(X,Y)Z,U).$$

Then by using (3.2), (3.11) and (3.12) in (6.2), we have

$$\begin{aligned}\widetilde{W}(X,Y,Z,U) &= W(X,Y,Z,U)-[a+(n-2)b]\{\alpha(Y,Z)g(X,U)\\ &\quad -\alpha(X,Z)g(Y,U)+g(Y,Z)\alpha(X,U)-g(X,Z)\alpha(Y,U)\}\\ &\quad -2[b(trace(\alpha))-\frac{(n-1)trace(\alpha)}{n}(\frac{a}{n-1}+2b)]\{g(Y,Z)g(X,U)\\ &\quad -g(X,Z)g(Y,U)\},\end{aligned} \tag{6.3}$$

where W denotes the quasi-conformal curvature tensor with respect to Levi-Civita connection, that is,

$$\begin{aligned}W(X,Y,Z,U) &= g(W(X,Y)Z,U)\\ &= aK(X,Y,Z,U)+b\{S(Y,Z)g(X,U)-S(X,Z)g(Y,U)\\ &\quad +g(Y,Z)S(X,U)-g(X,Z)S(Y,U)\}\\ &\quad -\frac{r}{n}[\frac{a}{n-1}+2b]\{g(Y,Z)g(X,U)-g(X,Z)g(Y,U)\}.\end{aligned}$$

If $a=(2-n)b$ then (6.3) becomes

$$\widetilde{W}(X,Y,Z,U)=W(X,Y,Z,U).$$

Hence we can state

Theorem 6.1. *Let M be an n-dimensional Lorentzian para-Sasakian manifold. Then the quasi-conformal curvature tensor of M with respect to the semi-symmetric metric connection is equal to the quasi-conformal curvature tensor of M with respect to the Levi-Civita connection provided that $a=(2-n)b$.*

Theorem 6.2. *Let M be an n-dimensional $(n>3)$ Lorentzian para-Sasakian manifold and $a=(2-n)b$. Then the manifold M is quasi-conformally flat with respect to the semi-symmetric metric connection if and only if it is quasi-conformally flat with respect to the Levi-Civita connection.*

Theorem 6.3. *Let M be an n-dimensional $(n>3)$ Lorentzian para-Sasakian manifold with $a=(2-n)b$. If $a=1$, then quasi-conformally flatness of M with respect to the semi-symmetric metric connection is equal to conformally flatness of M.*

It is known that

Theorem 6.4. *[17] A quasi-conformally flat Lorentzian para-Sasakian manifold is locally isometric to a unit Lorentzian sphere.*

Thus we have

Theorem 6.5. *If $a=(2-n)b$, then quasi-conformally flat Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection is locally isometric to a unit Lorentzian sphere.*

7 Deformation algebra $II(M, \nabla, \widetilde{\nabla})$

Let ∇^1 and ∇^2 be two linear connections in a Riemannian manifold. We define the product of two vector fields X and Y by

$$X \circ Y = \nabla^2_X Y - \nabla^1_X Y. \tag{7.1}$$

The module $\chi(M)$ of all differentiable vector fields becomes an algebra over the ring $F(M)$ all real functions over M. This algebra is called the deformation algebra [22] of the ordered pair of connection (∇^1, ∇^2) and it is denoted by $II(M, \nabla, \widetilde{\nabla})$ (see also [9]). An element $X \in \chi(M)$ is called a characteristic vector field if there is a $\lambda \in F(M)$ such that

$$X \circ X = \lambda X.$$

Here by using (7.1), we obtain

$$\begin{aligned} X \circ X &= \widetilde{\nabla}_X X - \nabla_X X \\ &= \eta(X)X - g(X, X)\xi. \end{aligned}$$

Thus we have

Theorem 7.1. *In the deformation algebra $II(M, \nabla, \widetilde{\nabla})$, every element associated to the null cone is a characteristic vector field.*

References

[1] N.S. Agashe, M.R. Chafle, *A semi-symmetric non-metric connection in a Riemannian manifold*, Indian J. Pure Appl. Math. 23 (1992), 399-409.

[2] S.C. Biswas, U.C. De, *On a type of semi-symmetric non-metric connection on a Riemannian manifold*, Ganita. 48 (1997), 91-94.

[3] B.B. Chaturvedi, P.N. Pandey, *Semi-symmetric non-metric connections on a Kaehler manifold*, Diff. Geom. Dyn. Syst., 10 (2008), 86-90.

[4] U.C. De, D. Kamilya, *Hypersurfaces of a Riemannian manifold with semi-symmetric non-metric connection*, J. Indian Inst. Sci. 75 (1995), 707-710.

[5] U.C. De, J. Sengupta, *On a type of semi-symmetric metric connection on an almost contact metric manifold*, Facta Universitatis Ser. Math. Inform. 16 (2001), 87-96.

[6] L.P. Eisenhart, *Riemannian Geometry*, Princeton University Press, Princeton, N. J., 1949.

[7] A. Friedmann, J.A. Schouten, *Über die Geometric der holbsymmetrischen Übertragurgen*, Math. Z. 21 (1924), 211-233.

[8] H.A. Hayden, *Subspaces of space with torsion*, Proc. London Math. Soc. 34 (1932), 27-50.

[9] I.E. Hirica, L. Nicolescu, *Associative deformation algebras on Weyl Manifolds*, Diff. Geom. Dyn. Syst., 8 (2006), 112-119.

[10] K. Matsumoto, *On Lorentzian Paracontact manifolds*, Bull. Yamagata Univ. Natur. Sci. 12(2) (1989), 151-156.

[11] K. Matsumoto, I. Mihai, *On a certain transformation in a Lorentzian para-Sasakian manifold*, Tensor (N. S.) 47 (1988), 189-197.

[12] I. Mihai, R. Rosca, *On Lorentzian P-Sasakian manifolds*, Classical Analysis, World Scientific Publ., Signapore, (1992), 155-169.

[13] I. Mihai, A.A. Shaikh, U.C. De, *On Lorentzian para-Sasakian manifolds,* Rendiconti del Seminario Matematico di Messina, Serie II, 1999.

[14] A.A. Shaikh, S. Biswas, *On LP-Sasakian manifolds*, Bull. of the Malaysian Math. Sci. Soc. 27 (2004), 17-26.

[15] S. Sharfuddin, S.I. Husain, *Semi-symmetric metric connexions in almost contact manifolds*, Tensor. 30 (1976), 133-139.

[16] J. Sengupta, U.C. De , T.Q. Binh, *On a type of semi-symmetric non-metric connection on a Riemannian manifold*, Indian J. Pure Appl. MAth. 31(12) (2000), 1659-1670.

[17] M. Tarafdar, A. Bhattacharyya, *On Lorentzian para-Sasakian manifolds*, Steps in Diff. Geom, Proccedings of the Collquium on Diff. Geom. 25-30 July, 2000, Debrecen, Hungary.

[18] M.M. Tripathi, *On a semi-symmetric metric connection in a Kenmotsu manifold*, J. Pure Math. 16 (1999), 67-71.

[19] M.M. Tripathi, N. Nakkar, *On a semi-symmetric non-metric connection in a Kenmotsu manifold*, Bull. Cal. Math. Soc. 16 (2001), no.4, 323-330.

[20] M.M. Tripathi, *A new connection in a Riemannian manifold*, International Electronic Journal of Geometry, 1, 1 (2008), 15-24.

[21] M.M. Tripathi, U.C. De, *Lorentzian almost paracontact manifolds and their submanifolds*, Journal of the Korean Society of Mathematical Education, 2 (2001), 101-125.

[22] I. Vaisman, *Sur quelques formulas du calcul de Ricci global*, Comm. Math. Helvv. 4 (1966-67), 73-87.

[23] K. Yano, *On semi-symmetric metric connections*, Revue Roumain de Math. Pures Appl. 15 (1970), 1579-1586.

[24] K. Yano, S. Sawaki, *Riemannian manifolds admitting a conformal transformation group*, J. Differential Geometry 2 (1968), 161-184.

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