The geometry of $(2,0)-$jet bundles

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Abstract. In this note we make an introduction to higher order holomorphic geometry. Two types of holomorphic jet bundles are compared. The (2,0)-jet bundles present some resemblances with the well known real osculator bundle. For this we study the main topics of this geometry: the fibre structure, its complexified tangent bundle, the decomposition by a nonlinear complex connection, $N-$linear complex connection.

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1 Holomorphic jet bundles

The *complex jet bundles* notion was firstly introduced by Green and Griffiths, [9], with the aimed to get a natural filtration of the sheaf of $k-$jet differentials.

There are known two different structures of jet bundles. According to Stoll and Wong, [20], one of the structures is called the full jet bundle (or holomorphic tangent bundle of order k) and the second is called the restricted jet bundle, which has a natural geometric meaning which we shall emphasize in the next section.

In a set of papers, [20, 8, 21], are studied the problems of algebraic geometry for a holomorphic jet bundle. The holomorphic jet bundle has a natural structure of complex manifold, whose total space will be further studied by the same schedule as for the real case for the known $k-$ osculator bundle, topic intensively investigated in the last decade, especially by the Romanian geometers, [12, 13, 7, 18, 6], etc.

In this introductory section we briefly revise the building of both the mentioned above holomorphic structures and will point out their differences.

Let M be a complex manifold, $\dim_C M = n$, and let $(z^i)_{i=\overline{1,n}}$ be the complex coordinates in a local chart (U,φ) near $z \in U$. Consider the differential operator $D_i = \frac{\partial}{\partial z^i}$ and its conjugate $D_{\bar{\text{i}}} = \frac{\partial}{\partial \bar{z}^i}$. The set of germs of holomorphic $k-$jets, denoted $\mathcal{T}^k M$ is the subsheaf of the sheaf of homomorphisms $Hom_C(\mathcal{O}_M, \mathcal{O}_M)$ built on the operators $\sum\limits_{I+J=1}^{k} D_I \circ D_{\bar{J}}$, where $D_I = D_{i_1} \circ D_{i_2} \circ ... \circ D_{i_p}$, $I = \{1 \leq i_1 \leq i_2 \leq ... \leq i_p \leq k\}$ and $D_{\bar{J}} = D_{\bar{j}_1} \circ D_{\bar{j}_2} \circ ... \circ D_{\bar{j}_q}$, $J = \{1 \leq j_1 \leq j_2 \leq ... \leq j_q \leq k\}$.

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Of course, this means that any element of $\mathcal{T}^k M$ is locally of the form $\sum_{I+J=1}^{k} a_{I\bar{J}} D_I \circ D_{\bar{J}}$, where $a_{I\bar{J}}$ are symmetric functions in all indices of I, respectively of $\bar{J}$. The operators $D_I \circ D_{\bar{J}}$ will be called to be of $(I, \bar{J})-$type. $\mathcal{T}^k M$ is locally free and $\mathcal{T}^{k-1} M$ injects into $\mathcal{T}^k M$. There exists an isomorphism $\mu : D_I \circ D_{\bar{J}} \to D_{i_1} \odot ... \odot D_{i_p} \odot D_{\bar{j}_1} \odot ... \odot D_{\bar{j}_q}$ between $\mathcal{T}^k M$ and the symmetric product $\odot^k TM = \odot^p T'M \otimes \odot^q T''M$. Generally $\mathcal{T}^k M$ is not a holomorphic vector bundle, but $\odot^p T'M$ is a holomorphic one, and via this isomorphism $\mathcal{T}^{(k,0)} M \equiv \odot^k T'M$ is a holomorphic vector bundle as well, called the $k-$tangent holomorphic bundle. For the changes of holomorphic coordinates on $\mathcal{T}^{(k,0)} M$, we refer to in [20, formula (1)].

The second kind of holomorphic bundle is related to $k-$th order jets differentials, being introduced by Green and Griffiths in [9]. We consider $\mathcal{H}_{z_0},\ z_0 \in M$, the sheaf of germs of holomorphic curves $\{f : \Delta_r \to M,\ f \in \mathcal{H}_{z_0},\ f(0) = z_0\}$. Let $(z^i)_{i=\overline{1,n}}$ be the complex coordinates in a local chart (U, φ) and $(z'^i)_{i=\overline{1,n}}$ complex coordinates in (U', φ') from $z_0 \in M$, with the changes

$$z'^{i} = z'^i(z^1, ..., z^n)\ ,\ \det(\frac{\partial z'^i}{\partial z^j}) \neq 0. \tag{1.1}$$

Since M is a complex manifold, we have $\frac{\partial z'^i}{\partial \bar{z}^j} = 0,\ \forall i, j = \overline{1,n}$.

We denote by $f^i = z^i \circ f$, $\forall i = \overline{1,n}$, $f \in \mathcal{H}_{z_0}$ and consider a complex parameter $\theta \in \Delta_r \subset \mathbb{C}$. According to [20], $f, g \in \mathcal{H}_{z_0}$ are said to *be $k-$equivalent*, and denoted as $f \overset{k}{\sim} g$, iff $f^i(0) = g^i(0)$, and $\dfrac{d^p f^i}{d\theta^p}(0) = \dfrac{d^p g^i}{d\theta^p}(0)$, $\forall i = \overline{1,n},\ p = \overline{1,k}$. The class of f is $[f]_{\underset{\sim}{k}}$ and the set of all classes is $J^{(k,0)} M = \cup_{z_0 \in M} \mathcal{H}_{z_0}/_{\underset{\sim}{k}}$. By $j^k f(0) = \left(f(0), \frac{df}{d\theta}(0), ..., \frac{d^p f}{d\theta^p}(0)\right)$we denote the $k-$jet of $f \in [f]_{\underset{\sim}{k}}$.

Let $\pi^{(k,0)} : J^{(k,0)} M \to M$ be the canonical projection. Then we immediately check that $\left(J^{(k,0)} M, \pi^{(k,0)}\right)$ has a fibre bundle structure, named in [20] the restricted $k-$jet bundle, and in [8] the parametrized $k-$jet bundle. Actually, $J^{(k,0)} M$ has the following geometric meaning. Let $c : \Delta_r \to M$ be a holomorphic curve parametrically given by $z^i = z^i(\theta)$, $i = \overline{1,n}$, with $z^i(0) = z_0$ and $\frac{dz^i}{d\theta}(0) = 0$. Then $\overset{*}{c}$ given by

$$\overset{*}{z^i}(0) = z^i(0) + \frac{dz^i}{d\theta}(0)\theta + \frac{d^2 z^i}{d\theta^2}(0)\frac{\theta^2}{2!} + ... + \frac{d^k z^i}{d\theta^k}(0)\frac{\theta^k}{k!} \tag{1.2}$$

span the class $[f]_{\underset{\sim}{k}}$ if we set $f^i = \overset{*}{z^i} \circ f$. Therefore the terminology of parametrized $k-$jet bundle is justifiable.

$J^{(k,0)} M$ has indeed a structure of a differentiable manifold, which we subsequently mark out. If $(z^i)_{i=\overline{1,n}}$ are complex coordinates in (U, φ) near $z \in U$, then $Z = \left(z^i = z^i(0),\ \overset{(1)}{\eta^i} = \frac{dz^i}{d\theta}(0),\ \overset{(2)}{\eta^i} = \frac{1}{2!}\frac{d^2 z^i}{d\theta^2}(0), ...,\ \overset{(k)}{\eta^i} = \frac{1}{k!}\frac{d^k z^i}{d\theta^k}(0)\right)$ will be complex coordinates in (U, Φ) from $J^{(k,0)} M$. Let us note that $\left(Z^i\right)$ are holomorphic functions, since:

$\frac{\partial z^i}{\partial \bar{z}^j}(0) = 0$; $\frac{\partial \overset{(1)}{\eta^i}}{\partial \bar{z}^j} = \frac{\partial}{\partial \bar{z}^j}\left(\frac{dz^i}{d\theta}\right)(0) = \frac{d}{d\theta}\left(\frac{\partial z^i}{\partial \bar{z}^j}\right)(0) = 0$;

$\frac{\partial \overset{(2)}{\eta^i}}{\partial \bar{z}^j} = \frac{1}{2}\frac{\partial}{\partial \bar{z}^j}\left(\frac{d\overset{(1)}{\eta^i}}{d\theta}\right)(0) = \frac{1}{2}\frac{d}{d\theta}\left(\frac{\partial \overset{(1)}{\eta^i}}{\partial \bar{z}^i}\right)(0) = 0$; Analogue, $\frac{\partial \overset{(2)}{\eta^i}}{\partial \overset{(1)}{\eta^j}}(0) = 0$, etc.

Moreover, due to the holomorphy requirement, we have:

$$\begin{aligned} \overset{(1)}{\eta^i} &= \frac{dz^i}{d\theta} \\ \overset{(2)}{\eta^i} &= \frac{1}{2}\frac{d}{d\theta}(\overset{(1)}{\eta^i}) = \frac{1}{2}\frac{\partial \overset{(1)}{\eta^i}}{\partial z^j}\overset{(1)}{\eta^j} \\ &\dots\dots\dots\dots \\ \overset{(k)}{\eta^i} &= \frac{1}{k!}\frac{d}{d\theta}(\overset{(k-1)}{\eta^i}) = \frac{1}{k!}\frac{\partial \overset{(k-1)}{\eta^i}}{\partial z^j}\overset{(1)}{\eta^j}. \end{aligned} \tag{1.3}$$

If $g \in j^k f(0)$, we easy infer that $\sum_{\alpha_1,..,\alpha_p=1}^{k} (D_I f)\, \overset{(\alpha_1)}{\eta^{i_1}} \dots \overset{(\alpha_p)}{\eta^{i_n}} =$

$\sum_{\alpha_1,..,\alpha_p=1}^{n} (D_I g)\, \overset{(\alpha_1)}{\eta^{i_1}} \dots \overset{(\alpha_p)}{\eta^{i_n}}$, $\forall I \subset \{1 \le i_1 \le ... \le i_p \le n\}$ and $D_{\bar{J}} f = D_{\bar{J}} g = 0$.

Let (U', Φ') be another chart over Z from $J^{(k,0)}M$, with $Z = (z'^i, \overset{(1)}{\eta'^i}, ..., \overset{(k)}{\eta'^i})$. Then the changes on the complex manifold $J^{(k,0)}M$ are given by:

$$\begin{aligned} z'^i &= z'^i(z^j) \\ \overset{(1)}{\eta'^i} &= \frac{\partial z'^i}{\partial z^j}\overset{(1)}{\eta^j} \\ 2\,\overset{(2)}{\eta'^i} &= \frac{\partial \overset{(1)}{\eta'^i}}{\partial z^j}\overset{(1)}{\eta^j} + 2\frac{\partial \overset{(1)}{\eta'^i}}{\partial \overset{(1)}{\eta^j}}\overset{(2)}{\eta^j} \\ &\dots\dots\dots\dots \\ k\,\overset{(k)}{\eta'^i} &= \frac{\partial \overset{(k-1)}{\eta'^i}}{\partial z^j}\overset{(1)}{\eta^j} + 2\frac{\partial \overset{(k-1)}{\eta'^i}}{\partial \overset{(1)}{\eta^j}}\overset{(2)}{\eta^j} + \dots + k\frac{\partial \overset{(k-1)}{\eta'^i}}{\partial \overset{(k-1)}{\eta^j}}\overset{(k)}{\eta^j}. \end{aligned} \tag{1.4}$$

Some remarks are appropriate at this place.

Firstly, $J^{(k,0)}M$ has not a vector bundle structure aside from $k = 1$, and then it is identified with $T'M$, the holomorphic tangent bundle.

Further, the described structure for $J^{(k,0)}M$ depends on the choice of the parameter θ on the curves $j^k f$. If we consider the $\mathbb{C}^*-$action $f_\lambda(\theta) = f(\lambda\theta)$, $\lambda \in \mathbb{C}^*$, then $j^k f_\lambda(0) = \left(f(0), \lambda\frac{df}{d\theta}(0), ..., \lambda^k \frac{d^p f}{d\theta^p}(0)\right)$ and hence, $J^{(k,0)}M$ we assume that it is well defined up to this $\mathbb{C}^*-$action.

We note that the rank of the fibre bundle $J^{(k,0)}M$ is kn, while the dimension of the complex manifold structure is $(k+1)n$.

As proved in [20], the map locally given by

$$j^k f(0) \to \frac{d^k f^{j_1}}{d\theta^k}(0)D_{j_1} + \frac{d^{k-1} f^{j_1}}{d\theta^{k-1}}(0)\frac{d^{k-1} f^{j_2}}{d\theta^{k-1}}(0)D_{j_1}\circ D_{j_2} + ... + \frac{df^{j_1}}{d\theta}(0)\cdots\frac{df^{j_k}}{d\theta}(0)D_{j_1}\circ....\circ D_{j_k}$$

is a holomorphic (but not biholomorphic) map from $J^{(k,0)}M$ onto $\mathcal{T}^kM$. The bundle $J^{(k-1,0)}M$ is not injected into $J^{(k,0)}M$, like is the case for $\mathcal{T}^kM$. There are some other differences between these bundles, such as their ranks. In exchange, $J^{(k,0)}M$ is a fibre bundle over M but also over $J^{(h,0)}M$, for $1\leq h < k$, $\pi^{(k,h)} : J^{(k,0)}M \to J^{(h,0)}M$, locally by restricting to the coordinates $(z^i, \overset{(1)}{\eta^j}, ..., \overset{(h)}{\eta^j})$.

A significant remark is that (1.2) provides a particular jet, let us say of $(k,0)-$type for f. In general, a $(p,q)-$jet on M can be spanned by $\frac{\partial f}{\partial \theta}(0)$, $\frac{\partial f}{\partial \bar{\theta}}(0)$, $\frac{\partial^2 f}{\partial \theta^2}(0)$, $\frac{\partial^2 f}{\partial \theta \partial \bar{\theta}}(0)$, $\frac{\partial^2 f}{\partial \bar{\theta}^2}(0), ..\therefore$, where $f \in \mathcal{F}(M)$, is not necessary holomorphic in $z_0 = f(0)$. In this position $J^{(p,q)}M$ is not always holomorphic. Certainly, if f is in $\mathcal{H}_{z_0}$ then $\frac{\partial f}{\partial \bar{\theta}}(0) = 0$, and it shows that $J^{(p,0)}M$ is a holomorphic subbundle of $J^{(p,q)}M$.

Finally, we specify that the corresponding real $k-$jet bundle has a special geometric name, that of $k-$osculator bundle, Osc^kM bundle (e.g., [12, 13, 7, 18]). The real and imaginary parts of $J^{(k,0)}M$ define both of them Osc^kM bundles, while this does not generally hold for $J^{(p,q)}M$.

2 The geometry of holomorphic $J^{(2,0)}M$ bundles

In this section for the sake of simplicity we limit ourselves to the case of the $J^{(2,0)}M$ space of $(2,0)-$holomorphic functions, while the general case $J^{(k,0)}M$ can be immediately derived.

There exits in our approach a strongly similitude to the study of the real case for the Osc^kM bundle.

Let M be a complex manifold, $T_CM = T'M \oplus T''M$, the complexified tangent bundle of $(1,0)-$ and of $(0,1)-$type vectors, respectively. If $(z^i)_{i=\overline{1,n}}$ are complex coordinates, then T'_zM is spanned by $\left\{\frac{\partial}{\partial z^i}\right\}_{i=\overline{1,n}}$ and T''_zM is spanned by $\left\{\frac{\partial}{\partial \bar{z}^i}\right\}_{i=\overline{1,n}}$, moreover $T'M$ is a holomorphic vector bundle.

A similar construction can be made for the $T_C(J^{(2,0)}M)$ bundle, still mentioning that the corresponding tangent subspaces do not admit a vector bundle structure.

Indeed, let $Z = (z^i, \overset{(1)}{\eta^i}, \overset{(2)}{\eta^i})$ be local coordinates in the chart (U, Φ) from $J^{(2,0)}M$, with the (1.4) changes for $k = 2$. Further, for simplifying the formalism, we shall use the following notations $Z = (z^i, \eta^i =: \overset{(1)}{\eta^i}, \zeta^i =: \overset{(2)}{\eta^i})$.

A local basis in $T'_Z(J^{(2,0)}M)$ is $\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \zeta^i}\}_{i=\overline{1,n}}$ and in $T''_Z(J^{(2,0)}M)$ theirs conjugates $\{\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{\eta}^i}, \frac{\partial}{\partial \bar{\zeta}^i}\}_{i=\overline{1,n}}$. Due to holomorphic changes on $J^{(2,0)}M$, that is all of $\frac{\partial z'^i}{\partial \bar{z}^j}, \frac{\partial \eta'^i}{\partial \bar{z}^j}, \frac{\partial \zeta'^i}{\partial \bar{z}^j}, \frac{\partial \eta'^i}{\partial \bar{\eta}^j}, \frac{\partial \zeta'^i}{\partial \bar{\eta}^j}, \frac{\partial \zeta'^i}{\partial \bar{\zeta}^j}$ are vanishing, and also theirs conjugates, it follows that local bases from $T'_Z(J^{(2,0)}M)$ change w.r.t. the transformations (1.4) by the rules:

$$
(2.1)\qquad
\begin{aligned}
\frac{\partial}{\partial z^j} &= \frac{\partial z'^i}{\partial z^j}\frac{\partial}{\partial z'^i} + \frac{\partial \eta'^i}{\partial z^j}\frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^i}{\partial z^j}\frac{\partial}{\partial \zeta'^i}\\
\frac{\partial}{\partial \eta^j} &= \frac{\partial \eta'^i}{\partial \eta^j}\frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^i}{\partial \eta^j}\frac{\partial}{\partial \zeta'^i}\\
\frac{\partial}{\partial \zeta^j} &= \frac{\partial \zeta'^i}{\partial \zeta^j}\frac{\partial}{\partial \zeta'^i}.
\end{aligned}
$$

In view of (1.4) we immediately infer that $\frac{\partial z'^i}{\partial z^j} = \frac{\partial \eta'^i}{\partial \eta^j} = \frac{\partial \zeta'^i}{\partial \zeta^j}$, but in change $\frac{\partial \eta'^i}{\partial z^j} = \frac{\partial \zeta'^i}{\partial \eta^j}$ contain the second order derivatives of z'^i, while $\frac{\partial \zeta'^i}{\partial z^j}$ contains even the 3-th derivatives of z'^i. Thus the nonlinear type of the changes (2.1) it is obvious. By conjugation everywhere in (2.1) we obtain the changes of local bases $\{\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{\eta}^i}, \frac{\partial}{\partial \bar{\zeta}^i}\}$ from $T''_Z(J^{(2,0)}M)$.

On $T_C(J^{(2,0)}M)$ the natural complex structure $J^2 = -I$ acts as follows:

$$(2.2)\qquad \begin{aligned} J\left(\frac{\partial}{\partial z^j}\right) &= i\frac{\partial}{\partial z^j}\ ;\ J\left(\frac{\partial}{\partial \eta^j}\right) = i\frac{\partial}{\partial \eta^j}\ ;\ J\left(\frac{\partial}{\partial \zeta^j}\right) = i\frac{\partial}{\partial \zeta^j} \\ J\left(\frac{\partial}{\partial \bar{z}^j}\right) &= -i\frac{\partial}{\partial \bar{z}^j}\ ;\ J\left(\frac{\partial}{\partial \bar{\eta}^j}\right) = -i\frac{\partial}{\partial \bar{\eta}^j}\ ;\ J\left(\frac{\partial}{\partial \bar{\zeta}^j}\right) = -i\frac{\partial}{\partial \bar{\zeta}^j} \end{aligned}$$

but also the almost second order tangent structure $F^3 = 0$,

$$(2.3)\qquad F\left(\frac{\partial}{\partial z^j}\right) = \frac{\partial}{\partial \eta^j}\ ;\ F\left(\frac{\partial}{\partial \eta^j}\right) = \frac{\partial}{\partial \zeta^j}\ ;\ F\left(\frac{\partial}{\partial \zeta^j}\right) = 0$$

and analogously for the conjugates.

The structure F is globally defined - as follows from the previous remarks concerning the (2.1) changes. Moreover, F commutes with J and $rank\ F = 2n,\ rank\ J = 3n$.

Consider $W = Ker\ F^2$ and $V = Ker\ F$. We infer that $V \subset W$ and $F(W) = V$ are locally spanned by $W = < \frac{\partial}{\partial \eta^j}, \frac{\partial}{\partial \zeta^j} >$ and $V = < \frac{\partial}{\partial \zeta^j} >$.

The distribution V will be called vertical. Like in real case, we can consider the following well defined Liouville fields on $T_C(J^{(2,0)}M)$,

$$(2.4)\qquad \overset{(1)}{C} = \eta^i\frac{\partial}{\partial \zeta^i} \quad\text{and}\quad \overset{(2)}{C} = \eta^i\frac{\partial}{\partial \eta^i} + 2\zeta^i\frac{\partial}{\partial \zeta^i}$$

and theirs conjugates.

We say that a complex field $S \in T'(J^{(2,0)}M)$ is a complex 2−spray iff $F \circ S = \overset{(2)}{C}$.

Proposition 2.1. *Any complex 2−spray is locally written as* $S = \eta^i\frac{\partial}{\partial z^i} + 2\zeta^i\frac{\partial}{\partial \eta^i} - 3G^i(z,\eta,\zeta)\frac{\partial}{\partial \zeta^i}$*, where* G^i *are the coefficients of the spray and they change by the rule*

$$3G'^i = 3\frac{\partial z'^i}{\partial z^j}G^j - \left(\eta^j\frac{\partial \zeta'^i}{\partial z^j} + 2\zeta^j\frac{\partial \zeta'^i}{\partial \eta^j}\right).$$

Let $W(J^{(2,0)}M)$ be the fibre bundle defined by the distribution W at any point $Z \in J^{(2,0)}M$.

Definition 2.2. A complex nonlinear connection on $J^{(2,0)}M$, (in brief, c.n.c.), is a complex subbundle $H(J^{(2,0)}M)$, supplementary to $W(J^{(2,0)}M)$ in $T'(J^{(2,0)}M)$.

This notion has a special role in the "linearizing" of the geometry of $T_C(J^{(2,0)}M)$. Obviously, we have $\dim H(J^{(2,0)}M) = n$ and by conjugation we get

$$(2.5)\qquad T_C(J^{(2,0)}M) = H(J^{(2,0)}M) \oplus W(J^{(2,0)}M) \oplus \overline{H}(J^{(2,0)}M) \oplus \overline{W}(J^{(2,0)}M).$$

It has been pointed out that $\pi^{(2,0)} : J^{(2,0)}M \to M$ admits a fibre bundle structure over $J^{(1,0)}M$, identified with $T'M$. Let $\pi^{*(2,0)}M$ be the pull-back bundle of $T'M$ onto $T'(J^{(2,0)}M)$. The sections of $\pi^{*(2,0)}M$ are identified with the vectors of $T'(J^{(2,0)}M)$ of the form $\{\frac{\partial}{\partial z^j}\}$. Denoting by $l^h : T'M \to H(J^{(2,0)}M) \subset T'(J^{(2,0)}M)$ the isomorphism which provides $\pi^{*(2,0)} \circ l^h = Id_{T'M}$.

Proposition 2.3. *The following sequence is exact:*

$$0 \to W(J^{(2,0)}M) \xrightarrow{i} T'(J^{(2,0)}M) \xrightarrow{p} \pi^{*(2,0)}M \to 0.$$

A splitting $\mathcal{C}$ of this sequence determines a (c.n.c.) on $J^{(2,0)}M$ and $Ker\ \mathcal{C} = H(J^{2,0)}M)$. Let be $\frac{\delta}{\delta z^i} =: l^h(\frac{\partial}{\partial z^i})$, and then the vectors $\{\frac{\delta}{\delta z^i}\}_{i=\overline{1,n}}$ determine a local basis in the horizontal bundle $H(J^{2,0)}M)$, called *adapted base* to the (c.n.c). Let be

$$\frac{\delta}{\delta z^j} = \frac{\partial}{\partial z^j} - \overset{(1)}{N}{}^i_j \frac{\partial}{\partial \eta^i} - \overset{(2)}{N}{}^i_j \frac{\partial}{\partial \zeta^i}. \tag{2.6}$$

Since the horizontal lift l^h is an isomorphism of fibres, it follows that the adapted basis $\{\frac{\delta}{\delta z^i}\}$, which spans the horizontal distribution, satisfies:

$$\frac{\delta}{\delta z^j} = \frac{\partial z'^i}{\partial z^j} \frac{\delta}{\delta z'^i}. \tag{2.7}$$

Theorem 2.4. *The functions $\overset{(1)}{N}{}^i_j, \overset{(2)}{N}{}^i_j$, called the coefficients of the (c.n.c.), change by the rules:*

$$\begin{aligned} \overset{(1)}{N'}{}^i_k \frac{\partial z'^k}{\partial z^j} &= \frac{\partial z'^i}{\partial z^k} \overset{(1)}{N}{}^k_j - \frac{\partial \eta'^i}{\partial z^j} \\ \overset{(2)}{N'}{}^i_k \frac{\partial z'^k}{\partial z^j} &= \frac{\partial z'^i}{\partial z^k} \overset{(2)}{N}{}^k_j + \frac{\partial \eta'^i}{\partial z^k} \overset{(1)}{N}{}^k_j - \frac{\partial \zeta'^i}{\partial z^j}. \end{aligned} \tag{2.8}$$

The proof relies on replacing the changes (2.1) in (2.7).

Summarizing, $T'(J^{(2,0)}M)$ is a direct sum of the $H(J^{(2,0)}M)$ and $W(J^{(2,0)}M)$ bundles, of rank n and respectively $2n$, locally spanned by $\{\frac{\delta}{\delta z^i}\}$ and respectively $\{\frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \zeta^i}\}$. Moreover, the bundle $V(J^{(2,0)}M)$ defined by the vertical distribution and called the vertical bundle of $(2,0)-$jets, is a holomorphic subbundle of $W(J^{(2,0)}M)$ and hence of $T'(J^{(2,0)}M)$. Now we can split $W(J^{(2,0)}M)$ by means of the almost tangent structure F into $V(J^{(2,0)}M)$ and some supplement $H_1(J^{(2,0)}M) =: F(H(J^{(2,0)}M))$.

From (2.6) and the definition (2.3) of the F structure, it results that the distributions of $H_1(J^{(2,0)}M)$ are spanned by a set of vectors of the form:

$$\frac{\delta}{\delta \eta^j} =: F(\frac{\delta}{\delta z^j}) = \frac{\partial}{\partial \eta^j} - \overset{(1)}{N}{}^i_j \frac{\partial}{\partial \zeta^i}, \tag{2.9}$$

where $\overset{(1)}{N}{}^i_j$ satisfies the same change rule as in (2.8) and $\frac{\delta}{\delta \eta^j} = \frac{\partial z'^i}{\partial z^j} \frac{\delta}{\delta \eta'^i}$. Hence the following decomposition is obtained, $T'(J^{(2,0)}M) = H(J^{(2,0)}M) \oplus H_1(J^{(2,0)}M) \oplus$

$V(J^{(2,0)}M)$, each of these of rank n, and by conjugation is obtained a decomposition for $T''(J^{(2,0)}M)$. Finally a decomposition as a sum of six subbundles for $T_C(J^{(2,0)}M)$ is obtained.

Let $\{dz^i, \delta\eta^i, \delta\zeta^i\}_{i=\overline{1,n}}$ be the dual of the obtained adapted basis $\{\frac{\delta}{\delta z^i}, \frac{\delta}{\delta \eta^i}, \frac{\partial}{\partial \zeta^i}\}_{i=\overline{1,n}}$ by using a (c.n.c.). If $\delta\eta^i = d\eta^i + \overset{(1)}{M}{}^i_j\, dz^j$ and $\delta\zeta^i = d\zeta^i + \overset{(1)}{M}{}^i_j\, d\eta^j + \overset{(2)}{M}{}^i_j\, dz^j$, then

Proposition 2.5. $\overset{(1)}{M}{}^i_j = \overset{(1)}{N}{}^i_j$ *and* $\overset{(2)}{M}{}^i_j = \overset{(2)}{N}{}^i_j + \overset{(1)}{N}{}^i_k \overset{(1)}{N}{}^k_j$.

The proof follows from the definition of the dual basis: $dz^i(\frac{\delta}{\delta z^j}) = \delta\eta^i(\frac{\delta}{\delta \eta^j}) = \delta\zeta^i(\frac{\delta}{\delta \zeta^j}) = \delta^i_j$, the Kronecher symbol, and zero for all the other choices.

We shall further use the following abbreviations: $\delta_{0j} := \frac{\delta}{\delta z^j}$; $\delta_{1j} := \frac{\delta}{\delta \eta^j}$; $\delta_{2j} = \frac{\partial}{\partial \zeta^j}$ and $\delta_{0\bar j}$, $\delta_{1\bar j}$, $\delta_{2\bar j}$ for their conjugates.

Proposition 2.6. *The brackets of the adapted basis of a (c.n.c) are given by:*

$$
\begin{aligned}
[\delta_{0j}, \delta_{0k}] &= \overset{(1)}{A}{}^i_{(jk)}\, \delta_{1i} + \overset{(2)}{A}{}^i_{(jk)}\, \delta_{2i} \textit{ where } \overset{(\alpha)}{A}{}^i_{(jk)} = \overset{(\alpha)}{A}{}^i_{jk} - \overset{(\alpha)}{A}{}^i_{kj} := \delta_{0k}\, \overset{(\alpha)}{N}{}^i_j - \delta_{0j}\, \overset{(\alpha)}{N}{}^i_k \\
[\delta_{0j}, \delta_{0\bar k}] &= \overset{(1)}{A}{}^i_{j\bar k}\, \delta_{1i} - \overset{\overline{(1)}}{A}{}^{\bar i}_{\bar k j}\, \delta_{1\bar i} + \overset{(2)}{A}{}^i_{j\bar k}\, \delta_{2i} - \overset{\overline{(2)}}{A}{}^{\bar i}_{\bar k j}\, \delta_{2\bar i} \textit{ where } \overset{(\alpha)}{A}{}^i_{j\bar k} := \delta_{0\bar k}\, \overset{(\alpha)}{N}{}^i_j \,;\, \overset{\overline{(\alpha)}}{A}{}^{\bar i}_{\bar j k} = \overline{\overset{(\alpha)}{A}{}^i_{j\bar k}} \\
[\delta_{0j}, \delta_{1k}] &= -\overset{(1)}{A}{}^i_{kj}\, \delta_{2i} + \overset{(1)}{B}{}^i_{jk}\, \delta_{1i} + \overset{(2)}{B}{}^i_{jk}\, \delta_{2i} \textit{ where } \overset{(\alpha)}{B}{}^i_{jk} := \delta_{1k}\, \overset{(\alpha)}{N}{}^i_j \;;\; \alpha = 1,2 \\
[\delta_{0j}, \delta_{1\bar k}] &= -\overset{\overline{(1)}}{A}{}^{\bar i}_{\bar k j}\, \delta_{2\bar i} + \overset{(1)}{B}{}^i_{j\bar k}\, \delta_{1i} + \overset{(2)}{B}{}^i_{j\bar k}\, \delta_{2i} \textit{ where } \overset{(\alpha)}{B}{}^i_{j\bar k} := \delta_{1\bar k}\, \overset{(\alpha)}{N}{}^i_j \\
[\delta_{0j}, \delta_{2k}] &= \overset{(1)}{C}{}^i_{jk}\, \delta_{1i} + \overset{(2)}{C}{}^i_{jk}\, \delta_{2i} \textit{ where } \overset{(\alpha)}{C}{}^i_{jk} := \delta_{2k}\, \overset{(\alpha)}{N}{}^i_j \\
[\delta_{0j}, \delta_{2\bar k}] &= \overset{(1)}{C}{}^i_{j\bar k}\, \delta_{1i} + \overset{(2)}{C}{}^i_{j\bar k}\, \delta_{2i} \textit{ where } \overset{(\alpha)}{C}{}^i_{j\bar k} := \delta_{2\bar k}\, \overset{(\alpha)}{N}{}^i_j \\
[\delta_{1j}, \delta_{1k}] &= \overset{(1)}{B}{}^i_{(jk)}\, \delta_{2i} := (\overset{(1)}{B}{}^i_{jk} - \overset{(1)}{B}{}^i_{kj})\delta_{2i} \\
[\delta_{1j}, \delta_{1\bar k}] &= \overset{(1)}{B}{}^i_{j\bar k}\, \delta_{2i} - \overset{\overline{(1)}}{C}{}^{\bar i}_{\bar k j}\, \delta_{2\bar i} \\
[\delta_{1j}, \delta_{2k}] &= \overset{(1)}{C}{}^i_{jk}\, \delta_{2i} \;;\; [\delta_{1j}, \delta_{2\bar k}] = \overset{\overline{(1)}}{C}{}^{\bar i}_{\bar k j}\, \delta_{2\bar i} \\
[\delta_{2j}, \delta_{2k}] &= 0 \;;\; [\delta_{2j}, \delta_{2\bar k}] = 0.
\end{aligned}
$$

We note that the horizontal distribution H is integrable if $\overset{(\alpha)}{A}{}^i_{jk} = \overset{(\alpha)}{A}{}^i_{j\bar k} = 0$ for all $\alpha = 1,2$, and with respect to the adapted bases of a (c.n.c) we have:

$$J(\delta_{\beta j}) = i\delta_{\beta j} \;\;;\;\; J(\delta_{\beta \bar j}) = -i\delta_{\beta \bar j} \;\;,\;\; \beta = 0,1,2$$

and

$$
\begin{aligned}
F(\delta_{0j}) &= \delta_{1j} \;;\; F(\delta_{1j}) = \delta_{2j} \;;\; F(\delta_{2j}) = 0 \;; \\
F(\delta_{0\bar j}) &= \delta_{1\bar j} \;;\; F(\delta_{1\bar j}) = \delta_{2\bar j} \;;\; F(\delta_{2\bar j}) = 0.
\end{aligned} \tag{2.10}
$$

Hence we may consider the following structure:

$$\begin{aligned} F^*(\delta_{2j}) &= \delta_{1j}\ ;\ F^*(\delta_{1j}) = \delta_{0j}\ ;\ F^*(\delta_{0j}) = 0\ ; \\ F^*(\delta_{2\bar{j}}) &= \delta_{1\bar{j}}\ ;\ F^*(\delta_{1\bar{j}}) = \delta_{0\bar{j}}\ ;\ F^*(\delta_{0\bar{j}}) = 0. \end{aligned} \tag{2.11}$$

Let h, h_1, v and $\bar{h}, \bar{h}_1, \bar{v}$ be the projectors on the six distributions determined by a (c.n.c.) on $J^{(2,0)}M$ and the almost second order tangent structure F.

Proposition 2.7. *F^* is a well defined almost second order tangent structure, $F^{*3} = 0$, called adjoint to F, and its restriction to $T'(J^{(2,0)}M)$ satisfies:*

i) $F^*h = 0,\ FF^* = v + h_1\ ;\ FF^* = h + h_1\ ;$

ii) $vF^* = 0\ ;\ F^*h = 0\ ;\ (h_1 + v)F^{*2} = 0\ ;$

iii) $F^2F^* = vF\ ;\ F^{*2}F = hF^*\ ;\ F^2F^{*2} = v\ ;\ F^{*2}F^2 = h.$

Similar formulas are fulfilled on $T''(J^{(2,0)}M)$.

We shall further introduce a special type of derivation law acting on $T_C(J^{(2,0)}M)$, named $N-$complex linear connection of $J^{(2,0)}M$, in brief $N-(c.l.c.)$.

Any derivation law D on $T_C(J^{(2,0)}M)$ admits the decomposition $D = D' + D''$. Moreover, D is is called *distinguished,* $d-$(c.l.c.), if it preserves the six distributions of $T_C(J^{(2,0)}M)$ defined by a (c.n.c.).

Proposition 2.8. *D is a $d-$(c.l.c.) if and only if $DJ = 0$ and $Dh = Dh_1 = Dv = D\bar{h} = D\bar{h}_1 = D\bar{v} = 0$.*

Proof. $DJ = 0$ implies D' is of $(1,0)$ type and D'' is of $(0,1)$ type. Further, $Dh = 0$ means that $D_X(hY) = hD_XY$, and hence D preserves the horizontal distribution. Similarly, $Dh_1 = 0$ means that D preserves H_1, etc. □

A $d-$(c.l.c.) admits the decomposition $D = D^h + D^{h_1} + D^v + D^{\bar{h}} + D^{\bar{h}_1} + D^{\bar{v}}$ as follows:

$$\begin{aligned} D^h_{\delta_{0k}}\delta_{0j} &= \overset{0}{L}{}^i_{jk}\ \delta_{0i}\ ;\ D^h_{\delta_{0k}}\delta_{1j} = \overset{1}{L}{}^i_{jk}\ \delta_{1i}\ ;\ D^h_{\delta_{0k}}\delta_{2j} = \overset{2}{L}{}^i_{jk}\ \delta_{2i} \\ D^h_{\delta_{1k}}\delta_{0j} &= \overset{0}{F}{}^i_{jk}\ \delta_{0i}\ ;\ D^h_{\delta_{1k}}\delta_{1j} = \overset{1}{F}{}^i_{jk}\ \delta_{1i}\ ;\ D^h_{\delta_{1k}}\delta_{2j} = \overset{2}{F}{}^i_{jk}\ \delta_{2i} \\ D^h_{\delta_{2k}}\delta_{0j} &= \overset{0}{C}{}^i_{jk}\ \delta_{0i}\ ;\ D^h_{\delta_{2k}}\delta_{1j} = \overset{1}{C}{}^i_{jk}\ \delta_{1i}\ ;\ D^h_{\delta_{2k}}\delta_{2j} = \overset{2}{C}{}^i_{jk}\ \delta_{2i} \end{aligned}$$

and

$$\begin{aligned} D^h_{\delta_{0k}}\delta_{0\bar{j}} &= \overset{0}{L}{}^{\bar{1}}_{\bar{j}k}\ \delta_{0\bar{1}}\ ;\ D^h_{\delta_{0k}}\delta_{1\bar{j}} = \overset{1}{L}{}^{\bar{1}}_{\bar{j}k}\ \delta_{1\bar{1}}\ ;\ D^h_{\delta_{0k}}\delta_{2\bar{j}} = \overset{2}{L}{}^{\bar{1}}_{\bar{j}k}\ \delta_{2\bar{1}} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

with $\overline{D_XY} = D_{\bar{X}}\bar{Y}$.

The coefficients of the $d-$(c.l.c.) D chang by the rules:

$$\overset{\alpha}{L}{}'^i_{jk} = \frac{\partial z'^i}{\partial z^r}\frac{\partial z^p}{\partial z'^j}\frac{\partial z^q}{\partial z'^k}\overset{\alpha}{L}{}^r_{pq} + \frac{\partial z'^i}{\partial z^p}\frac{\partial^2 z^p}{\partial z'^j\partial z'^k}\ ;\ \alpha = 0,1,2$$

and all the other coefficients are $d-$ complex tensors, i.e.,

$$\begin{aligned}
\overset{\alpha}{L}{}'^{\bar{i}}_{\bar{j}k} &= \frac{\partial \bar{z}'^{i}}{\partial \bar{z}^{r}} \frac{\partial \bar{z}^{p}}{\partial \bar{z}'^{j}} \frac{\partial z^{q}}{\partial z'^{k}} \overset{\alpha}{L}{}^{\bar{r}}_{\bar{p}q}\ ; \quad \alpha = 0,1,2 \\
\overset{\alpha}{F}{}'^{i}_{jk} &= \frac{\partial z'^{i}}{\partial z^{r}} \frac{\partial z^{p}}{\partial z'^{j}} \frac{\partial z^{q}}{\partial z'^{k}} \overset{\alpha}{F}{}^{r}_{pq}\ ; \ \overset{\alpha}{F}{}'^{\bar{i}}_{\bar{j}k} = \frac{\partial \bar{z}'^{i}}{\partial \bar{z}^{r}} \frac{\partial \bar{z}^{p}}{\partial \bar{z}'^{j}} \frac{\partial z^{q}}{\partial z'^{k}} \overset{\alpha}{F}{}^{\bar{r}}_{\bar{p}q}; \\
\overset{\alpha}{C}{}'^{i}_{jk} &= \frac{\partial z'^{i}}{\partial z^{r}} \frac{\partial z^{p}}{\partial z'^{j}} \frac{\partial z^{q}}{\partial z'^{k}} \overset{\alpha}{C}{}^{r}_{pq}\ ; \ \overset{\alpha}{C}{}'^{\bar{i}}_{\bar{j}k} = \frac{\partial \bar{z}'^{i}}{\partial \bar{z}^{r}} \frac{\partial \bar{z}^{p}}{\partial \bar{z}'^{j}} \frac{\partial z^{q}}{\partial z'^{k}} \overset{\alpha}{C}{}^{\bar{r}}_{\bar{p}q}\ .
\end{aligned}$$

Such connection generally infer heavy computation.

We say that D *is a* $N-$*(c.l.c)* if and only if $\overset{0}{L}{}^{i}_{jk}=\overset{1}{L}{}^{i}_{jk}=\overset{2}{L}{}^{i}_{jk}$; $\overset{0}{L}{}^{\bar{i}}_{\bar{j}k}=\overset{1}{L}{}^{\bar{i}}_{\bar{j}k}=\overset{2}{L}{}^{\bar{i}}_{\bar{j}k}$ and $\overset{0}{F}{}^{i}_{jk}=\overset{1}{F}{}^{i}_{jk}=\overset{2}{F}{}^{i}_{jk}$; $\overset{0}{F}{}^{\bar{i}}_{\bar{j}k}=\overset{1}{F}{}^{\bar{i}}_{\bar{j}k}=\overset{2}{F}{}^{\bar{i}}_{\bar{j}k}$; $\overset{0}{C}{}^{i}_{jk}=\overset{1}{C}{}^{i}_{jk}=\overset{2}{C}{}^{i}_{jk}$; $\overset{0}{C}{}^{\bar{i}}_{\bar{j}k}=\overset{1}{C}{}^{\bar{i}}_{\bar{j}k}=\overset{2}{C}{}^{\bar{i}}_{\bar{j}k}$.

Theorem 2.9. *D is a $N-$(c.l.c) if and only if $DJ = DF = DF^{*} = 0$.*

Proof. $DJ = 0$ implies that D preserves both distributions $T'(J^{(2,0)}M)$ and $T''(J^{(2,0)}M)$.

If $DF = DF^{*} = 0$, then in view of Proposition 2.5 it follows $Dh = Dh_1 = Dv = D\bar{h} = D\bar{h}_1 = D\bar{v} = 0$, and hence D is a d-(c.l.c). Expressing $D_X(FY) = FD_XY$ for $X = \delta_{0k}$ and $Y = \delta_{0j}$ we infer $\overset{0}{L}{}^{i}_{jk}=\overset{1}{L}{}^{i}_{jk}$. Taking X and Y as choices in the set $\{\delta_{\alpha k}, \delta_{\alpha \bar{k}}\}$, one can prove the remaining claims. □

The torsions and the curvatures of a $N-$(c.l.c.) are deduced by direct computation, replacing X, Y, Z with the fields of the adapted frame $\{\delta_{\alpha k}, \delta_{\alpha \bar{k}}\}_{\alpha=0,1,2}$.

References

[1] M. Abate, G. Patrizio, *Finsler Metrics - A Global Approach*, Lecture Notes in Math. 1591, Springer-Verlag, 1994.

[2] T. Aikou, *Finsler geometry on complex vector bundles*, Riemannian Finsler Geometry, MSRI Pub. 50, 2004.

[3] G. Atanasiu, *New aspects in differential geometry of the second order*, Seminarul de Mecanica, Univ. din Timisoara, 2001, 1-81.

[4] G. Atanasiu, M. Neagu, *Canonical nonlinear connections in the multi-time Hamilton Geometry*, Balkan J. Geom. Appl., 14, 2 (2009), 1-12.

[5] D. Bao, S.S. Chern, Z. Shen, *An Introduction to Riemannian Finsler Geometry*, Graduate Texts in Math., Springer-Verlag, 2000.

[6] N. Branzei, *Geodesics and Jacobi fields in second order geometry*, Ed. Univ. "Transilvania" Brasov, 2007.

[7] I. Bucataru, *Characterisation of the nonlinear connection in the higher order geometry*, Balkan J. Geom. Appl., 2, 2 (1997), 13-22.

[8] K. Chandler, P-M. Wong, *Finsler geometry of holomorphic jet bundles*, Riemann-Finsler geometry, MSRI Publ., 50 (2004), 107-196.

[9] M. Green, P. Griffiths, *Two aplications of algebraic geometry to entire holomorphic mappings*, The Chern Symp. Berkeley, 1979, Springer, New York, 1980.

[10] A. Manea, *A decomposition of the bundle of second order jets on a complex manifold,* Analle St. Univ. "Al. I. Cuza", Iasi, LVI (2010), 151-162.

[11] I.M. Masca, V.S. Sabau, H. Shimada, *The L-dual of an* (α, β) *Finsler space of order two*, Balkan J. Geom. Appl., 12, 1 (2007), 85-99.

[12] R. Miron, *The Geometry of Higher Order Lagrange Spaces. Aplications to Mechanics and Physics*, FTPH 82, Kluwer Acad. Publ., 1997.

[13] R. Miron, G. Atanasiu, *Higher-order Lagrange spaces*, Rev. Roumaine Math. Pures et Appl., 41, 3-4 (1996), 251-263.

[14] F. Munteanu, *Second order partial differential equations (SOPDEs) and nonlinear connections on the tangent bundle of* k^1*-velocities of a manifold*, Diff. Geom. Dyn. Syst., 8 (2006), 166-180.

[15] G. Munteanu, *Complex spaces in Finsler, Lagrange and Hamilton geometries*, Kluwer Acad. Publ., FTPH 141, 2004.

[16] G. Munteanu, *On almost tangent structure of the second order*, Bul. Univ. "Transilvania" Brasov 31, 1989, 23-28.

[17] G. Munteanu, G. Atanasiu, *On Miron connections on Lagrange space of second order*, Tensor N.S. 56 (1995), 166-174.

[18] M. Roman, *Special Higher Order Lagrange Spaces. Aplications*, Ph.D. Thesis, Univ. Al. I. Cuza, Iasi, 2001.

[19] D.J. Saunders, *The Geometry of Jet Bundles*, London Math. Soc., Lecture Note Series, 142, Cambridge U.P. 1989.

[20] W. Stoll, P.-M. Wong, *On holomorphic jet bunldes*, preprint, arxiv: math/0003226v1/2000.

[21] P.-M. Wong, *A survey of complex Finsler Geometry*, Advanced Studies of Pure Math. 48, Math. Soc. Japan (Finsler Geometry in memory of Matsumoto 2005), (2007), 375-433.

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