Higher order Hessian structures on Fréchet manifolds

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Abstract. The concept of higher order Hessian structures on finite dimensional manifolds is introduced by Kumar in [12]. Following that formalism, we establish a one-to-one correspondence between third-order Hessian structures and a special class of connections on the second-order tangent bundle for a wide class of Fréchet manifolds i.e. those which can be considered as projective limits of Banach manifolds. These connections are called special second-order connections. Moreover the second-order geodesics of special second-order connections are introduced and they are generalized to the category of those Fréchet manifolds which can be considered as projective limits of Banach manifolds.

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1 Introduction

There is an increasing interest in the study of infinite dimensional manifolds because of their applications in jet fields, Lagrangians, Finsler structure, theoretical physics, etc. (for example see [1], [4], [5], [11], and [17]).
The presence of main difficulties in Fréchet manifolds (even Fréchet spaces) and also the fact that every Fréchet space is a projective limit of Banach spaces [18], leads us to use the algebraic tool *projective limit* which is compatible under certain assumptions with differential tools, to study this type of manifolds.
The most important obstacles in Fréchet case are pathological structure of the general linear group $GL(\mathbb{F})$ for a Fréchet space $\mathbb{F}$ and the lack of a general solvability-uniqueness theory for ordinary differential equations on these spaces [11]. Our approach shows that we can overcome these obstacles if we restrict ourselves to those Fréchet manifolds which are obtained as projective limits of Banach manifolds ([2], [3], [4], [5] and [8]).

In our last works with C.T.J. Dodson and G.N. Galanis ([2], [3]), by using the bundle of accelerations, we proposed an alternative way to study the second order ordinary differential equations on infinite dimensional non-Banach modelled manifolds and we generalized some geometric structures to projective limit manifolds.

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In this paper the concepts of the third-order Hessian structures and an appropriate second order tangent bundle are extended to this category of manifolds. Moreover for the second order special connections on projective limit manifolds we obtain a system of ordinary differential equations for the corresponding geodesics and we propose a way to study these systems in spite of the fact that there is no general solvability-uniqueness theory for ordinary differential equations on such manifolds. Finally as an application we associate to any second order special connection an ordinary differential equation on a trivial bundle which translates the geometric structures to ordinary differential equations.

2 Preliminaries

Let M be a Hausdorff paracompact smooth manifold modelled on a Banach space $\mathbb{E}$ (possibly infinite dimensional). For $m \in M$, let $G_m := \{f \in C^\infty(U_m) : U_m$ is a neighborhood of $m\}$, $G^0_m := \{f \in G_m : f(m)=0\}$ and $G^0_{m,n} := \{\sum_{i=1}^k f^i_1 f^i_2 ... f^i_n : f^i_j \in G^0_m,\ 1 \le j \le n,\ k \in \mathbb{N}\}$. Then $f \in G^0_{m,n}$ if and only if for any chart (U, φ), $f_\varphi{:=}f \circ \varphi^{-1}$ has vanishing partial derivatives up to order $(n-1)$.

First we present a short description of a wide class of Fréchet manifolds i.e. those which can be considered as projective limits of Banach manifolds. Let $\{M_i, \varphi_{ji}\}_{i,j\in\mathbb{N}}$ be a projective system of Banach manifolds modelled on the Banach spaces $\{\mathbb{E}_i\}_{i\in\mathbb{N}}$ respectively. The models $\{\mathbb{E}_i\}_{i\in\mathbb{N}}$ also form a projective system with connecting morphisms $\rho_{ji} : \mathbb{E}_j \longrightarrow \mathbb{E}_i$ for $j \ge i$, and $\mathbb{F} := \varprojlim \mathbb{E}_i$. Furthermore, suppose that for each $m{=}(m_i)_{i\in\mathbb{N}} \in M{:=}\varprojlim M_i$ there exists a projective system of local charts $\{(U_i, \varphi_i)\}_{i\in\mathbb{N}}$ such that $m_i \in U_i$ and the corresponding limits $\varprojlim U_i$, $\varprojlim \varphi_i(U_i)$ are open in M and $\mathbb{F}$ respectively. By these means $M{=}\varprojlim M_i$ turns out to be a Fréchet manifold modelled on $\mathbb{F}$ [1]. To introduce the second-order tangent bundle on Fréchet manifolds we define $\mathbb{F}^{(2)}{:=}\varprojlim \mathbb{E}^{(2)}_i$ where, for any $i \in \mathbb{N}$, $\mathbb{E}^{(2)}_i{=}\mathbb{E}_i \oplus (\mathbb{E}_i \otimes \mathbb{E}_i)$ and for $j \ge i$ the connecting morphisms $\rho^{(2)}_{ji} : \mathbb{E}^{(2)}_j \longrightarrow \mathbb{E}^{(2)}_i$ are given by $\rho^{(2)}_{ji}{:=}\rho_{ji} \oplus (\rho_{ji} \otimes \rho_{ji}){:=}\rho_{ji} \oplus \rho'_{ji}$.

3 second order tangent bundles

Let $\Omega^n(M)$ be the space of all covariant tensors of n-th order on M. If $f \in G^0_{n,m}$, then the n-th order derivative of f is well-defined. Here we restate the following definition from [12].

Definition 3.1. An n-th order Hessian structure H^n on M is a mapping $H^n : C^\infty(M) \longrightarrow \Omega^n(M)$, $f \longmapsto H^n f$, such that (i) H^n is real linear in f (ii) if $f \in G^0_{n,m}$ then $H^n f(m) = D^n f(m)$.

Now let ∇ be a connection on the tangent bundle TM of M. For $f \in C^\infty(M)$, define $\nabla f = df$ and $\nabla^n f = \nabla(\nabla^{n-1} f)$ where the action of ∇ is the covariant derivative acting on covariant tensors defining the Hessian structures H^n. Kumar in [12] showed that if H^3 is a third order Hessian structure on M, then there exists a connection on the second-order tangent bundle, say $\tilde{\nabla}$, such that H^3 arises from $\tilde{\nabla}$. In [2] we considered the case $n = 2$ and generalized these concepts to Fréchet manifolds.

To develop the case $n = 3$ for Fréchet manifolds we need the concepts of Hessian structures and second-order connections on the corresponding spaces.

Theorem 3.1. *The limit of a projective system of n-th order Hessian structures on* $\{M_i\}_{i\in\mathbb{N}}$ *is an n-th order Hessian structure on* $M = \varprojlim M_i$.

To introduce the second-order tangent bundle on Fréchet manifolds, we first construct it for the Banach case. Let $\mathbb{E}\otimes\mathbb{E}$ denote the symmetric tensor product of $\mathbb{E}$ with itself. One can identify the symmetric bilinear maps from $\mathbb{E}\times\mathbb{E}$ to $\mathbb{E}$ with linear maps from $\mathbb{E}\otimes\mathbb{E}$ to $\mathbb{E}$. Each typical element of the space $\mathbb{E}^{(2)}{:=}\mathbb{E}\oplus(\mathbb{E}\otimes\mathbb{E})$ is denoted by $e\oplus x$. For any chart (U,φ) at $m\in M$, consider the triples of the form $(U,\varphi,e\oplus x)$. Triples $(U,\varphi,e\oplus x)$ and $(V,\psi,\bar{e}\oplus\bar{x})$ are called equivalent if $\bar{e}{=}DF(u).e + D^2F(u).x$ and $\bar{x}{=}[DF(u)\otimes DF(u)].x$, where $F{=}\psi\circ\varphi^{-1}$ and $u{=}\varphi m$. It is not difficult to show that this is an equivalence relation. Each equivalence class is called a second-order tangent vector at m. The set of all equivalence classes are denoted by $T^{(2)}_m M$ and its typical element is denoted by t. For $(U,\varphi,e\oplus x)\in t$ let $t_\varphi{=}e\oplus x$ and $t{=}[U,\varphi,e\oplus x]$. If $f\in C^\infty(M)$, then define $t(f)(m){:=}D^2f_\varphi(\varphi m).x + Df_\varphi(\varphi m)e$. It is clear that if $f\in G^0_{m,n}$ then $t(f)(m){=}0$ (see also [12]).

Theorem 3.2. $T^{(2)}M$ *is a vector bundle on* M *with structural group* $GL(\mathbb{E}^{(2)})$.

Proof. Let $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}$ be an atlas for M and $\pi : T^{(2)}M \longrightarrow M$ be the canonical projection. For each $\alpha\in I$ consider the map

$$\begin{array}{rcl} \tau_\alpha:\pi^{-1}(U_\alpha) & \longrightarrow & U_\alpha\times\mathbb{E}^{(2)} \\ {[U_\beta,\varphi_\beta,e_\beta\oplus x_\beta]} & \longmapsto & \big(m, DF(u).e_\beta + D^2F(u).x_\beta \oplus [DF(u)\otimes DF(u)].x_\beta\big) \end{array}$$

where $(U_\alpha,\varphi_\alpha)$ is a chart of M around m, $F{=}\varphi_\alpha\circ\varphi_\beta^{-1}$, $u=\varphi_\beta m$ and $[U_\beta,\varphi_\beta,e_\beta\oplus x_\beta]\in T^{(2)}M$. We claim that τ_α is a local trivialization for $T^{(2)}M$. To show that τ_α is well defined, let $(U_\gamma,\varphi_\gamma)$ and (U_β,φ_β) be two charts at m, with $[U_\beta,\varphi_\beta,e_\beta\oplus x_\beta]{=}[U_\gamma,\varphi_\gamma,e_\gamma\oplus x_\gamma]$. Then $e_\gamma = DG(u).e_\beta + D^2G(u).x_\beta$ and $x_\gamma = [DG(u)\otimes DG(u)].x_\beta$ where $G{=}\varphi_\gamma\circ\varphi_\beta^{-1}$. If $H{=}\varphi_\alpha\circ\varphi_\gamma^{-1}$ and $w{=}\varphi_\gamma m$, then

$$\begin{array}{rcl} DH(w).e_\gamma + D^2H(w).x_\gamma & = & DH(w)[DG(u).e_\beta + D^2G(u).x_\beta] \\ & & +D^2H(w)[DG(u)\otimes DG(u)].x_\beta \\ & = & DF(u).e_\beta + D^2F(u).x_\beta. \end{array}$$

Moreover

$$\begin{array}{rcl} [DH(w)\otimes DH(w)].x_\gamma & = & [DH(w)\otimes DH(w)][DG(u)\otimes DG(u)].x_\beta \\ & = & [DF(u)\otimes DF(u)]x_\beta \end{array}$$

which yields that τ_α, for any $\alpha\in I$, is well defined. τ_α is injective since

$$\tau_\alpha([U_\beta,\varphi_\beta,e_\beta\oplus x_\beta]) = \tau_\alpha([U_\gamma,\varphi_\gamma,e_\gamma\oplus x_\gamma])$$

gives $[DF(u)\otimes DF(u)].x_\beta = [DH(w)\otimes DH(w)].x_\gamma$ and $DF(u).e_\beta + D^2F(u).x_\beta = DH(w).e_\gamma + D^2H(w).x_\gamma$. Setting $v{=}\varphi_\alpha m$ and $x_\beta = (x^1_\beta, x^2_\beta)$, we get

$$\begin{array}{rcl} [DG(u)\otimes DG(u)].x_\beta & = & [D(H^{-1}\circ F)(u)\otimes D(H^{-1}\circ F)(u)].x_\beta \\ & = & [DH^{-1}(v)\otimes DH^{-1}(v)][DF(u)\otimes DF(u)].x_\beta = x_\gamma \end{array}$$

and

$$\begin{aligned} DG(u).e_\beta + D^2G(u).x_\beta &= DH^{-1}(v)DF(u).e_\beta + DH^{-1}(v)D^2F(u)(x^1_\beta, x^2_\beta) \\ &\quad + D^2H^{-1}(v)(DF(u).x^1_\beta, DF(u)x^2_\beta) \\ &= DH^{-1}(v)[DH(w).e_\gamma + D^2H(w)x_\gamma] \\ &\quad + D^2H^{-1}(v)[DH(w)\otimes DH(w)].x_\gamma = e_\gamma. \end{aligned}$$

Moreover, we claim that τ_α is surjective. In fact for an arbitrary element $(m, e_\beta \oplus x_\beta) \in U_\alpha \times \mathbb{E}^{(2)}$ we have

$$\begin{aligned} &\tau_\alpha([U_\beta, \varphi_\beta, DF^{-1}(v).e_\beta + D^2F^{-1}(v).x_\beta \oplus [DF^{-1}(v)\otimes DF^{-1}(v)].x_\beta]) \\ &= (m, DF(u).[DF^{-1}(v).e_\beta + D^2F^{-1}(v).x_\beta] \\ &\quad + D^2F(u).[DF^{-1}(v)\otimes DF^{-1}(v)].x_\beta \\ &\quad \oplus [DF(u)\otimes DF(u)][DF^{-1}(v)\otimes DF^{-1}(v)].x_\beta) \\ &= (m, DF(u)DF^{-1}(v).e_\beta + DF(u)D^2F^{-1}(v).x_\beta) \\ &\quad + D^2F(u).[DF^{-1}(v)\otimes DF^{-1}(v)].x_\beta \oplus \\ &\quad [DF(u)\otimes DF(u)][DF^{-1}(v)\otimes DF^{-1}(v)].x_\beta) \\ &= (m, e_\beta \oplus x_\beta) \end{aligned}$$

which completes the proof.

□

Note that for any $\alpha, \alpha' \in I$ with $U_\alpha \cap U_{\alpha'} \neq \varnothing$, the transition functions of $T^{(2)}M$ are:

$$\tau_\alpha \circ {\tau_{\alpha'}}^{-1} : (U_\alpha \cap U_{\alpha'}) \times \mathbb{E}^{(2)} \longrightarrow (U_\alpha \cap U_{\alpha'}) \times \mathbb{E}^{(2)}$$

$$(m, e_\beta \oplus x_\beta) \longmapsto (m, [DF(u).e_\beta + D^2F(u).x_\beta] \oplus [DF(u)\otimes DF(u)].x_\beta\big);\ \alpha \in I.$$

The following theorem paves the way to endow $T^{(2)}M$ with a differentiable Fréchet manifold structure as well as a generalized vector bundle structure.

Theorem 3.3. *Let $\{M_i\}_{i\in\mathbb{N}}$ be a projective system of manifolds. Then $\{T^{(2)}M_i\}_{i\in\mathbb{N}}$ is also a projective system with the limit (set-theoretically) isomorphic to $T^{(2)}M = \varprojlim T^{(2)}M_i$.*

Proof. For any $i, j \in \mathbb{N}$, with $j \geq i$, we define the map

$$\begin{aligned} s_{ji} : T^{(2)}M_j &\longrightarrow T^{(2)}M_i \\ [U_j, \varphi_j, e_j \oplus x_j]_j &\longmapsto [U_i, \varphi_i, e_i \oplus x_i]_i, \end{aligned}$$

where $U_i = \varphi_{ji}(U_j)$ and $e_i \oplus x_i = \rho^{(2)}_{ji}(e_j \oplus x_j)$. To show that s_{ji} is well defined we consider two charts (U_j, φ_j) and $(\bar{U}_j, \bar{\varphi}_j)$ at $m_j \in M_j$ and suppose that $[U_j, \varphi_j, e_j \oplus x_j]_j = [\bar{U}_j, \bar{\varphi}_j, \bar{e_j} \oplus \bar{x_j}]_j$. Then $\bar{e_j} = DF_j(u_j).e_j + D^2F_j(u_j).x_j$ and

$\bar{x_j} = [DF_j(u_j) \otimes DF_j(u_j)]\ .x_j$, where $F_j = \bar{\varphi_j} \circ \varphi_j^{-1}$ and $u_j = \varphi_j m_j$. Let $F_i = \bar{\varphi_i} \circ \varphi_i^{-1}$ and $x_j = (x_j^1 \otimes x_j^2)$. Then $DF_i(\varphi_i m_i).e_i + D^2 F_i(\varphi_i m_i).x_i = \rho_{ji}\bar{e_j} = \bar{e}_i$ and

$$\begin{aligned}
[DF_i(\varphi_i m_i) \otimes DF_i(\varphi_i m_i)].x_j &= [DF_i(\varphi_i \varphi_{ji} m_j) \otimes DF_i(\varphi_i \varphi_{ji} m_j)].x_j \\
&= [DF_i(\rho_{ji}\varphi_j m_j) \otimes DF_i(\rho_{ji}\varphi_j m_j)].x_j \\
&= [\rho_{ji} DF_j(\varphi_j m_j) \otimes \rho_{ji} DF_j(\varphi_j m_j)].(x_j^1 \otimes x_j^2) \\
&= [\rho_{ji} DF_j(\varphi_j m_j).x_j^1 \otimes \rho_{ji} DF_j(\varphi_j m_j).x_j^2] \\
&= (\rho_{ji}\bar{x}_j^1 \otimes \rho_{ji}\bar{x}_j^2) = (\bar{x}_i^1 \otimes \bar{x}_i^2) = \bar{x}_i.
\end{aligned}$$

As a consequence we have

$$s_{ji}([U_j, \varphi_j, e_j \oplus x_j]_j) = [U_i, \varphi_i, e_i \oplus x_i]_i = [\bar{U}_i, \bar{\varphi}_i, \bar{e}_i \oplus \bar{x}_i]_i = s_{ji}([\bar{U}_j, \bar{\varphi}_j, \bar{e_j} \oplus \bar{x_j}]_j).$$

Furthermore, for $k \leq i \leq j$, $s_{ik} \circ s_{ji} = s_{jk}$ which yields that $\{T^{(2)}M_i, s_{ji}\}_{i,j\in\mathbb{N}}$ is a projective system i.e. $\varprojlim T^{(2)}M_i$ exists. For any $i \in \mathbb{N}$ define

$$\begin{aligned}
S_i : T^{(2)}M &\longrightarrow T^{(2)}M_i \\
[U = \varprojlim U_i, \varphi = \varprojlim \varphi_i, e \oplus x = (e_i \oplus x_i)_{i\in\mathbb{N}}] &\longmapsto [U_i, \varphi_i, e_i \oplus x_i]_i.
\end{aligned}$$

It is easily seen that $s_{ji} \circ S_j = S_i$ holds for any $j \geq i$. The map

$$\begin{aligned}
S = \varprojlim S_i : T^{(2)}M &\longrightarrow \varprojlim T^{(2)}M_i \\
[U, \varphi, e \oplus x] &\longmapsto ([U_i, \varphi_i, e_i \oplus x_i]_i)_{i\in\mathbb{N}}
\end{aligned}$$

is well defined. This is also a one-to-one mapping since $S([U, \varphi, e \oplus x]) = S([\bar{U}, \bar{\varphi}, \bar{e} \oplus \bar{x}])$ yields

$$[U_i, \varphi_i, e_i \oplus x_i] = [\bar{U}_i, \bar{\varphi}_i, \bar{e}_i \oplus \bar{x}_i],\ i \in \mathbb{N},$$

and consequently

$$\begin{aligned}
&[U, \varphi, e \oplus x] = [\varprojlim U_i, \varprojlim \varphi_i, (e_i \oplus x_i)_{i\in\mathbb{N}}] = \varprojlim [U_i, \varphi_i, e_i \oplus x_i] \\
&= \varprojlim [\bar{U}_i, \bar{\varphi}_i, \bar{e}_i \oplus \bar{x}_i] = [\varprojlim \bar{U}_i, \varprojlim \bar{\varphi}_i, (\bar{e}_i \oplus \bar{x}_i)_{i\in\mathbb{N}}] = [\bar{U}, \bar{\varphi}, \bar{e} \oplus \bar{x}].
\end{aligned}$$

Moreover, S is surjective since for every $([U_i, \varphi_i, e_i \oplus x_i]_i)_{i\in\mathbb{N}} \in \varprojlim T^{(2)}M_i$, $S(b) = ([U_i, \varphi_i, e_i \oplus x_i]_i)_{i\in\mathbb{N}}$ where $b = [\varprojlim U_i, \varprojlim \varphi_i, (e_i \oplus x_i)_{i\in\mathbb{N}}]$. □

Theorem 3.4. $T^{(2)}M = \varprojlim T^{(2)}M_i$ *possesses a generalized Fréchet vector bundle structure on* $M = \varprojlim M_i$.

Proof. Let $\{(U_\alpha = \varprojlim U_\alpha^i, \varphi_\alpha = \varprojlim \varphi_\alpha^i)\}_{\alpha\in I}$ be a projective limit atlas for M. For α, $\beta \in I$, we define a vector bundle structure with fiber type ${\mathbb{E}_i}^{(2)}$ on $T^{(2)}M_i$ over M_i with the following local trivialization

$$\begin{aligned}
{\tau_\alpha}^i : {\pi_i}^{-1}({U_\alpha}^i) &\longrightarrow {U_\alpha}^i \times {\mathbb{E}_i}^{(2)} \\
[U_\beta^i, \varphi_\beta^i, x_\beta^i \oplus e_\beta^i] &\longmapsto (m_i, [DF_i(u_i).e_\beta^i + D^2F_i(u_i).x_\beta^i] \oplus [DF_i(u_i) \otimes DF_i(u_i)].x_\beta^i),
\end{aligned}$$

where π_i is the natural projection sending $[U^i_\beta, \varphi^i_\beta, x^i_\beta \oplus e^i_\beta] \in T^{(2)}_{m_i} M_i$ to m_i. Consider the families $\{s_{ji}\}_{i,j\in\mathbb{N}}$, $\{\varphi_{ji}\}_{i,j\in\mathbb{N}}$ and $\{\rho^{(2)}_{ji}\}_{i,j\in\mathbb{N}}$ which are the connecting morphisms of the projective systems $T^{(2)}M = \varprojlim T^{(2)}M_i$, $M = \varprojlim M_i$ and $\mathbb{E}^{(2)} = \varprojlim \mathbb{E}^{(2)}_i$ respectively. One can check that for any $i, j \in \mathbb{N}$ with $j \geq i$, $\varphi_{ji}\pi_j = \pi_i s_{ji}$ which implies that $\pi := \varprojlim \pi_i : T^{(2)}M \longrightarrow M$ exists and is a surjection. Moreover for the family of trivializations $\{\tau_i\}_{i\in\mathbb{N}}$ we have $(\varphi_{ji} \times \rho^{(2)}_{ji})\tau^j_\alpha = \tau^i_\alpha s_{ji}$ which means that for any $\alpha \in I$

$$\tau_\alpha := \varprojlim \tau^i_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \varprojlim \mathbb{E}^{(2)}_i$$

exists and $pr_1 \circ \tau_\alpha = \pi$. Furthermore

$$\tau_{\alpha,m} := pr_2 \circ \tau_\alpha|_{\pi^{-1}(m)} : \varprojlim (\pi_i{}^{-1}(m)) \longrightarrow \varprojlim \mathbb{E}^{(2)}_i$$

is a linear isomorphism. If $\{(U_\alpha{}^i, \tau_\alpha{}^i)\}_{i\in\mathbb{N}}$ and $\{(U_\beta{}^i, \tau_\beta{}^i)\}_{i\in\mathbb{N}}$ are local trivializations of M_i, for any $\alpha, \beta \in I$ with $U_{\alpha\beta} := U_\alpha \cap U_\beta$, we define the transition functions

$$\begin{array}{rcl} \zeta_{U_{\alpha\beta}} : U_\alpha \cap V_\beta & \longrightarrow & \mathcal{L}(\mathbb{E}^{(2)}) \\ m & \longmapsto & \tau_{\alpha,m} \circ \tau^{-1}{}_{\beta,m} \end{array}$$

where $\tau_\alpha = \varprojlim \tau_\alpha{}^i$ and $\tau_\beta = \varprojlim \tau_\beta{}^i$. To complete the proof it suffices to check that $\zeta_{U_{\alpha\beta}}$ is smooth. Note that here the pathological general linear group $GL(\mathbb{E}^{(2)})$ is replaced with the topological group (and even a generalized Lie group)

$$\mathcal{H}_0(\mathbb{E}^{(2)}) = \{(f_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} GL(\mathbb{E}^{(2)}_i) : \varprojlim f_i \ exists\}.$$

where $\mathcal{H}_0(\mathbb{E}^{(2)}) = \varprojlim \mathcal{H}_0{}^i(\mathbb{E}^{(2)})$ is the projective limit of Banach Lie groups

$$\mathcal{H}_0{}^i(\mathbb{E}^{(2)}) = \{(f_1, f_2, ..., f_i)_{i\in\mathbb{N}} \in \prod_{k=1}^{i} GL(\mathbb{E}^{(2)}_k) : \rho^{(2)}_{jk} \circ f_j = f_k \circ \rho^{(2)}_{jk}, (k \leq j \leq i)\}.$$

For any $i \in \mathbb{N}$ define

$$\begin{array}{rcl} (\zeta^*{}_{U_{\alpha\beta}})_i : U_\alpha \cap U_\beta & \longrightarrow & H_i(\mathbb{E}^{(2)}) \\ m & \longmapsto & (\tau^1_{\alpha,m} \circ (\tau^1{}_{\beta,m})^{-1}, ..., \tau^i_{\alpha,m} \circ (\tau^i{}_{\beta,m})^{-1}). \end{array}$$

Then $\{(\zeta^*{}_{U_{\alpha\beta}})_i\}_{i\in\mathbb{N}}$ is a projective system of smooth maps and $\varprojlim (\zeta^*{}_{U_{\alpha\beta}})_i : U_\alpha \cap U_\beta \longrightarrow H_i(\mathbb{F})$ is smooth. Moreover if ε is the natural inclusion

$$\begin{array}{rcl} \varepsilon : H(\mathbb{E}^{(2)}) & \longrightarrow & \mathcal{L}(\mathbb{E}^{(2)}) \\ (g_i) & \longmapsto & \varprojlim g_i \end{array}$$

then $\zeta_{U_{\alpha\beta}} = \varepsilon \circ (\zeta^*{}_{U_{\alpha\beta}})_i$ which implies that $\zeta_{U_{\alpha\beta}}$ is smooth. □

4 Second-order special connections

Let $\pi : E \longrightarrow M$ be a Banach vector bundle with fibers isomorphic to the Banach space $\mathbf{F}$. For $m \in M$ and a chart (U,φ) around m, let $\Phi : \pi^{-1}(U) \longrightarrow \varphi(U) \times \mathbf{F}$ be a local trivialization for π. If $\tau : U \longrightarrow E$ is a local section of π, then the principal part $\tau_\varphi : \varphi(U) \longrightarrow \mathbf{F}$ with respect to the trivialization Φ is given by $(\Phi \circ \tau \circ \varphi^{-1})(\varphi m)=(\varphi m, \tau_\varphi(\varphi m))$. Let (V,ψ) be another chart around m with trivialization Ψ, then a transition function is given by

$$\begin{aligned} \gamma_{\Psi\Phi} : (\varphi(U) \cap \psi(V)) \times \mathbf{F} &\longrightarrow (\varphi(U) \cap \psi(V)) \times \mathbf{F} \\ (u,f) &\longmapsto (\psi \circ \varphi^{-1}(u), pr_2 \circ \Psi \circ \Phi^{-1}(u,f)). \end{aligned}$$

Consider $\zeta : (\varphi(U) \cap \psi(V)) \longrightarrow L(\mathbf{F},\mathbf{F})$ and define $\zeta(u)f{:=}pr_2 \circ \gamma_{\Psi\Phi}(u,f)$. As a consequence if $v{=}\psi m$ and $u{=}\varphi m$ then $\tau_\psi(v){=}\zeta(u)\tau_\varphi(u)$. Let $\chi_E(M)$ be the space of smooth sections of the vector bundle $\pi : E \longrightarrow M$. Here we state a definition of **connections on vector bundles** based on [6].

Definition 4.1. A connection $\tilde{\nabla}$ on a vector bundle $\pi : E \longrightarrow M$ with fiber $\mathbf{F}$ is a mapping

$$\begin{aligned} \tilde{\nabla} : \chi(M) \times \chi_E(M) &\longrightarrow \chi_E(M) \\ (X,\tau) &\longmapsto \tilde{\nabla}_X \tau \end{aligned}$$

such that for any local chart (U,φ), there exists a smooth mapping $\tilde{\Gamma}_\varphi : \varphi U \longrightarrow L^2(\mathbb{E} \times \mathbf{F}, \mathbf{F})$ with

$$(\tilde{\nabla}_X \tau)(\varphi m) = D\tau_\varphi(\varphi m).X_\varphi(\varphi m) - \tilde{\Gamma}_\varphi(\varphi m)(X_\varphi(\varphi m), \tau_\varphi(\varphi m));\ \forall m \in U.$$

Here $L^2(\mathbb{E} \times \mathbf{F}, \mathbf{F})$ is the space of all bilinear maps from $\mathbb{E} \times \mathbf{F}$ to $\mathbf{F}$.

In [12] Kumar considered the vector bundle $T^{(2)}M$ with fiber $\mathbb{E}^{(2)}$. Then by using the definition of $\tilde{\nabla}$ on $T^{(2)}M$, he found the transformation formula of $\tilde{\Gamma}$. In [2], [5] and [8] it is shown that the projective limit $\tilde{\nabla} = \varprojlim \tilde{\nabla}_i$ of a system of connections on $M = \varprojlim M_i$ is a connection. Consequently we can find the transformation property of $\tilde{\Gamma}{=}\varprojlim \tilde{\Gamma}_i$ by considering the connection $\tilde{\nabla}{=}\varprojlim \tilde{\nabla}_i$ on the vector bundle $T^{(2)}M{=}\varprojlim T^{(2)}M_i$ with fibers isomorphic to $\mathbb{E}^{(2)}{=}\varprojlim \mathbb{E}_i^{(2)}$. Let $(U{=}\varprojlim U_i, \varphi{=}\varprojlim \varphi_i)$ and $(V{=}\varprojlim V_i, \psi{=}\varprojlim \psi_i)$ be two charts at $m{=}(m_i)_{i\in\mathbb{N}}$ with trivializations Φ and Ψ respectively. Moreover suppose that $F{:=}\varprojlim F_i = \varprojlim(\psi_i \circ \varphi_i^{-1})$, $\varphi m = \varprojlim \varphi_i m_i = \varprojlim u_i = u$ and $\psi m = \varprojlim \psi_i m_i = \varprojlim v_i = v$. If $\tau{:=}\varprojlim \tau_i : U \longrightarrow T^{(2)}M$ is a local section of $T^{(2)}M$, then its principal part is $\tau_\varphi{=}\varprojlim \tau_{\varphi_i} : \varphi U \longrightarrow \mathbb{E}^2$. If $\tau_\varphi{=}\varprojlim \tau_{\varphi_i}{=}\varprojlim(V_{\varphi_i} \oplus S_{\varphi_i})$, where

$$V_{\varphi_i} : \varphi_i(U_i) \longrightarrow \mathbb{E}_i, \quad S_{\varphi_i} : \varphi_i(U_i) \longrightarrow \mathbb{E}_i \otimes \mathbb{E}_i$$

and $\zeta{=}\varprojlim \zeta_i : \varprojlim(\varphi_i U_i \cap \psi_i V_i) \longrightarrow H(\mathbb{E}^{(2)}, \mathbb{E}^{(2)})$, then

$$\tau_\psi(v) = \varprojlim \tau_{\psi_i}(v_i) = \varprojlim \zeta_i(u_i)\tau_{\varphi_i}(u_i).$$

By using the definition of second-order tangent vector we have

$$V_\psi(v) = \varprojlim V_{\psi_i}(v_i) = \varprojlim [DF_i(u_i).v_{\varphi_i}(u_i) + D^2F_i(u_i).S_{\varphi_i}(u_i)]$$

and

$$S_\psi(v) = \varprojlim S_{\psi_i}(v_i) = \varprojlim[[DF_i(u_i)\otimes DF_i(u_i)].S_{\varphi_i}(u_i)].$$

Suppose that $\tilde{\Gamma}_\varphi(X_\varphi,\tau_\varphi) = \varprojlim \tilde{\Gamma}_{\varphi_i}(X_{\varphi_i},\tau_{\varphi_i}) = \varprojlim[\tilde{\Gamma}^1_{\varphi_i}(X_{\varphi_i},\tau_{\varphi_i})\oplus\tilde{\Gamma}^2_{\varphi_i}(X_{\varphi_i},\tau_{\varphi_i})]$, where $\tilde{\Gamma}^1_{\varphi_i}(X_{\varphi_i},\tau_{\varphi_i})\in\mathbb{E}_i$ and $\tilde{\Gamma}^2_{\varphi_i}(X_{\varphi_i},\tau_{\varphi_i})\in\mathbb{E}_i\otimes\mathbb{E}_i$. Then according to the definition of $\tilde{\nabla}$, locally on a chart (U,φ) of M we have

$$\begin{aligned}(\tilde{\nabla}_X\tau)_\varphi &= \varprojlim(\tilde{\nabla}_{X_i}\tau_i)_{\varphi_i} = \varprojlim[D\tau_{\varphi_i}.X_{\varphi_i} - \tilde{\Gamma}_{\varphi_i}(X_{\varphi_i},\tau_{\varphi_i})]\\ &= \varprojlim[(DV_{\varphi_i}.X_{\varphi_i} - \tilde{\Gamma}^1_{\varphi_i}(X_{\varphi_i},\tau_{\varphi_i}))\oplus(DS_{\varphi_i}.X_{\varphi_i} - \tilde{\Gamma}^2_{\varphi_i}(X_{\varphi_i},\tau_{\varphi_i}))].\end{aligned}$$

Now

$$\begin{aligned}[DV_\psi.X_\psi].(v) &= \varprojlim[[DV_{\psi_i}.X_{\psi_i}].(v_i)]\\ &= \varprojlim[D^2F_i(u_i)(X_{\varphi_i}(u_i)\otimes V_{\varphi_i}(u_i)) + DF_i(u_i).DV_{\varphi_i}(u_i).X_{\varphi_i}(u_i)\\ &\quad +D^3F_i(u_i)(X_{\varphi_i}(u_i)\otimes S_{\varphi_i}(u_i)) + D^2F_i(u_i).DS_{\varphi_i}(u_i).X_{\varphi_i}(u_i)]\end{aligned}$$

and

$$\begin{aligned}[DS_\varphi.X_\psi](v) &= [DS_{\varphi_i}.X_{\psi_i}](v_i) = \varprojlim[DS_{\psi_i}(v_i).DF_i(u_i).X_{\varphi_i}(u_i)]\\ &= \varprojlim[D(DF_i\otimes DF_i)(u_i)(X_{\varphi_i}(u_i)\otimes S_{\varphi_i}(u_i))\\ &\quad +(DF_i\otimes DF_i)(u_i).DS_{\varphi_i}(u_i).X_{\varphi_i}(u_i)]\end{aligned}$$

which implies that

$$\begin{aligned}&[DV_\psi.X_\psi](v)\oplus[DS_\psi.X\psi](v) = \varprojlim[[DV_{\psi_i}.X_{\psi_i}](v_i)\oplus[DS_{\psi_i}.X\psi_i](v_i)]\\ &= \varprojlim\Big([\zeta_i(u_i).[DV_{\varphi_i}(u_i).X_{\varphi_i}(u_i)\oplus DS_{\varphi_i}(u_i).X_{\varphi_i}(u_i)]]\\ &\quad +\{[D^3F_i(u_i)(X_{\varphi_i})(u_i)\otimes S_{\varphi_i}(u_i) + D^2F_i(u_i)(X_{\varphi_i}(u_i)\otimes V_{\varphi_i}(u_i))]\oplus\\ &\quad [D(DF_i\otimes DF_i)(u_i)(X_{\varphi_i}(u_i)\otimes S_{\varphi_i}(u_i))]\}\Big).\end{aligned}$$

As a result one can check that

$$\begin{aligned}\Gamma_\psi(X_\psi,\tau_\psi) &= \varprojlim\Gamma_{\psi_i}(X_{\psi_i},\tau_{\psi_i})\\ &= [DF.\tilde{\Gamma}^1_\varphi(X_\varphi,\tau_\varphi) + D^2F.\tilde{\Gamma}^2_\varphi(X_\varphi,\tau_\varphi) + [D^3F(X_\varphi\otimes S_\varphi) + D^2F(X_\varphi\otimes V_\varphi)]\\ &\quad \oplus[(DF\otimes DF).\tilde{\Gamma}^2_\varphi(X_\varphi,\tau_\varphi) + D(DF\otimes DF)(X_\varphi\otimes S_\varphi)].\end{aligned}$$

which is the transformation formula of $\tilde{\Gamma}=\varprojlim\tilde{\Gamma}_i$ on different charts. If we define

$$\begin{aligned}&A_\varphi(X_\varphi,.) := \varprojlim A_{\varphi_i}(X_{\varphi_i},.) : \mathbb{E}\longrightarrow\mathbb{E},\\ &R_\varphi(X_\varphi,.) := \varprojlim R_{\varphi_i}(X_{\varphi_i},.) : \mathbb{E}\otimes\mathbb{E}\longrightarrow\mathbb{E},\\ &B_\varphi(X_\varphi,.) := \varprojlim B_{\varphi_i}(X_{\varphi_i},.) : \mathbb{E}\longrightarrow\mathbb{E}\otimes\mathbb{E},\\ &C_\varphi(X_\varphi,.) := \varprojlim C_{\varphi_i}(X_{\varphi_i},.) : \mathbb{E}\otimes\mathbb{E}\longrightarrow\mathbb{E}\otimes\mathbb{E},\end{aligned}$$

by

$$\begin{aligned}
A_\varphi(X_\varphi, V_\varphi) &= \varprojlim A_{\varphi_i}(X_{\varphi_i}, V_{\varphi_i}) = \varprojlim [pr_1 \circ \tilde{\Gamma}_{\varphi_i}(X_{\varphi_i}, V_{\varphi_i} \oplus 0)],\\
R_\varphi(X_\varphi, S_\varphi) &= \varprojlim R_{\varphi_i}(X_{\varphi_i}, S_{\varphi_i}) = \varprojlim [pr_1 \circ \tilde{\Gamma}_{\varphi_i}(X_{\varphi_i}, 0 \oplus S_{\varphi_i})],\\
B_\varphi(X_\varphi, V_\varphi) &= \varprojlim B_{\varphi_i}(X_{\varphi_i}, V_{\varphi_i}) = \varprojlim [pr_2 \circ \tilde{\Gamma}_{\varphi_i}(X_{\varphi_i}, V_{\varphi_i} \oplus 0)],\\
C_\varphi(X_\varphi, S_\varphi) &= \varprojlim C_{\varphi_i}(X_{\varphi_i}, S_{\varphi_i}) = \varprojlim [pr_2 \circ \tilde{\Gamma}_{\varphi_i}(X_{\varphi_i}, 0 \oplus S_{\varphi_i})],
\end{aligned}$$

then we can write $\tilde{\Gamma}{=}\varprojlim \tilde{\Gamma}_i$ as

$$\tilde{\Gamma}_\varphi(X_\varphi, \tau_\varphi) = [A_\varphi(X_\varphi, V_\varphi) + R_\varphi(X_\varphi, S_\varphi)] \oplus [B_\varphi(X_\varphi, V_\varphi) + C_\varphi(X_\varphi, S_\varphi)],$$

which means that

$$\begin{aligned}
\tilde{\Gamma}^1_\varphi(X_\varphi, \tau_\varphi) &= A_\varphi(X_\varphi, V_\varphi) + R_\varphi(X_\varphi, S_\varphi),\\
\tilde{\Gamma}^2_\varphi(X_\varphi, \tau_\varphi) &= B_\varphi(X_\varphi, V_\varphi) + C_\varphi(X_\varphi, S_\varphi).
\end{aligned}$$

Hence the translation formula for $\tilde{\Gamma}$ is given by

$$\begin{aligned}
&\tilde{\Gamma}_\psi(X_\psi, \tau_\psi) = [A_\psi(X_\psi, V_\psi) + R_\psi(X_\psi, S_\psi)] \oplus [B_\psi(X_\psi, V_\psi) + C_\psi(X_\psi, S_\psi)]\\
&= [DF.(A_\psi(X_\psi, V_\psi) + R_\psi(X_\psi, S_\psi)) + D^2F.(B_\psi(X_\psi, V_\psi) + C_\psi(X_\psi, S_\psi))\\
&\quad + D^3F(X_\varphi \otimes S_\varphi) + D^2F(X_\varphi \otimes V_\varphi)] \oplus [(DF \otimes DF).(B_\psi(X_\psi, V_\psi)\\
&\quad + C_\psi(X_\psi, S_\psi)) + D(DF \otimes DF)(X_\varphi \otimes S_\varphi)].
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
A_\psi(X_\psi, V_\psi) + R_\psi(X_\psi, S_\psi) \quad = \quad & DF.(A_\psi(X_\psi, V_\psi) + R_\psi(X_\psi, S_\psi))\\
& + D^2F.(B_\psi(X_\psi, V_\psi) + C_\psi(X_\psi, S_\psi))\\
& + D^3F(X_\varphi \otimes S_\varphi) + D^2F(X_\varphi \otimes V_\varphi)
\end{aligned}$$

and

$$\begin{aligned}
B_\psi(X_\psi, V_\psi) + C_\psi(X_\psi, S_\psi) \quad = \quad & (DF \otimes DF).(B_\psi(X_\psi, V_\psi) + C_\psi(X_\psi, S_\psi))\\
& + D(DF \otimes DF)(X_\varphi \otimes S_\varphi).
\end{aligned}$$

Here we state the definition of special second-order connection $\tilde{\nabla}$ from [12] which is susceptible to be generalized for a wide class of Fréchet manifolds.

Definition 4.2. In definition 4.1, let $\tilde{\Gamma}_\varphi(X_\varphi, \tau_\varphi) = R_\varphi(X_\varphi, S_\varphi) \oplus [(-X_\varphi \otimes V_\varphi) + C_\varphi(X_\varphi, S_\varphi)]$. The connection with this Christoffel symbol is called the special second-order connection. In fact R_φ and C_φ are stated as the Christoffel symbols of this connection.

Theorem 4.1. *The projective limit $\tilde{\nabla} = \varprojlim \tilde{\nabla}_i$ of a system of special second-order connections $\{\tilde{\nabla}_i\}_{i\in\mathbb{N}}$ is a special second-order connection on $M = \varprojlim M_i$.*

Proof. For any $i, j \in \mathbb{N}$ with $j \geq i$, let (U_j, φ_j) be a chart of M_j around m_j and consider (U_i, φ_i) as a chart of M_i around $\varphi_{ji}(m_j) = m_i$. Also for any $i \in \mathbb{N}$, suppose that $X_{\varphi_i} : \varphi_i U_i \longrightarrow \mathbb{E}_i$ and $\tau_{\varphi_i} : \varphi_i U_i \longrightarrow \mathbb{E}_i^{(2)}$. It suffices to show that for $j \geq i$,

$$\rho_{ji}^{(2)} \nabla_{X_{\varphi_j}} \tau_{\varphi_j} = \nabla_{X_{\varphi_i}} \tau_{\varphi_i} \rho_{ji}.$$

Our proof has four steeps.
Step 1:

$$\begin{aligned} \rho_{ji} DV_{\varphi_j}(\varphi_j m_j) X_{\varphi_j}(\varphi_j m_j) &= \rho_{ji} \frac{d}{dt} V_{\varphi_j}(\varphi_j m_j + t X_{\varphi_j}(\varphi_j m_j)) \\ &= \frac{d}{dt} V_{\varphi_i}(\varphi_i m_i + t X_{\varphi_i}(\varphi_i m_i)) \\ &= DV_{\varphi_i}(\varphi_i m_i) X_{\varphi_i}(\varphi_i m_i). \end{aligned}$$

Step 2:

$$\begin{aligned} \rho_{ji} R_{\varphi_j}(X_{\varphi_j}(\varphi_j m_j), S_{\varphi_j}(\varphi_j m_j)) &= \rho_{ji}(pr_1 \circ \tilde{\Gamma}_{\varphi_j}(\varphi_j m_j))(X_{\varphi_j}(\varphi_j m_j), S_{\varphi_j}(\varphi_j m_j)) \\ &= (pr_1 \circ \tilde{\Gamma}_{\varphi_i}(\varphi_i m_i))(X_{\varphi_i}(\varphi_i m_i), S_{\varphi_i}(\varphi_i m_i)) \\ &= R_{\varphi_i}(X_{\varphi_i}(\varphi_i m_i), S_{\varphi_i}(\varphi_i m_i)). \end{aligned}$$

Step 3:

$$\begin{aligned} \rho'_{ji} DS_{\varphi_j}(\varphi_j m_j).X_{\varphi_j}(\varphi_j m_j) &= \rho'_{ji} \frac{d}{dt} S_{\varphi_j}((\varphi_j m_j) + t X_{\varphi_j}(\varphi_j m_j)) \\ &= \frac{d}{dt} S_{\varphi_i} \rho_{ji}((\varphi_j m_j) + t X_{\varphi_j}(\varphi_j m_j)) \\ &= DS_{\varphi_i}(\varphi_i m_i).X_{\varphi_i}(\varphi_i m_i). \end{aligned}$$

Step 4:

$$\begin{aligned} \rho'_{ji} C_{\varphi_j}((X_{\varphi_i}(\varphi_i m_i), S_{\varphi_i}(\varphi_i m_i)) &= \rho'_{ji}(pr_2 \circ \tilde{\Gamma}_{\varphi_j}(\varphi_j m_j))(X_{\varphi_j}(\varphi_j m_j), S_{\varphi_j}(\varphi_j m_j)) \\ &= (pr_2 \circ \tilde{\Gamma}_{\varphi_i}(\varphi_i m_i))(X_{\varphi_i}(\varphi_i m_i), S_{\varphi_i}(\varphi_i m_i)) \\ &= C_{\varphi_i}(X_{\varphi_i}(\varphi_i m_i), S_{\varphi_i}(\varphi_i m_i)). \end{aligned}$$

As a consequence of these steps we have

$$\begin{aligned} \rho_{ji}^{(2)} \nabla_{X_{\varphi_j}} \tau_{\varphi_j}(\varphi_j m_j) &= [DV_{\varphi_i}(\varphi_i m_i) - R_{\varphi_i}(X_{\varphi_i}(\varphi_i m_i), S_{\varphi_i}(\varphi_i m_i))] \\ &\quad \oplus [DS_{\varphi_i}(\varphi_i m_i).X_{\varphi_i}(\varphi_i m_i) - X_{\varphi_i}(\varphi_i m_i) \otimes V_{\varphi_i}(\varphi_i m_i) \\ &\quad - C_{\varphi_i}(X_{\varphi_i}(\varphi_i m_i), S_{\varphi_i}(\varphi_i m_i))] \\ &= \nabla_{X_{\varphi_i}} \tau_{\varphi_i} \rho_{ji}(\varphi_j m_j)). \end{aligned}$$

□

Here we present a one-to-one correspondence between third order Hessian structures and special second-order connections on the Fréchet manifolds. However, we should consider just the smooth functions and the smooth vector fields

$$\mathcal{F}(M) = \{(f_i)_{i \in \mathbb{N}} : \forall i \in \mathbb{N},\ f_i : M_i \longrightarrow \mathbb{R} \text{ is continuous and } \varprojlim f_i \text{ exists}\}$$

and

$$\mathcal{G}(M) = \{(X_i)_{i\in\mathbb{N}} : \forall i \in \mathbb{N},\ X_i \text{ is a vector field on} M_i \text{ and } \varprojlim X_i \text{ exists}\}$$

respectively.

Theorem 4.2. *There is a one-to-one correspondence between the third order Hessian structures $H^3 = \varprojlim H_i^3$ on $M = \varprojlim M_i$ and the special second-order connection $\tilde{\nabla} = \varprojlim \tilde{\nabla}_i$ on $T^2M = \varprojlim T^2M_i$ via*

$$H^3 f(X,Y,Z) = D^3 f_\varphi(X_\varphi, Y_\varphi, Z_\varphi) + D^2 f_\varphi.C_\varphi(X_\varphi, Y_\varphi, Z_\varphi) + Df_\varphi.R_\varphi(X_\varphi, Y_\varphi, Z_\varphi),$$

where $(U{=}\varprojlim U_i, \varphi{=}\varprojlim \varphi_i)$ is any local chart on M; $X, Y, Z \in \mathcal{G}(U)$ and $f \in \mathcal{F}(U)$.

Proof. Let $H^3 = \varprojlim H_i^3$ be a third order Hessian structure on M. Considering the Banach case mentioned in [12], for any $i \in \mathbb{N}$,

$$\begin{aligned} H_i^3 f_i(X_i, Y_i, Z_i) &= D^3 f_{i\varphi_i}(X_{\varphi_i}, Y_{\varphi_i}, Z_{\varphi_i}) + D^2 f_{i\varphi_i}.C_{\varphi_i}(X_{\varphi_i}, Y_{\varphi_i}, Z_{\varphi_i}) \\ &\quad + Df_{i\varphi_i}.R_{\varphi_i}(X_{\varphi_i}, Y_{\varphi_i}, Z_{\varphi_i}) \end{aligned}$$

defines a special second-order connection $\tilde{\nabla}_i$ on $T^{(2)}M_i$. According to the theorem 4.1, the projective limit of a system of special second-order connections is a (generalized) special second-order connection. Therefore, $H^3 = \varprojlim H_i^3$ defines a special second-order connection $\tilde{\nabla}$ on $T^{(2)}M$. Conversely, let $\tilde{\nabla}$ be a special second-order connection on $T^{(2)}M$ with the Christoffel symbols of the form $R_\varphi{=}\varprojlim R_{\varphi_i}$ and $C_\varphi{=}\varprojlim C_{\varphi_i}$. Then for any $i \in \mathbb{N}$,

$$D^3 f_{i\varphi_i}(X_{\varphi_i}, Y_{\varphi_i}, Z_{\varphi_i}) + D^2 f_{i\varphi_i}.C_{\varphi_i}(X_{\varphi_i}, Y_{\varphi_i}, Z_{\varphi_i}) + Df_{i\varphi_i}.R_{\varphi_i}(X_{\varphi_i}, Y_{\varphi_i}, Z_{\varphi_i})$$

determines a Hessian structure on M (for more details see [12]) and theorem 3.1 completes the proof. □

5 Second-order Geodesics on Fréchet Manifolds

For any $i \in \mathbb{N}$, let $\gamma_i : (-\varepsilon, \varepsilon) \longrightarrow M_i$ be a smooth curve with $\gamma_i(0) = m_i$. Moreover suppose that for $f \in \mathcal{F}(M)$, $\gamma''(f){:=}\varprojlim (f_i \circ \gamma_i)''(0)$ where $\gamma{:=}\varprojlim \gamma_i$. Then locally on a chart $(U{=}\varprojlim U_i, \varphi{=}\varprojlim \varphi_i)$ we have

$$\gamma''(f) = \varprojlim \gamma_i''(f_i) = \varprojlim [D^2 f_{i\varphi_i}(\varphi_i m_i)(\gamma'_{i\varphi_i}(0), \gamma'_{i\varphi_i}(0)) + Df_{i\varphi_i}(\varphi_i m_i).\gamma''_{i\ \varphi_i}(0)]$$

where $\gamma_{i\varphi_i}$ is a smooth curve in $\mathbb{E}_i$. Suppose that

$$G^0_{m,n} := \{(f_i)_{i\in\mathbb{N}} : f_i \in G^{0,i}_{m,n}, \varprojlim m_i = m \text{ and } \varprojlim f_i \text{ exists}\}.$$

This definition implies that if $f{=}\varprojlim f_i \in G^0_{m,n}$, then $\gamma''(f) = 0$ and $\gamma'' \in T^{(2)}M = \varprojlim T^{(2)}M_i$. Therefore we can define

$$[\gamma'']_\varphi := \varprojlim [\gamma_i'']_{\varphi_i} = \varprojlim [\gamma''_{\varphi_i} \oplus (\gamma'_{\varphi_i} \otimes \gamma'_{\varphi_i})].$$

Moreover, if $\tilde{\nabla}=\varprojlim \tilde{\nabla}_i$ is a special second-order connection with the Christoffel symbols $R_\varphi=\varprojlim R_{\varphi_i}$ and $C_\varphi=\varprojlim C_{\varphi_i}$, then

$$\begin{aligned}(\tilde{\nabla}_{\gamma'}\gamma'')_\varphi &= \varprojlim(\tilde{\nabla}_{\gamma'_i}\gamma''_i)_{\varphi_i} \\ &= \varprojlim\Big([\gamma'''_{i\ \varphi_i} - R_{\varphi_i}(\gamma'_{\varphi_i},\gamma'_{\varphi_i},\gamma'_{\varphi_i})] \oplus [C_{\varphi_i}(\gamma'_{\varphi_i},\gamma'_{\varphi_i},\gamma'_{\varphi_i}) - 3\gamma'_{\varphi_i}\otimes\gamma''_{\varphi_i}]\Big).\end{aligned}$$

Definition 5.1. $\gamma = \varprojlim \gamma_i$ is a second-order geodesic of $\tilde{\nabla} = \varprojlim \tilde{\nabla}_i$ if and only if for any $i \in \mathbb{N}$
i) $\gamma'''_{i\ \varphi_i} = R_{\varphi_i}(\gamma'_{\varphi_i},\gamma'_{\varphi_i},\gamma'_{\varphi_i})$,
ii) $C_{\varphi_i}(\gamma'_{\varphi_i},\gamma'_{\varphi_i},\gamma'_{\varphi_i}) = 3\gamma'_{\varphi_i}\otimes\gamma''_{\varphi_i}$.

Let $\nabla=\varprojlim \nabla_i$ be a connection on $TM=\varprojlim TM_i$. For $f \in \mathcal{F}(M)$, $\nabla^3 f$ determines a Hessian structure on $M=\varprojlim M_i$, called Kf by Kumar [12], which is in a correspondence with a special second-order connection $\tilde{\nabla}=\varprojlim \tilde{\nabla}_i$ on $T^{(2)}M=\varprojlim T^{(2)}M_i$. Because of this $\tilde{\nabla}$ is called the connection induced by ∇. It is not difficult to show that if $\nabla=\varprojlim \nabla_i$ is a connection on $M=\varprojlim M_i$ then for $f \in \mathcal{F}(M)$, $\nabla^n f=\varprojlim \nabla^n_i f_i$ is an n-th order Hessian structure on M.

Proposition 5.1. *Let $\nabla=\varprojlim \nabla_i$ be a connection with the Christoffel symbols $\{\Gamma_\varphi = \varprojlim \Gamma_{\varphi_i}\}$. Then the Christoffel symbols of the induced connection $\tilde{\nabla} = \varprojlim \tilde{\nabla}_i$ on $T^{(2)}M = \varprojlim T^{(2)}M_i$ are*

$$\begin{aligned}C_\varphi(X_\varphi,Y_\varphi,Z_\varphi) &= \varprojlim C_{\varphi_i}(X_{\varphi_i},Y_{\varphi_i},Z_{\varphi_i}) \\ &= \varprojlim[X_{\varphi_i}\otimes\Gamma_{\varphi_i}(Y_{\varphi_i},Z_{\varphi_i}) + Y_{\varphi_i}\otimes\Gamma_{\varphi_i}(X_{\varphi_i},Z_{\varphi_i}) \\ &\quad +Z_{\varphi_i}\otimes\Gamma_{\varphi_i}(X_{\varphi_i},Y_{\varphi_i}))]\end{aligned}$$

and

$$\begin{aligned}R_{\varphi_i}(X_\varphi,Y_\varphi,Z_\varphi) &= \varprojlim R_{\varphi_i}(X_{\varphi_i},Y_{\varphi_i},Z_{\varphi_i}) \\ &= \varprojlim[D(\Gamma_{\varphi_i}(Y_{\varphi_i},Z_{\varphi_i}).X_{\varphi_i}) - \Gamma_{\varphi_i}((\nabla_{X_i}Y_i)_{\varphi_i},Z_{\varphi_i}) \\ &\quad -\Gamma_{\varphi_i}(Y_{\varphi_i},(\nabla_{X_i}Z_i)_{\varphi_i})].\end{aligned}$$

Proof. Locally on a chart $(U = \varprojlim U_i, \varphi = \varprojlim \varphi_i)$ around $m = (m_i)_{i\in\mathbb{N}}$, $\nabla = \varprojlim \nabla_i$ takes the form $(\nabla_X Y)_\varphi = \varprojlim(\nabla_{X_i}Y_i)_{\varphi_i} = \varprojlim[DY_{\varphi_i}X_{\varphi_i} - \Gamma_{\varphi_i}(X_{\varphi_i},Y_{\varphi_i})]$. For $f \in \mathcal{F}(M)$ and $X,Y,Z \in \mathcal{G}(M)$, $\nabla^3 f$ has the following expression:

$$\begin{aligned}\nabla^3 f(X,Y,Z) &= \varprojlim \nabla^3_i f_i(X_i,Y_i,Z_i) = \varprojlim[(\nabla_{X_i}(\nabla^2_i f_i))(Y_i,Z_i)] \\ &= \varprojlim[\nabla_{X_i}(\nabla^2_i f_i(Y_i,Z_i) - \nabla^2_i f_i(\nabla_{X_i}Y_i,Z_i) - \nabla^2_i f_i(Y_i,\nabla_{X_i}Z_i)] \\ &= \varprojlim[X_i.(Y_i.Z_i.f_i - \nabla_{Y_i}Z_i.f_i) - (\nabla_{X_i}Y_i.Z_i.f_i - \nabla_{\nabla_{X_i}Y_i}Z_i.f_i) \\ &\quad -(Y_i.\nabla_{X_i}Z_i.f_i - \nabla_{Y_i}\nabla_{X_i}Z_i.f_i)]\ \varprojlim[D^3 f_{i\varphi_i}(X_{\varphi_i},Y_{\varphi_i},Z_{\varphi_i}) \\ &\quad +\big(D^2 f_{i\varphi_i}(X_{\varphi_i},\Gamma_\varphi(Y_{\varphi_i},Z_{\varphi_i})) + D^2 f_{i\varphi_i}(Y_{\varphi_i},\Gamma_\varphi(X_{\varphi_i},Z_{\varphi_i})) \\ &\quad +D^2 f_{i\varphi_i}(Z_{\varphi_i},\Gamma_\varphi(X_{\varphi_i},Y_{\varphi_i}))\big) + \big(Df_{i\varphi_i}.D\Gamma_{\varphi_i}(Y_{\varphi_i},Z_{\varphi_i})\big).X_{\varphi_i} \\ &\quad -Df_{i\varphi_i}.\Gamma_{\varphi_i}((\nabla_{X_i}Y_i)_{\varphi_i},Z_{\varphi_i})] - Df_{i\varphi_i}.\Gamma_{\varphi_i}((\nabla_{X_i}Z_i)_{\varphi_i},Y_{\varphi_i}).\end{aligned}$$

Hence

$$\begin{aligned}\nabla^3 f(X,Y,Z) &= D^3 f_\varphi(X_\varphi, Y_\varphi, Z_\varphi)\\ &= D^2 f_\varphi . \varprojlim [X_{\varphi_i} \otimes \Gamma_{\varphi_i}(Y_{\varphi_i}, Z_{\varphi_i})\\ &\quad + Y_{\varphi_i} \otimes \Gamma_{\varphi_i}(X_{\varphi_i}, Z_{\varphi_i}) + Z_{\varphi_i} \otimes \Gamma_{\varphi_i}(X_{\varphi_i}, Y_{\varphi_i}))]\\ &\quad + Df_\varphi . \varprojlim [D(\Gamma_{\varphi_i}(Y_{\varphi_i}, Z_{\varphi_i}).X_{\varphi_i})\\ &\quad - \Gamma_{\varphi_i}((\nabla_{X_i} Y_i)_{\varphi_i}, Z_{\varphi_i}) - \Gamma_{\varphi_i}(Y_{\varphi_i}, (\nabla_{X_i} Z_i)_{\varphi_i})].\end{aligned}$$

Comparison of C_φ and R_φ, as in theorem 4.2, completes the proof. □

Corollary 5.2. *Every third order Hessian structure on Fréchet manifolds is not necessarily ∇^3 of a connection $\nabla = \varprojlim \nabla_i$. (In [2] it is proved that every second order Hessian structure on a projective limit Fréchet manifolds takes the form ∇^2 for a unique connection $\nabla = \varprojlim \nabla_i$.)*

For the Fréchet space $\mathbb{E} = \varprojlim \mathbb{E}_i$ let $\{\| \, . \, \|_i\}_{i\in\mathbb{N}}$ be a countable family of seminorms which determine the topology of $\mathbb{E}$ and for any $i \in \mathbb{N}$;

$$\rho_i : \mathbb{E} \longrightarrow \mathbb{E}_i \ ; \quad (x_i)_{i\in\mathbb{N}} \longmapsto x_i$$

be the canonical projection.

Let $\bar{\mathbb{E}} = \varprojlim \bar{\mathbb{E}}_i$ be another Fréchet space. A map $g : \mathbb{E} \longrightarrow \bar{\mathbb{E}}$ is called projective limit map of the family of maps $\{g_i : \mathbb{E}_i \longrightarrow \bar{\mathbb{E}}_i\}_{i\in\mathbb{N}}$ whenever for any $i \in \mathbb{N}$, $\rho_i \circ g = g_i \circ \rho_i$ and $\rho_{ji} \circ g_j = g_i \circ \rho_{ji}$ for any $j \in \mathbb{N}$ with $j \geq i$. In this case we say that the limit of the family $\{g_i\}_{i\in\mathbb{N}}$ exists and we denote it by $g = \varprojlim g_i$ (see also [1] and [9]).

Definition 5.2. A projective limit map $g : \mathbb{E} \longrightarrow \mathbb{E}$ is called k-Lipschitz, $k > 0$, if

$$\| \rho_i \circ g(x) - \rho_i \circ g(y) \|_i \leq k \, \| \rho_i(x) - \rho_i(y) \|_i$$

for any $x, y \in \mathbb{F}$ and $i \in \mathbb{N}$.

Theorem 5.3. *Let $\mathbb{E}$ be a Fréchet space and $g : \mathbb{R} \times \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{E}$ a k-Lipschitz projective limit mapping. For the second order differential equation*

$$x'' = g(t, x, x'); \tag{5.1}$$

with the initial condition $(t^0, x^0, y^0) \in \mathbb{R} \times \mathbb{E} \times \mathbb{E}$, if there exists a constant $\tau \in \mathbb{R}^+$ such that

$$M = sup\{\big(\| (y_i^0) \|_i^2 + \| g(t, x_i^0, y_i^0) \|_i^2 \big)^{1/2}; \ i \in \mathbb{N}, t \in [t_0 - \tau, t_0 + \tau]\} < \infty$$

and $a = min\{\tau, \frac{1}{M+k}\}$, then (5.1) has a unique solution on $I = [t^0 - a, t^0 + a]$.

Proof. see [2], appendix. □

Proposition 5.4. *Let $\tilde{\nabla} = \varprojlim \tilde{\nabla}_i$ be a second-order connection on $T^{(2)}M = \varprojlim T^{(2)}M_i$ induced from a connection $\nabla = \varprojlim \nabla_i$ on $M = \varprojlim M_i$ with k-Lipschitz Christoffel symbols. For $x_0 \in M$ and $y_0 \in T_{x_0}M$ suppose that (U, φ) is a chart around x_0 and $M_\varphi = \{\big(\| x_0 \|_i^2 + \| \Gamma_\varphi(x_0)(y_0, y_0) \|_i^2 \big)^{1/2}; \ i \in \mathbb{N}\} < \infty$. If γ is a ∇-geodesic with the initial conditions $\gamma(0) = x_0$ and $\gamma'(0) = y_0$, then γ is also a second-order geodesic of $\tilde{\nabla}$.*

Proof. According to [2] theorem 7.2, there exists a unique ∇-geodesic $\gamma = \varprojlim \gamma^i : (-\epsilon, \epsilon) \longrightarrow M$ s.t. $\gamma(0) = x_0$, $\gamma'(0) = y_0$ and $\epsilon > 0$. Then for any chart $(U = \varprojlim U_i, \varphi = \varprojlim \varphi_i)$, ${\gamma_i''}_{\varphi_i} = \Gamma_{\varphi_i}(\gamma'_{i\varphi_i}, \gamma'_{i\varphi_i})$, $(i \in \mathbb{N})$ [2]. Using this fact and considering our calculation in proposition 5.1, for any $i \in \mathbb{N}$, $C_{\varphi_i}(\gamma'_{i\varphi_i}, \gamma'_{i\varphi_i}, \gamma'_{i\varphi_i}) = 3\gamma'_{i\varphi_i} \otimes \Gamma_{\varphi_i}(\gamma'_{i\varphi_i}, \gamma'_{i\varphi_i}) = 3\gamma'_{i\varphi_i} \otimes {\gamma_i''}_{\varphi_i}$ and $R_{\varphi_i}(\gamma'_{i\varphi_i}, \gamma'_{i\varphi_i}, \gamma'_{i\varphi_i}) = {\gamma_i'''}_{\varphi_i}$. Therefore, γ is a second-order geodesic of $\tilde{\nabla}$. □

Note that for any $i \in \mathbb{N}$, γ_i is a geodesic with respect to ∇_i and its lifetime $(-\varepsilon_i, \varepsilon_i)$ possibly depends on i and $\varepsilon_i \longrightarrow 0$ as $i \longrightarrow \infty$. In this case the solution is trivial. To avoid it we needed some Lipschitz condition on local components of the connections to be sure that the domain of $\gamma = \varprojlim \gamma_i$ is not trivial. (See appendix for another criterion to determine the lifetime of solutions.)

6 Applications

In [19] it is shown that there is a one-to-one correspondence between linear ordinary differential equations and connections on the trivial bundle with the solution ζ being the horizontal section of this connection. Now consider the special second-order connection $\tilde{\nabla}$ on the trivial bundle $L = (\mathbb{R} \times \mathbb{E}^{(2)}, \mathbb{R}, pr_1)$ with the total chart and the Christoffel symbols

$$\Gamma : \mathbb{R} \longrightarrow L^2(\mathbb{R} \times \mathbb{E}^{(2)}, \mathbb{E}^{(2)})$$

by $\Gamma_i(t)(x,s) = R_i(t)(x,s) \oplus -x \otimes V_i + C_i(t)(x,s)$. Let $A(t) := \Gamma(t)(1, .)$ where $t \in \mathbb{R}$ and 1 is the unit of $\mathbb{R}$. By the above construction there is a one-to-one correspondence between special second-order connections and linear ordinary differential equations of the form $dx/dt = A(t)x$, on the trivial bundle L. Here we generalize these concepts to the case of Fréchet vector bundles and connections. Let $\mathbb{F} = \varprojlim(\mathbb{E}_j^{(2)}, \rho_{ji}^{(2)})$ and $L = (\mathbb{R} \times \mathbb{F}, \mathbb{R}, pr_1)$. Consider the special second-order connection $\tilde{\nabla}_i$ over $L_i = (\mathbb{R} \times \mathbb{E}_i^{(2)}, \mathbb{R}, pr_1)$ with the Christoffel symbols $\Gamma_i(t)(x,s) = R_i(t)(x,s) \oplus -x \otimes V_i + C_i(t)(x,s)$ and the connecting morphisms $\rho_{ji}^{(2)}$ for $j \geq i$. Then by theorem 4.1, $\tilde{\nabla} = \varprojlim \tilde{\nabla}_i$ is a special second-order connection on $L = \varprojlim L_i$ characterized by the Christoffel symbols

$$\Gamma : \mathbb{R} \longrightarrow L^2(\mathbb{R} \times \mathbb{F}, \mathbb{F})$$

with $\Gamma := \Gamma(X, \tau) = R(t)(X, S) \oplus -X \otimes V + C(X, S)$, where $X, V : \mathbb{R} \longrightarrow \mathbb{E} = \varprojlim \mathbb{E}_i$ and $S : \mathbb{R} \longrightarrow \mathbb{E} \otimes \mathbb{E}$. Here we have the following proposition deduced from the above discussion.

Proposition 6.1. *There is a one-to-one correspondence between special second-order connections $\tilde{\nabla} = \varprojlim \tilde{\nabla}_i$ and linear ordinary differential equations $dx(t)/dt = A(t)x(t)$ where $x(t) = (x_i(t))_{i \in \mathbb{N}} \in \mathbb{F}$ and $A = \varprojlim A_i$. Suppose that $(t_0, f_0) \in \mathbb{R} \times \mathbb{E}^{(2)}$ be an arbitrary element and Γ be a k-Lipschitz mapping. If there exists an $\alpha > 0$ such that*

$$M = sup\{p_i(\Gamma(t)(f_0, 1)),\ i \in \mathbb{N},\ t \in [t_0 - \alpha, t_0 + \alpha]\} < \infty,$$

then the equation $dx(t)/dt = A(t)x(t)$ has a unique solution $\zeta_p : (t_0 - \epsilon, t_0 + \epsilon) \longrightarrow \mathbb{E}^{(2)}$ where ζ_p is the principal part of the horizontal global section of L, $\zeta_p(t_0) = f_0$ and $\epsilon = min\{\alpha, \frac{1}{M+k}\}$.

Proof. By the preceding argument we have $\tilde{\nabla} = \varprojlim \tilde{\nabla}_i$ and by [19], $\tilde{\nabla}_i$ corresponds to a linear ordinary differential equation $dx_i/dt = A_i(t)x_i$. Moreover, every solution $dx_i/dt = A(t)x_i$ is the principal part of the horizontal global section of $\tilde{\nabla}_i$ which we call ζ_i. For any $i \in \mathbb{N}$, $A_i(t) = \Gamma_i(t)(.,1)$, where Γ_i's are the Christoffel symbols of $\tilde{\nabla}_i$ over L_i with respect to the chart $(\mathbb{E}_i^{(2)}, id_{\mathbb{E}_i^{(2)}})$. Since $\tilde{\nabla} = \varprojlim \tilde{\nabla}_i$, $\Gamma(t) = \varprojlim \Gamma_i(t)(.,1)$ and consequently we can define $A(t) = \varprojlim A_i(t)$. Therefore, $dx/dt = A(t)x$ is a linear ordinary differential equation on $\mathbb{F}$. Let $(t_0, f_0) = (t_0, (f_0^i)_{i\in\mathbb{N}}) \in \mathbb{R} \times \mathbb{E}^{(2)}$ be an arbitrary element. For $f_0^i \in \mathbb{E}_i^{(2)}$, let ζ_p^i be the unique solution for $dx/dt = A_i(t)x$ which satisfies $\zeta_p^i(t_0) = f_0^i$. We will show that $\{\zeta_p^i\}_{i\in\mathbb{N}}$ is a projective system of maps with $\zeta_p = \varprojlim \zeta_p^i$ such that ζ_p is the solution of $dx/dt = A(t)x$ and $\zeta_p(t_0) = f_0$. First we show that for $i \leq j$, $\rho_{ji}^{(2)} \circ \zeta_p^j = \zeta_p^i$. Note that

$$\begin{aligned}(\rho_{ji}^{(2)} \circ \zeta_p^j)'(t) &= \rho_{ji}^{(2)} \circ (\zeta_p^j)'(t) = \rho_{ji}^{(2)} \circ [A_j(t)](\zeta_p^j(t)) \\ &= [\rho_{ji}^{(2)} \circ A_j(t)](\zeta_p^j(t)) = [A_i(t) \circ \rho_{ji}^{(2)}](\zeta_p^j(t)) \\ &= [A_i(t)](\zeta_p^j(t)).\end{aligned}$$

Moreover, for any $i \in \mathbb{N}$, $\zeta_p^i(t_0) = f_0^i$ and $(\rho_{ji}^{(2)} \circ \zeta_p^j)(t_0) = \rho_{ji}^{(2)}(f_0^j) = f_0^i$ for $j \geq i$. Based on the existence and uniqueness theorem for ordinary differential equations on Banach spaces with initial conditions; $\rho_{ji}^{(2)} \circ \zeta_p^j = \zeta_p^i$ and hence $\zeta_p = \varprojlim \zeta_p^i$ exists. Consequently,

$$\zeta_p'(t) = (\zeta_p^i(t))_{i\in\mathbb{N}} = (A_i(t))(\zeta_p^i(t))_{i\in\mathbb{N}} = A(t)\zeta_p(t)$$

is the unique solution of the desired differential equation. Moreover the solution ζ_p is not trivial (i.e. its domain is not the single point t_0) since according to the assumption and [9] we have $\zeta_p : (t_0 - \epsilon, t_0 + \epsilon) \longrightarrow \mathbb{E}^{(2)}$. By similar calculation we deduce that $\zeta_p = \varprojlim \zeta_p^i$ is also the unique horizontal global section of $\tilde{\nabla}$ as a projective limit of global sections. □

Let $\tilde{\nabla} = \varprojlim \tilde{\nabla}$ and $\tilde{\nabla}' = \varprojlim \tilde{\nabla}'$ be two special second-order connections over L, such that for each $i \in \mathbb{N}$, $\tilde{\nabla}_i$ and $\tilde{\nabla}_i'$ are g_i-related on L_i and $g = \varprojlim g_i$.

Theorem 6.2. *Two special second-order connections $\tilde{\nabla} = \varprojlim \tilde{\nabla}$ and $\tilde{\nabla}' = \varprojlim \tilde{\nabla}'$ on L are $(g, id_{\mathbb{R}})$-related if and only if their corresponding differential equations given by $dx/dt = A(t)x$ and $dy/dt = A(t)y$, are equivalent i.e. there exists a smooth transformation $Q : \mathbb{R} \longrightarrow \mathcal{H}^0(\mathbb{F})$ such that $x(t) = Q(t)y(t)$ or equivalently,*

$$C(t) = Q^{-1}(t) \circ (A(t) \circ Q(t) - \dot{Q}(t))$$

for each $t \in \mathbb{R}$.

Proof. By [19], $\tilde{\nabla}_i$ and $\tilde{\nabla}_i'$ are g_i-related on L_i if and only if $dx_i/dt = A_i(t)x_i$ and $dy_i/dt = A_i(t)y_i$, are equivalent i.e.

$$C_i(t) = Q_i^{-1}(t) \circ (A_i(t) \circ Q_i(t) - \dot{Q}_i(t)) : \quad i \in \mathbb{N}, \tag{6.1}$$

where $Q = \epsilon \circ Q^*$, $Q^* = (Q_i)_{i\in\mathbb{N}}$ and ϵ is the natural morphism

$$\epsilon : \mathcal{H}^0 \longrightarrow \mathcal{L}(\mathbb{F})$$
$$(l_i)_{i\in\mathbb{N}} \longmapsto \varprojlim l_i.$$

Therefore, (6.1) implies that $\tilde{\nabla}_i$ and $\tilde{\nabla}'_i$ are g_i-related if and only if $x_i(t) = Q_i(t)y_i(t)$. □

In other branches like [7] and [14] we may find applications for our theory.

7 Appendix: Existence and uniqueness theorems for solution of Ordinary Differential Equations on Fréchet spaces

Let $\mathbb{F} = \varprojlim \mathbb{E}_i$ be a Fréchet space and $\{\| \, . \, \|_i\}_{i\in\mathbb{N}}$ be a countable family of seminorms which determine the topology of $\mathbb{F}$ and for any $i \in \mathbb{N}$;

$$\rho_i : \mathbb{F} \longrightarrow \mathbb{E}_i \ ; \quad (x_i)_{i\in\mathbb{N}} \longmapsto x_i$$

be the canonical projection.

Another useful criterion to determine the lifetime of solutions, as well as the uniqueness and differentiability of a solution with respect to the initial value, in Fréchet spaces is due to Nel [15]. We start with two modified definitions from [15].

Definition 7.1. Let $\mathbb{F} = \varprojlim \mathbb{E}_i$ be a Fréchet space and $a \in \mathbb{F}$. For any $i \in \mathbb{N}$ and $\epsilon > 0$ put $a_i := \rho_i(a)$ and

$$B(a_i, \epsilon) := \{w \in \mathbb{E}_i \text{ s.t. } \| w - a_i \|_i < \epsilon\}; \quad i \in \mathbb{N}.$$

Then $B(a, \epsilon) := \varprojlim B(a_i, \epsilon)$ is called a quasiball in $\mathbb{F}$ centered at a and with radius ϵ.

Definition 7.2. Let $B(a, \epsilon_1) \subseteq \mathbb{F}$ and $B(\bar{a}, \epsilon_2) \subseteq \bar{\mathbb{F}}$ be two quasiballs and $g : B(a, \epsilon_1) \times B(\bar{a}, \epsilon_2) \longrightarrow \bar{\mathbb{F}}$ be a projective limit map. g is called a **projective limit-Lipschitz** (abbrev. PLL) in its second variable if there exist positive constants μ and ν s.t.

$$\| g_i(u_i, x_i) - g_i(u_i, y_i) \|_i \leq \mu \| x_i - y_i \|_i \quad \text{and} \quad \| g_i(u_i, x_i) \|_i \leq \nu$$

for all $i \in \mathbb{N}$, $u_i \in B(a_i, \epsilon_1)$ and $x_i, y_i \in B(a_i, \epsilon_2)$.

Now according to theorem 4b of [15] we can state the following (modified) uniqueness and existence theorem for ordinary differential equations on Fréchet spaces.

Theorem 7.1. *Let $\mathbb{F} = \varprojlim \mathbb{E}_i$, $B(a, \epsilon_1) \subseteq \mathbb{F}$ be a quasiball and $I = (-\epsilon, \epsilon)$ with ϵ, $\epsilon_1 > 0$. If $g : I \times B(a, \epsilon_1) \longrightarrow \mathbb{F}$ is a PLL map in its second variable then the initial value problem*

$$x'(t) = g(t, x(t)) \tag{7.1}$$

has a unique solution on $(-\bar{\epsilon}, \bar{\epsilon}) \subseteq (-\epsilon, \epsilon)$ which is continuous on the initial value and $0 < \bar{\epsilon} < min\{\epsilon, \frac{1}{\mu}, \frac{\epsilon}{2\nu + 2\epsilon\mu}\}$.

With the same method proposed in [2] this theorem can be proved for second order ordinary differential equations.

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