Conformal vector fields on a Kähler manifold

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Abstract. For a $2n$-dimensional Kähler manifold (M, J, g), using a conformal transformation $\varphi : M \rightarrow M$, $\overline{g} = \varphi^* g = e^{-2f} g$, we obtain a condition under which $(M, J, \overline{g})$ is a Kähler manifold and show that in general $(M, J, \overline{g})$ is not a Kähler manifold. The Hermitian manifold $(M, J, \overline{g})$ has a new structure which we call as Kähler-like manifold. We show that a Killing vector fields on the Kähler manifold (M, J, g) are the conformal vector fields on the Kähler-like manifold $(M, J, \overline{g})$. Similarly it is shown that a Killing vector field on the Kähler-like manifold $(M, J, \overline{g})$ is a conformal vector field on the Kähler manifold (M, J, g).

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1 Introduction

Conformal vector fields are important objects for the geometry of several kinds of manifolds. They have been studied quite extensively on Riemannian manifolds (cf. [1], [3], [4], [5], [9], [10], [12]). However, though the Kähler geometry is quite rich, conformal vector fields have not been studied that extensively on a Kähler manifold. This is perhaps due to the result that on a compact Kähler manifold a conformal vector field is Killing (cf. [7], [11]), however non-Killing conformal vector fields on noncompact Kähler manifold are in abundance. For example, consider the Euclidean space $\mathbb{C}^n$ of dimension $2n$ which is a Kähler manifold with natural canonical complex structure J and the position vector field ψ, then the vector field $\xi = \psi + J\psi$ is a conformal vector field on the Kähler manifold $(\mathbb{C}^n, J, \langle , \rangle)$ which is not Killing, where $\langle \cdot\,, \cdot \rangle$ is the Euclidean inner product on $\mathbb{C}^n$.

In this paper, first we obtain a condition on a conformal transformation φ of a Kähler manifold (M, J, g), $\overline{g} = \varphi^* (g) = e^{-2f} g$ that gives rise to the Kähler manifold $(M, J, \overline{g})$ and show that in general $(M, J, \overline{g})$ is not a Kähler manifold and indeed it has new structure which we call as Kähler-like manifold. Then we show that the Killing vector fields on (M, J, g) are conformal vector fields on the Kähler-like manifold $(M, J, \overline{g})$ if φ is not a homothety. Moreover we also show that a Killing vector field on the Kähler-like manifold $(M, J, \overline{g})$ is a conformal vector field on the Kähler manifold (M, J, g).

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2 Preliminaries

Let (M, J, g) be a $2n$-dimensional Kähler manifold with complex structure J and Hermitian metric g. We denote by ∇ the Levi Civita connection and by $\mathfrak{X}(M)$ the Lie-algebra of smooth vector fields on M. Then we have

$$\nabla_X JY = J\nabla_X Y, \quad g(JX, JY) = g(X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.1}$$

The curvature tensor field R and the Ricci tensor field Ric of a Kähler manifold (M, J, g) satisfy

$$R(JX, JY; JZ, JW) = R(X, Y; Z, W), \quad Ric(JX, JY) = Ric(X, Y), \tag{2.2}$$

$X, Y, Z, W \in \mathfrak{X}(M)$. A Kähler manifold of constant holomorphic sectional curvature c is called a complex space form and it is denoted by $M(c)$. The curvature tensor field of a complex space form $M(c)$ has the expression

$$\begin{aligned} R(X, Y)Z &= \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\ &- g(JX, Z)JY + 2g(X, JY)JZ\}. \end{aligned} \tag{2.3}$$

Recall that a smooth vector field $\xi \in \mathfrak{X}(M)$ is said to be a conformal vector field on M if its one-parameter group of transformations $\{\phi_t\}$ consists of conformal transformations ϕ_t of M, that is $\phi_t^* g = e^{-2f} g$ for a smooth function $f : M \to R$, for each $t \in R$. Moreover $\xi \in \mathfrak{X}(M)$ is a conformal vector field if and only if

$$\pounds_\xi(g) = 2hg, \tag{2.4}$$

for a smooth function $h : M \to R$, where $\pounds_\xi$ is the Lie derivative with respect to ξ. We call the smooth function h associated to conformal vector field ξ in above definition the potential function of ξ. If ξ is a conformal vector field on a Kähler manifold (M, J, g) and η is a 1- form dual to ξ, we define a skew-symmetric tensor field ϕ of type $(1, 1)$ on M by

$$d\eta(X, Y) = 2g(\phi X, Y), \quad X, Y \in \mathfrak{X}(M).$$

Then we have the following:

Lemma 2.1. *(cf. [5]) Let ξ be a conformal vector field on a Kähler manifold (M, J, g) with potential function h. Then*

$$\nabla_X \xi = hX + \phi X, \quad X \in \mathfrak{X}(M).$$

It immediately follow from above Lemma that the curvature tensor of the Kähler manifold satisfies

$$R(X, Y)\xi = X(h)Y - Y(h)X + (\nabla_X \phi)(Y) - (\nabla_Y \phi)(X), \quad X, Y \in \mathfrak{X}(M), \tag{2.5}$$

where $(\nabla_X \phi)(Y) = \nabla_X \phi Y - \phi(\nabla_X Y)$, consequently choosing a local orthonormal frame $\{e_1, ..., e_{2n}\}$, and using the fact that ϕ is skew-symmetric and $\sum g(\phi e_i, e_i) = 0$ in the equation (2.5), we get the following relation

$$Ric(X, \xi) = -(2n-1)X(h) - g\left(X, \sum_{i=1}^{2n} (\nabla_{e_i}\phi)(e_i)\right).$$

The Ricci operator Q is a symmetric $(1,1)$ tensor field defined by $Ric(X,Y) = g(Q(X), Y)$, $X, Y \in \mathfrak{X}(M)$. Thus above equation gives

$$Q(\xi) = -(2n-1)\nabla h - \sum_{i=1}^{2n} (\nabla_{e_i}\phi)(e_i), \tag{2.6}$$

where ∇h is the gradient of the potential function h.

3 Conformal transformations

Let (M, J, g) be a $2n$-dimensional Kähler manifold. If $\varphi : M \to M$ is a conformal transformation of (M, J, g), then in new metric $\overline{g} = e^{-2f}g$ which is a Hermitian metric, the manifold $(M, J, \overline{g})$ need not be Kähler manifold as the complex structure J need not be parallel with respect to the Riemannian connection $\overline{\nabla}$ corresponding to the metric $\overline{g}$. In this section first we find conditions under which $(M, J, \overline{g})$ is a Kähler manifold. First we define a curvature-like tensor field R_0 on a Riemannian manifold (M, g) by

$$R_0(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \quad X, Y, Z \in \mathfrak{X}(M).$$

It is well known that if ∇ and $\overline{\nabla}$ are Riemannian connections with respect to the conformally related Riemannian metrics $g, \overline{g}$; $\overline{g} = e^{-2f}g$, then

$$\overline{\nabla}_X Y = \nabla_X Y - \{X(f)Y + Y(f)X - g(X,Y)\nabla f\}, \quad X, Y \in \mathfrak{X}(M), \tag{3.1}$$

holds, where ∇f is the gradient of the function f with respect to the metric g.

Lemma 3.1. *Let $\varphi : M \to M$ be a conformal transformation of a Kähler manifold (M, J, g), with $\overline{g} = \varphi^* g = e^{-2f}g$ for a smooth function $f : M \to R$. Then*

$$\left(\overline{\nabla}_X J\right)(Y) = J\left(\overline{R}_0(X, \overline{\nabla} f)Y\right) - \overline{R}_0(X, \overline{\nabla} f)JY, \quad X, Y \in \mathfrak{X}(M),$$

where $\overline{\nabla} f$ is the gradient field of f with respect to the metric $\overline{g}$.

Proof. Let $\overline{\nabla} f$ be the gradient field of the function f with respect to the metric $\overline{g}$. Then it is easy to see that $\overline{\nabla} f = e^{2f}\nabla f$. Using equation (3.1) we compute

$$\begin{aligned}
\left(\overline{\nabla}_X J\right)(Y) &= \overline{\nabla}_X JY - J(\overline{\nabla}_X Y) \\
&= (\nabla_X J)(Y) - X(f)JY - JY(f)X + g(X, JY)\nabla f \\
&+ X(f)JY + Y(f)JX - g(X,Y)J\nabla f \\
&= -\overline{g}(\overline{\nabla} f, JY)X + \overline{g}(X, JY)\overline{\nabla} f + \overline{g}(\overline{\nabla} f, Y)JX - \overline{g}(X,Y)J\overline{\nabla} f \\
&= -\overline{R}_0(X, \overline{\nabla} f)JY + J\left(\overline{R}_0(X, \overline{\nabla} f)Y\right),
\end{aligned}$$

which proves the result. ☐

Recall that a conformal transformation $\varphi : M \to M$ of a Kähler manifold (M, J, g) with $\overline{g} = \varphi^* g = e^{-2f} g$ for a smooth function $f : M \to R$ is said to be homothety if the function f is a constant. Our next result is the following:

Theorem 3.2. *Let $\varphi : M \to M$ be a conformal transformation of a $2n$-dimensional connected Kähler manifold (M, J, g), with $\overline{g} = \varphi^* g = e^{-2f} g$ for a smooth function $f : M \to R$. Then $(M, J, \overline{g})$ is a Kähler manifold if either $n = 1$ or φ is a homothety.*

Proof. If $(M, J, \overline{g})$ is a Kähler manifold, then by Lemma 3.1, we have

$$\overline{R}_0(X, \overline{\nabla} f) JY = J\left(\overline{R}_0(X, \overline{\nabla} f) Y\right).$$

Taking a local orthonormal frame $\{e_1, ..., e_{2n}\}$ on $(M, J, \overline{g})$, choosing $X = e_i$ and taking inner product with e_i in above equation with respect to $\overline{g}$ and summing these $2n$ equations we arrive at

$$2(n-1)\overline{g}(\overline{\nabla} f, Y) = 0, \quad Y \in \mathfrak{X}(M),$$

which gives either $n = 1$ or else $\overline{\nabla} f = 0$. Since M is connected we get either $n = 1$ or else f is a constant. This proves the result. □

The above result suggests that if the conformal transformation $\varphi : M \to M$ of a Kähler manifold (M, J, g) with $\overline{g} = \varphi^* g = e^{-2f} g$ is not a homothety and $\dim M > 2$, then $(M, J, \overline{g})$ is not a Kähler manifold.

Definition 3.1. A Hermitian manifold (M, J, g) is said to be Kähler-like manifold if it satisfies

$$(\nabla_X J)(JY) + (\nabla_{JX} J)(Y) = 0, \quad X, Y \in \mathfrak{X}(M),$$

where ∇ is the Levi-Civita connection with respect to the Hermitian metric g.

We do not find the study of this class of Hermitian manifolds in the literature, though there are many examples of these manifolds as seen in the following.

Theorem 3.3. *Let $\varphi : M \to M$ be a conformal transformation of a $2n$-dimensional Kähler manifold (M, J, g), with $\overline{g} = \varphi^* g = e^{-2f} g$ for a smooth function $f : M \to R$. Then $(M, J, \overline{g})$ is a Kähler-like manifold.*

Proof. Using Lemma 3.1, we compute

$$(\nabla_X J)(JY) = J\left(\overline{R}_0(X, \overline{\nabla} f) JY\right) + \overline{R}_0(X, \overline{\nabla} f) Y,$$

and

$$(\nabla_{JX} J)(Y) = J\left(\overline{R}_0(JX, \overline{\nabla} f) Y\right) - \overline{R}_0(JX, \overline{\nabla} f) JY.$$

Adding these two equations and using the definition of $\overline{R}_0$, we get the result. □

4 Conformal vector fields

Let (M, J, g) be a Kähler manifold and $\varphi : M \to M$ be a conformal transformation that gives the Kähler-like manifold $(M, J, \overline{g})$, $\overline{g} = e^{-2f}g$. In this section we show that there is a duality between Killing and conformal vector fields on the manifolds (M, J, g) and $(M, J, \overline{g})$ respectively. We also show that a Killing vector field on a complex space form $M(c)$ (Kähler manifold of constant holomorphic sectional curvature c) is eigenvector of the Laplacian operator acting on smooth vector fields.

Theorem 4.1. *Let $\varphi : M \to M$ be a non-homothetic conformal transformation of a $2n$-dimensional Kähler manifold (M, J, g), with $\overline{g} = \varphi^* g = e^{-2f}g$ for a smooth function $f : M \to R$. If ξ is a Killing vector field on the Kähler manifold (M, J, g), then ξ is a conformal vector field on the Kähler-like manifold $(M, J, \overline{g})$.*

Proof. We use equation (3.1) to conclude

$$\overline{\nabla}_X \xi = \nabla_X \xi - X(f)\xi - \xi(f)X + g(X, \xi)\nabla f.$$

As f is non-constant function, we define a smooth function $h : M \to R$ by $h = -\overline{g}(\xi, \overline{\nabla} f)$. Now using the fact that ξ is a Killing vector field on the Kähler manifold (M, J, g) and above equation we arrive at

$$\left(\pounds_\xi \overline{g}\right)(X, Y) = 2h\overline{g}(X, Y), \quad X, Y \in \mathfrak{X}(M).$$

This proves that ξ is a conformal vector field on the Kähler-like manifold $(M, J, \overline{g})$. □

Theorem 4.2. *Let $\varphi : M \to M$ be a non-homothetic conformal transformation of a $2n$-dimensional Kähler manifold (M, J, g), with $\overline{g} = \varphi^* g = e^{-2f}g$ for a smooth function $f : M \to R$. If ξ is a Killing vector field on the Kähler-like manifold $(M, J, \overline{g})$, then ξ is a conformal vector field on the Kähler manifold (M, J, g).*

Proof. We use equation (3.1) to conclude

$$\overline{\nabla}_X \xi = \nabla_X \xi - X(f)\xi - \xi(f)X + g(X, \xi)\nabla f.$$

As f is non-constant function, we define a smooth function $h : M \to R$ by $h = g(\xi, \nabla f)$. Now using the fact that ξ is a Killing vector field on the Kähler-like manifold (M, J, g) and above equation we arrive at

$$\left(\pounds_\xi g\right)(X, Y) = 2hg(X, Y), \quad X, Y \in \mathfrak{X}(M).$$

This proves that ξ is a conformal vector field on the Kähler manifold (M, J, g). □

Recently García-Río et. al [6] have initiated the study of Laplacian operator acting on smooth vector fields on a Riemannian manifold and used it to characterize Euclidean spheres. Given a Riemannian manifold (M, g), the Laplacian operator $\Delta : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by

$$\Delta X = \sum_{i=1}^{n} \left(\nabla_{e_i}\nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X\right),$$

where ∇ is the Riemannian connection and $\{e_1, ..., e_n\}$ is a local orthonormal frame on (M, g), $n = \dim M$.

In [6], one of the important tasks was to construct eigenvectors of the Laplacian operator. Here we show that a Killing vector field on a complex space form $M(c)$ is indeed an eigenvector of the Laplacian operator acting on smooth vector field as seen in the following :

Theorem 4.3. *Let ξ be a Killing vector field on a complex space form $M(c)$. Then ξ is an eigenvector of the Laplacian operator acting on smooth vector fields on $M(c)$.*

Proof. Let ξ be a Killing vector field on a complex space form $M(c)$. Then by Lemma 2.1, we have

$$\nabla_X \xi = \phi X, \quad X \in \mathfrak{X}(M), \tag{4.1}$$

where ϕ is a skew-symmetric tensor field of type $(1,1)$ (as $h = 0$ for Killing vector field). Then the equation (2.6) gives

$$Q(\xi) = -\sum_{i=1}^{2n} (\nabla_{e_i}\phi)(e_i), \tag{4.2}$$

where $2n = \dim M$. Now using equation (2.3), we see that the Ricci tensor of $M(c)$ satisfies

$$Ric(X, Y) = \frac{c}{2}(n+1)g(X, Y).$$

Combining this equation with (4.2), we conclude that

$$\sum_{i=1}^{2n} (\nabla_{e_i}\phi)(e_i) = -\frac{c}{2}(n+1)\xi. \tag{4.3}$$

Using the definition of the Laplacian operator acting on smooth vector fields and equation (4.1), we have

$$\Delta\xi = \sum_{i=1}^{2n} \left(\nabla_{e_i}\nabla_{e_i}\xi - \nabla_{\nabla_{e_i}e_i}\xi\right) = \sum_{i=1}^{2n} (\nabla_{e_i}\phi)(e_i).$$

This equation together with the equation (4.3) gives

$$\Delta\xi = -\frac{c}{2}(n+1)\xi,$$

and this proves the result. □

As a direct consequence of the above theorem we have the following :

Corollary 4.4. *On the complex space form $\mathbb{C}^n$ Killing vector fields are harmonic.*

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