Hypersurfaces of a Riemannian manifold with Killing vector field

Haila Alodan

Abstract. In this paper we consider a Riemannian manifold $(\overline{M}, g)$ with a Killing vector field ξ and study the question of obtaining conditions under which its compact hypersurfaces admit a Killing vector field. The presence of the Killing vector field ξ on $\overline{M}$, naturally gives rise to two special vector fields μ, ν, a smooth function ρ and a $(1,1)$-tensor field ψ on a compact hypersurface M of $\overline{M}$. We prove that if the scalar curvatures of M and $\overline{M}$ are equal, $\overline{M}$ has positive Ricci curvature and the smooth function ρ is a constant then μ is a Killing vector field on the compact hypersurface M. Our other result states that if M is totally umbilical hypersurface such that the Rcci curvature of $\overline{M}$ in the direction of the unit normal vector field is a positive constant, then ξ is tangential to M and is therefore Killing vector field on M.

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1 Introduction

A vector field on a Riemannian manifold whose flow preserves the metric is called a Killing vector field. Flows generated by Killing vector fields are isometries of the manifold and moving each point on an object in the direction of the Killing vector field will not distort distances on the object. Killing vector fields play an important role in the geometry as well as topology of a Riemannian manifold, for instance it is known that the presence of a Killing vector field on a compact Riemannian manifold does not allow it to have all its Ricci curvatures to be negative. Also it is known that on an even dimensional positive curved Riemannian manifold a Killing vector field must have a zero. Recently, Berestovskii and Nikonorov (cf. [2]) have studied Riemannian manifolds admitting nontrivial Killing vector fields of constant length and obtained interesting results. The notion of Killing vector fields is generalized to 2-Killing vector fields in (cf. [9]). Recall that, to study geometric properties of a Riemannian manifold, a convenient way is to isometrically immerse it into some known Riemannian manifold as hypersurface (if it is possible) and then analyze the induced geometry. This method has been used to characterize spheres among compact

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connected Riemannian manifolds in (cf. [2], [4], [5], [6]). In this paper we shall consider a Riemannian manifold $(\overline{M}, g)$ that has a Killing vector field ξ and find conditions under which a compact hypersurface M of the Riemannian manifold $(\overline{M}, g)$ also possesses a Killing vector field. The main theorems of this paper are the following:

Theorem 1.1. *Let $(\overline{M}, g)$ be an $n+1$-dimensional Riemannian manifold of positive Ricci curvature and ξ be a Killing vector field on $\overline{M}$. If M is an orientable compact hypersurface of $\overline{M}$ with unit normal vector field N, such that $g(\xi, N)$ is a constant and $S = \overline{S}$, then M admits a Killing vector field, where S and $\overline{S}$ are the scalar curvatures of M and $\overline{M}$ respectively.*

Theorem 1.2. *Let $(\overline{M}, g)$ be an $n+1$-dimensional Riemannian manifold with Killing vector field ξ. If M is a totally umbilic hypersurface of $\overline{M}$ such that Ricci curvature of M is in the direction of unit normal is a positive constant, then ξ is tangent to M and it is Killing vector field on M.*

2 Preliminaries

Let $(\overline{M}, g)$ be an $(n+1)$-dimensional Riemannian manifold with Riemannian connection $\overline{\nabla}$, curvature tensor $\overline{R}$, Ricci curvature tensor $\overline{Ric}$ and scalar curvature $\overline{S}$. Suppose that there is a unit Killing vector field $\xi \in \mathfrak{X}(\overline{M})$, that is

$$(\mathfrak{L}_\xi g)(X, Y) = g(\overline{\nabla}_X \xi, Y) + g(\overline{\nabla}_Y \xi, X) = 0, \quad X, Y \in \mathfrak{X}(\overline{M}).$$

Now suppose that M is an orientable hypersurface of $\overline{M}$ with unit normal N and Weingarten map A. We denote the induced metric on M by the same letter g and the Riemannian connection by ∇.Then we have the following fundamental equations of Gauss and Weingarten,

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \overline{\nabla}_X N = -AX, \quad X, Y \in \mathfrak{X}(M), \tag{2.1}$$

and the Gauss formula

$$R(X, Y)Z = \overline{R}(X, Y)Z + g(AY, Z)AX - g(AX, Z)AY, \tag{2.2}$$

which gives

$$Ric(Y, Z) = \overline{Ric}(Y, Z) - \overline{R}(N, Y, Z, N) + n\alpha g(AY, Z) - g(AY, AZ), \tag{2.3}$$

and

$$S = \overline{S} - 2\overline{Ric}(N, N) + n^2\alpha^2 - \|A\|^2 . \tag{2.4}$$

Also, we have the following formula for hypersurface

$$\overline{R}(X, Y)N = (\nabla A)(Y, X) - (\nabla A)(X, Y). \tag{2.5}$$

Let η be 1-form dual to ξ, then define a a skew-symmetric operator $\varphi : \mathfrak{X}(\overline{M}) \to \mathfrak{X}(\overline{M})$ by

$$d\eta(X, Y) = 2g(\varphi(X), Y), \quad X \in \mathfrak{X}(\overline{M}). \tag{2.6}$$

Then using the Koszul's formula in the form

$$2g(\overline{\nabla}_X\xi, Y) = (\mathfrak{L}_\xi g)(X,Y) + d\eta(X,Y), \quad X,Y \in \mathfrak{X}(\overline{M})$$

we get

$$\overline{\nabla}_X\xi = \varphi(X), \quad X \in \mathfrak{X}(\overline{M}). \tag{2.7}$$

We can express the restriction of $\xi \in \mathfrak{X}(\overline{M})$ to M as

$$\xi = \mu + \rho N, \tag{2.8}$$

where $\mu \in \mathfrak{X}(M)$ and $\rho = g(\xi, N)$ is a smooth function on M. Also, we have

$$\varphi(X) = \psi(X) + \alpha(X)N, \quad X \in \mathfrak{X}(M), \tag{2.9}$$

where $\psi(X) = (\varphi(X))^T$ is the tangential component of φX and $\alpha(X) = g(v, X)$, where $\varphi(N) = -v$. Note that $g(\varphi(N), N) = 0$ as φ is a skew-symmetric operator which implies that $v \in \mathfrak{X}(M)$.

Next, to find the properties of these two special vector fields $\xi^T = \mu \in \mathfrak{X}(M)$ and $v = -\varphi(N) \in \mathfrak{X}(M)$ and the smooth function $\rho = g(\xi, N)$, which are summarized in the following Lemmas which are used in proving our results.

Lemma 2.1. *Let $(\overline{M}, g)$ be an $n+1$-dimensional Riemannian manifold and ξ be a Killing vector field on $\overline{M}$. If M is a hypersurface of $\overline{M}$, then*

$$\nabla_X\mu = \psi(X) + \rho AX, \quad X \in \mathfrak{X}(M), \tag{2.10}$$

and

$$\nabla\rho = v - A\mu, \tag{2.11}$$

where $\nabla\rho$ is the gradient of the smooth function ρ on M.

Proof. Observe that

$$\overline{\nabla}_X\xi = \varphi(X) = \overline{\nabla}_X\mu + \overline{\nabla}_X\rho N,$$

which together with equations (2.7), (2.8) and (2.9), gives

$$\psi(X) + \alpha(X)N = \nabla_X\mu - \rho AX + g(AX, \mu)N + X(\rho)N.$$

Equating normal and tangential components of the above equation, we get the desired results. □

Lemma 2.2. *Let $(\overline{M}, g)$ be an $n+1$-dimensional Riemannian manifold and ξ be a Killing vector field on $\overline{M}$. If M is a hypersurface of $\overline{M}$, then*

$$\overline{Ric}(Y,\xi) = -\sum_{i=1}^{n+1} g(Y, (\overline{\nabla}\varphi)(e_i, e_i)), Y \in \mathfrak{X}(\overline{M}) \tag{2.12}$$

and

$$\overline{Q}(\xi) = -\sum_{i=1}^{n+1} (\overline{\nabla}\varphi)(e_i, e_i), \tag{2.13}$$

where $\{e_1, e_2, ..., e_{n+1}\}$ is a linear orthonormal frame on $\overline{M}$ and $\overline{Q}$ is the Ricci operator on $\overline{M}$ satisfying $\overline{Ric}(X,Y) = \overline{g}(\overline{Q}X, Y)$, $X, Y \in X(\overline{M})$.

Proof. To prove (2.12), consider

$$\overline{R}(X,Y)\xi = \overline{\nabla}_X\overline{\nabla}_Y\xi - \overline{\nabla}_Y\overline{\nabla}_X\xi - \overline{\nabla}_{[X,Y]}\xi$$

and then use (2.7) to get that

$$\overline{R}(X,Y)\xi = (\overline{\nabla}\varphi)(X,Y) - (\overline{\nabla}\varphi)(Y,X).$$

Next use the above equation together with a local orthonormal frame $\{e_{1,}e_2,...,e_{n+1}\}$ on $\overline{M}$ to compute the desired value for $\overline{Ric}(Y,\xi)$. The other part of the lemma is obvious. ☐

Lemma 2.3. *Let $(\overline{M},g)$ be an $n+1$-dimensional Riemannian manifold and ξ be a Killing vector field on $\overline{M}$. If M is a hypersurface of $\overline{M}$, then*

$$div\,\mu = n\rho\alpha, \tag{2.14}$$

$$div\ v = -\overline{Ric}(\xi,N), \tag{2.15}$$

and

$$div\,A\mu = \rho\,\|A\|^2 + \sum_i g(\mu,(\nabla A)(e_i,e_i)), \tag{2.16}$$

where $\{e_{1,}e_2,...,e_n\}$ is a local orthonormal frame on M.

Proof. To prove the first equation, we use (2.10) and the fact that $\psi(X) = (\varphi(X))^T$ is a skew-symmetric operator and we immediately get the result. To prove the second equation, we use that $\varphi(N) = -v$ and compute

$$\begin{aligned}\text{div } v &= \sum_i e_i g(\varphi(e_i),N)\\ &= \sum_i g(\overline{\nabla}_{e_i}\varphi(e_i),N) - g(\varphi(e_i),Ae_i).\end{aligned}$$

Now, for a local orthornormal frame $\{e_{1,}e_2,...,e_n\}$ on M that diagonalizes A such that $Ae_i = \lambda_i e_i$, using the fact that φ is a skew-symmetric operator, we conclude that

$$\sum_i g(\varphi(e_i),Ae_i) = \sum_i \lambda_i g(\varphi(e_i),e_i) = 0,$$

which implies that

$$\begin{aligned}\text{div } v &= \sum_i g((\overline{\nabla}\varphi)(e_i,e_i) + \varphi(\overline{\nabla}_{e_i}e_i),N)\\ &= \sum_i g((\overline{\nabla}\varphi)(e_i,e_i),N) + \sum_i g(\varphi(g(Ae_i,e_i)N,N)\end{aligned}$$

Also, using the fact that $g(\varphi(N),N) = 0$, as φ is a skew-symmetric operator, we get

$$\text{div } v = \sum_i g((\overline{\nabla}\varphi)(e_i,e_i),N).$$

Now if $\{e_{1,}e_2, ..., e_n\}$ is a local orthonormal frame on M, it gives $\{e_{1,}e_2, ..., N\}$ a local orthonormal frame on $\overline{M}$ and using (2.13) in the previous equation yields

$$\operatorname{div} v = g(\overline{Q}(\xi), N) - g((\overline{\nabla}\varphi)(N, N), N). \tag{2.17}$$

Note that on $\overline{M}$ as φ is a skew-symmetric operator, we get

$$g((\overline{\nabla}\varphi)(X, Y), Z) = -g(Y, (\overline{\nabla}\varphi)(X, Z)) \quad X, Y, Z \in \mathfrak{X}(\overline{M}).$$

Now using $X = Y = Z = N$ in the above equation we get

$$g((\overline{\nabla}\varphi)(N, N), N) = 0$$

and using this in (2.17) proves the second equation. Finally, to prove the last equation, we have

$$\operatorname{div} A\mu = \sum_i e_i g(A\mu, e_i) = \sum_i g(\nabla_{e_i}\mu, Ae_i) + g(\mu, \nabla_{e_i} Ae_i),$$

which together with equation (2.10) gives

$$\operatorname{div} A\mu = \sum_i (g(\psi(e_i), Ae_i) + g(\mu, (\nabla A)(e_i, e_i))) + \rho \|A\|^2 .$$

Note that on $\overline{M}$, as $\psi(X) = (\varphi(X))^T$ is a skew-symmetric operator, we get

$$\operatorname{div} A\mu = \sum_i g(\mu, (\nabla A)(e_i, e_i))) + \rho \|A\|^2 ,$$

which completes the proof. □

3 Integral formulas

Lemma 3.1. *Let $(\overline{M}, g)$ be an $n+1$-dimensional Riemannian manifold and ξ be a Killing vector field on $\overline{M}$. If M is a hypersurface of $\overline{M}$, then*

$$\Delta\rho = -\rho\overline{Ric}(N, N) - \rho \|A\|^2 - n\mu(\alpha), \tag{3.1}$$

where Δ is the Laplacian operator on the hypersurface M.

Proof. Note that $\Delta\rho = \operatorname{div} \nabla\rho$ and by using (2.11), we get

$$\Delta\rho = \operatorname{div} v - \operatorname{div} A\mu,$$

which together with (2.15) and (2.16), gives

$$\Delta\rho = -\overline{Ric}(\xi, N) - \sum_i g(\mu, (\nabla A)(e_i, e_i))) - \rho \|A\|^2 . \tag{3.2}$$

We have,

$$nX(\alpha) = \sum_i g((\nabla A)(X, e_i), e_i) = \sum_i g((\nabla A)(e_i, X) - \overline{R}(X, e_i)N, e_i),$$

which implies that

$$ng(X,\alpha) = \sum_i g(X, (\nabla A)(e_i, e_i)) - \sum_i \overline{R}(e_i, X, N, e_i).$$

Taking a local orthonormal frame $\{e_1, e_2, ..., e_n\}$ on M so that $\{e_1, e_2, ..., N\}$ is a local orthonormal frame on $\overline{M}$, we get

$$\overline{Ric}(X,N) = \sum_i \overline{R}(e_i, X, N, e_i) + \overline{R}(N, X, N, N)$$

and since

$$ng(X,\alpha) = \sum_i g(X, (\nabla A)(e_i, e_i)) + \overline{Ric}(X, N),$$

we arrive at

$$n\mu(\alpha) = \sum_i g(\mu, (\nabla A)(e_i, e_i)) + \overline{Ric}(\mu, N)$$

and thus substituting the above equation in (3.2) we have

$$\Delta\rho = -\overline{Ric}(\xi, N) - n\mu(\alpha) - \rho \|A\|^2 + \overline{Ric}(\mu, N).$$

Now, substituting $\mu = \xi - \rho N$ in the above equation proves (3.1). □

Lemma 3.2. *Let $(\overline{M}, g)$ be an $n+1$-dimensional Riemannian manifold and ξ be a Killing vector field on $\overline{M}$. If M is a compact hypersurface of $\overline{M}$, then*

$$\int_M \left\{\rho\left(\|A\|^2 - n^2\alpha^2 + \overline{Ric}(N,N)\right)\right\} dv = 0. \tag{3.3}$$

Proof. Note that

$$\mu(\alpha) = \operatorname{div}(\alpha\mu) - \alpha \operatorname{div}\mu$$

and using equation (2.14) in the above equation we get

$$\mu(\alpha) = \operatorname{div}(\alpha\mu) - n\rho\alpha^2,$$

which gives

$$\Delta\rho - \operatorname{div}(n\alpha\mu) = -\rho\overline{Ric}(N,N) - \rho\|A\|^2 + \rho n^2\alpha^2.$$

Integrating the above equation gives the result. □

Lemma 3.3. *Let $(\overline{M}, g)$ be an $n+1$-dimensional Riemannian manifold and ξ be a Killing vector field on $\overline{M}$. If M is a totally umbilical hypersurface of $\overline{M}$, then*

$$\int_M \left\{g(\mu, v) - \alpha\|\mu\|^2 + n\alpha\rho^2\right\} dv = 0. \tag{3.4}$$

Proof. Using (2.11), (2.14) and the fact that M is a totally umbilical hypersurface of $\overline{M}$, that is, $A = \alpha I$, where α is a constant, to find the $\operatorname{div}(\rho\mu)$, we get

$$\operatorname{div}(\rho\mu) = g(v,\mu) - \alpha\|\mu\|^2 + n\alpha\rho^2.$$

Integrating the above equation gives the result. □

4 Main results

In this section we establish our main theorems stated in the introduction. Precisely, we start with the first main theorem.

Theorem 4.1. *Let $(\overline{M}, g)$ be an $n+1$-dimensional Riemannian manifold of positive Ricci curvature and ξ be a Killing vector field on $\overline{M}$. If M is a compact hypersurface of $\overline{M}$ such that $\rho = g(\xi, N)$ is a constant and $S = \overline{S}$, then M admits a Killing vector field.*

Proof. We rearrange equation (2.4) to get

$$\overline{Ric}(N,N) - n^2\alpha^2 + \|A\|^2 = \overline{S} - S - \overline{Ric}(N,N),$$

which, when substituted in the integral formula of Lemma 3.2 gives

$$\int_M \left\{\rho\left(\overline{S} - S - \overline{Ric}(N,N)\right)\right\} dv = 0.$$

Using $\rho = g(\xi, N)$ is a constant and $S = \overline{S}$, we get $\rho\int_M \overline{Ric}(N,N)dv = 0$, and since $\overline{Ric}(N,N) > 0$, above equation implies that $\rho = 0$ which shows that $\xi = \mu$ and $\nabla_X\mu = \psi(X)$, therefore $\mathfrak{L}_\mu g = 0$, which proves μ a Killing vector field of M. □

Next we state our second theorem.

Theorem 4.2. *Let $(\overline{M}, g)$ be an $n+1$-dimensional Riemannian manifold with Killing vector field ξ. If M is a totally umbilic hypersurface of $\overline{M}$, such that Ricci curvature of $\overline{M}$ in the direction of unit normal is a positive constant, then ξ is tangent to M and it is Killing vector field on M.*

Proof. Note that since M is a totally umbilical hypersurface of $\overline{M}$ then $A = \alpha I$, where α is a constant, this implies that $(\nabla A)(X,Y) = 0$, $X, Y \in \mathfrak{X}(M)$. Using this in (2.5) gives $\overline{R}(X,Y)N = 0$, $X, Y \in \mathfrak{X}(M)$, and this implies $\overline{Ric}(\mu, N) = 0$, $\|A\|^2 = n\alpha^2$ and $\mu(\alpha) = 0$. Using the above equation and (3.1) to calculate $\Delta\rho$, we get $\Delta\rho = -\rho\overline{Ric}(N,N) - n\rho\alpha^2$. If the Ricci curvature $\overline{Ric}$ of $\overline{M}$ in the direction of N is a constant c, then we have

$$\Delta\rho = -(n\alpha^2 + c)\rho. \tag{4.1}$$

This gives $\rho\Delta\rho = -(n\alpha^2 + c)\rho^2$, which implies that

$$\frac{1}{2}\Delta\rho^2 + \|\nabla\rho\|^2 = -(n\alpha^2 + c)\rho^2.$$

Now, since

$$\|\nabla\rho\|^2 = \|v - \alpha\mu\|^2 = \|v\|^2 + \alpha^2\|\mu\|^2 - 2\alpha g(\mu, v), \tag{4.2}$$

substituting the value of $\|\nabla\rho\|^2$ in (4.2) in the previous equation and then using the second integral formula (3.4) to get

$$\int_M \left\{\|v\|^2 - \alpha^2\|\mu\|^2 + (3n\alpha^2 + c)\rho^2\right\} dv = 0. \tag{4.3}$$

Also the equation (4.1) gives $\int_M \|\nabla\rho\|^2 dv = (n\alpha^2 + c)\int_M \rho^2 dv$, which implies

$$\int_M \left\{ \|v\|^2 + \alpha^2 \|\mu\|^2 - 2\alpha g(\mu, v) - (n\alpha^2 + c)\rho^2 \right\} dv = 0,$$

where we have used (4.2). Now using the second integral formula (3.4) in above equation we get

$$\int_M \left\{ \|v\|^2 - \alpha^2 \|\mu\|^2 + (n\alpha^2 - c)\rho^2 \right\} dv = 0. \tag{4.4}$$

Now subtracting (4.3) from (4.4) we have $\int_M \left\{ (2n\alpha^2 + 2c)\rho^2 \right\} dv = 0$. Since $c > 0$, $2n\alpha^2 + 2c$ is a positive constant and the above equation gives

$$(2n\alpha^2 + 2c)\int_M \rho^2 dv = 0,$$

that is $\rho = 0$ and consequently, ξ is a tangent vector field and $\xi = \mu$ is a Killing vector field on M. □

As a consequence of (4.1), we have the following (for a similar result see [5]).

Corollary 4.3. *Let $(\overline{M}, g)$ be an $n+1$-dimensional Riemannian manifold with Killing vector field ξ. If M is a totally umbilic hypersurface of $\overline{M}$ such that Ricci curvature of M in the direction of unit normal is a constant c, then the first nonzero eigenvalue λ_1 of the Laplacian operator Δ on M satisfies $\lambda_1 \leq 2(n\alpha^2 + c)$.*

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Author's address:

Haila Alodan
Department of Mathematics, King Saud University,
P.O. Box 2455, Riyadh 11451, Saudi Arabia,
E-mail: halodan1@ksu.edu.sa